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REGULARITY RESULTS FOR A CLASS OF HYPERBOLIC EQUATIONS WITH VMO COEFFICIENTS

MAÎTINE BERGOUNIOUX AND ERICA L. SCHWINDT

Abstract. In this note we show a regularity result for an hyperbolic system with discontinuous coefficients. More precisely, we deal with coefficients in the function space VMO and we prove the existence and uniqueness of a solution $u \in L^\infty(0, T; H^2(\Omega))$ with also suitable regularity for $\frac{\partial^2 u}{\partial t^2}$, $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^3 u}{\partial x^3}$.

1. Introduction

Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$ with $d \geq 3$. In the context of photoacoustic tomography process modelling [1], we are led to study the following wave equation

$$
\begin{cases}
\frac{\partial^2 p}{\partial t^2}(t, x) - \text{div}(v_s^2 \nabla p)(t, x) = f(t, x) & \text{in } (0, T) \times \Omega \\
p(t, x) = 0 & \text{on } (0, T) \times \partial \Omega \\
p(0, x) = \frac{\partial p}{\partial t}(0, x) = 0 & \text{in } \Omega,
\end{cases}
$$

where $p = p(t, x)$ is an acoustic pressure wave, $v_s = v_s(x)$ is the speed of sound, $f$ is a distributed source that comes from a lightning process and $\Omega$ is the domain where the wave propagates. The coefficient $v_s$ is generally unknown and not smooth. We are interested in establishing new results of regularity of the solution $p$ in the case of discontinuous coefficient $v_s$.

Hereafter we will assume that $\partial \Omega$ is of class $C^2$ and we consider the following initial/boundary value problem:

$$
\begin{cases}
\frac{\partial^2 u}{\partial t^2} + Lu = f & \text{in } (0, T) \times \Omega \\
u = 0 & \text{on } (0, T) \times \partial \Omega \\
u(0, x) = u_0, \quad \frac{\partial u}{\partial t}(0, x) = u_1 & \text{in } \Omega,
\end{cases}
$$

where $f : (0, T) \times \Omega \to \mathbb{R}$, $u_0, u_1 : \Omega \to \mathbb{R}$ are given and $L$ denotes a second order partial differential operator in the divergence form:


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(1.2) \[ Lu = - \sum_{i,j=1}^{d} (a_{ij}(x)u_{x_i}) x_j \]

where \( u_{x_i} \) denotes the partial derivative of \( u \) with respect to \( x_i \).

Systems of equations as (1.1) have been extensively studied. Classical results of well-posedness and regularity can be found in [9, §7.2]. In this reference a regularity result similar to our Theorem 3.1 is obtained under coefficient smoothness assumptions, namely \( a_{ij} \in C^1(\Omega) \) and \( \nabla(a_{ij}) \in [C^1(\Omega)]^d \).

In this work, we consider discontinuous coefficients \( a_{ij} \) such that \( a_{ij} \in VMO \cap L^\infty(\Omega) \) and \( \nabla a_{ij} \in [L^p(\Omega)]^d \) with \( p > d \). Assuming the coefficients \( a_{ij} \) belong to \( L^\infty(\Omega) \), it can proved that System (1.1) admits a unique solution \( u \in C^0(0,T; H^1_0(\Omega)) \) with \( \frac{\partial u}{\partial t} \in C^0(0,T; L^2(\Omega)) \) (see first part of the proof of Theorem 3.1).

Roughly speaking, the improved regularity, with respect to space, of the solution \( u \) is associated with the elliptic regularity of the equation for almost every \( t \in [0,T] \), that is, with the regularity of \( Lu(t) = f(t) - \frac{\partial^2 u}{\partial t^2}(t) \). Several regularity results for elliptic operator \( L \) have been obtained with more general elliptic operators of type \( \tilde{L}u = - \sum_{i,j=1}^{d} a_{ij}(t,x)u_{x_i,x_j} + \sum_{i=1}^{d} b_{ij}(t,x)u_{x_i} + c(t,x)u \), and there exists a non-exhaustive list of papers devoted to results of regularity associated with the operator \( \tilde{L} \) with different hypothesis on the coefficients \( a_{ij}, b_{ij} \) and \( c \) (see for example [4, 5, 6, 8, 7, 9, 14, 15] and references therein). Other results for parabolic equations with VMO coefficients can be found in [2, 11].

In Section 2 we introduce some definitions and notations and the variational formulation of System (1.1). Section 3 is devoted to the proof of Theorem 3.1 which is based on the regularity results obtained in [14].

2. Preliminaries

In the sequel, \( L^p(\Omega) \) is the space of measurable functions \( u \) on \( \Omega \) such that \( \int_\Omega |u|^p < +\infty \) for \( 1 \leq p < \infty \). \( L^\infty(\Omega) \) is the space of essentially bounded functions on \( \Omega \). \( C^k(\Omega) \) is the set of all functions \( k \)-times continuously differentiable and its derivates of order \( |\alpha| \) are continuous for all multiindex \( \alpha \) such that \( |\alpha| \leq k \). \( C^\infty_c(\Omega) \) denote the subspace of all functions \( u \) infinitely differentiable with compact support in \( \Omega \). We will denote \( H^k(\Omega) \) the usual Sobolev space of all functions \( u \) such that \( D^\alpha u \) exists in the distributional sense and belongs to \( L^2(\Omega) \) for all multiindex \( \alpha \) with \( |\alpha| \leq k \). The subspace \( H^1_0(\Omega) \) is the closure of \( C^\infty_c(\Omega) \) in \( H^1(\Omega) \) and the subspace \( H^{-1}(\Omega) \) denotes the dual subspace to \( H^1_0(\Omega) \). Let \( X \) be a Banach space: we will denote by \( L^p(0,T;X) \) the space of all measurable functions \( u \) such that \( u : [0,T] \to X \) defined by \( u(t)(x) = u(t,x) \) (by abuse of notation) satisfies

\[
\|u\|_{L^p(0,T;X)} = \left( \int_0^T \|u(t)\|^p_X \, dt \right)^{1/p} < +\infty, \quad \text{if } p \in [1, +\infty)
\]

and

\[
\|u\|_{L^\infty(0,T;X)} = \text{ess sup} \sup_{0 \leq t \leq T} \|u(t)\|_X < +\infty, \quad \text{if } p = +\infty.
\]
The space $W^{1,p}(0,T;X)$ denotes all the functions $u \in L^p(0,T;X)$ such that $\frac{\partial u}{\partial t} \in L^p(0,T;X)$. For simplicity, we will use often the notation $W^{1,p}(X)$ instead of $W^{1,p}(0,T;X)$.

Recall that the partial differential operator $L$ is elliptic if there exists a constant $\kappa > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \kappa |\xi|^2$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^d$. Moreover, we assume

$$a_{ij} = a_{ji}$$

and $0 < a_{\min} \leq a_{ij} \leq a_{\max}$, for all $i,j \in \{1,2,\ldots,d\}$.

so that the operator defined by (1.2) is elliptic.

2.1. Elliptic regularity results. Here, we recall the results obtained in [14]. We first introduce useful functional spaces.

**Definition 2.1.** A function $u$ is a bounded mean oscillation (BMO) function, if $u$ is a real-valued function whose mean oscillation is bounded (finite). This function space is also called John–Nirenberg space. More precisely, we say that a locally integrable function $u$ is a BMO function if

$$\sup_{B} \int_{B} |u(x) - u_B| \, dx =: \|u\|_{\ast} < +\infty$$

where $B$ ranges in the class of the balls of $\mathbb{R}^d$ and $u_B = \int_{B} u(x) \, dx = \frac{1}{|B|} \int_{B} u(x) \, dx$.

If $u$ a BMO function and $r > 0$ we set

$$\eta(r) = \sup_{\rho \leq r} \int_{B_{\rho}} |u(x) - u_{B_{\rho}}| \, dx$$

where $B_{\rho}$ ranges in the class of the balls with radius $\rho$ less than or equal to $r$.

**Definition 2.2.** A function $u$ is a vanishing mean oscillation (VMO) function, if $u$ belongs to the subspace of the BMO functions whose BMO norm over a ball vanishes as the radius of the ball tends to zero:

$$\lim_{r \to 0} \eta(r) = 0.$$
and the associated Dirichlet problem

\begin{equation}
\begin{cases}
\tilde{L}u = \tilde{f} \\
u \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega), \quad \tilde{f} \in L^q(\Omega).
\end{cases}
\tag{2.4}
\end{equation}

**Theorem 2.1.** Let $\Omega$ be $C^{1,1}$. Assume $\bar{a}_{ij} = \bar{a}_{ji}$, $\bar{a}_{ij} \in VMO \cap L^\infty(\Omega)$ and that there exists $\lambda > 0$ such that

$$\forall \xi \in \mathbb{R}^d \quad \lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^{d} \bar{a}_{ij}(x) \xi_i \xi_j \leq \lambda |\xi|^2 \text{ a.e. in } \Omega.$$ 

We also suppose $\bar{b}_i \in L^s(\Omega)$, $s > d$ for $1 < q \leq d$, $s = q$ for $q > d$, and $\bar{c} \in L^r(\Omega)$ with $r = \begin{cases} d & \text{if } 1 < q \leq d \\
q & \text{if } q > d \end{cases}$ and $\bar{c} \leq 0$ a.e. in $\Omega$. Then the Dirichlet problem (2.4) has a unique solution $u$. Furthermore there exists a positive constant $C$ such that

$$\|u\|_{W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)} \leq C\|\tilde{f}\|_{L^q(\Omega)}$$

where the constant $C$ depend on $d$, $\partial \Omega$, $\lambda$, on the VMO modulus of $\bar{a}_{ij}$, on the $L^s$ and $L^d$ norms respectively of $\bar{b}_i$ and $\bar{c}$ and their AC modulus (see [14] for definition of AC modulus).

Here $W^{k,\beta}(\Omega)$ denotes the space of all functions $u$ such that $D^\alpha u \in L^q(\Omega)$ for all multiindex $\alpha$ with $|\alpha| \leq k$ and $1 \leq q \leq +\infty$.

**Remark 2.1.** In this work, we will use Theorem 2.1 with no lower order term ($\bar{c} = 0$).

2.2. **Variational formulation of (1.1).** Let $u \in C^2([0, T] \times \Omega)$ be a classical solution of (1.1), (i.e., $u$ satisfies equation (1.1) at any $(t, x) \in (0, T) \times \Omega$). Multiplying the main equation of (1.1) by $\phi \in C^\infty_c(\Omega)$ and integrate by parts, we obtain

\begin{equation}
\frac{\partial^2 u}{\partial t^2}(t, x) \phi(x) \, dx + \int_{\Omega} a_{ij}(x) \nabla u(t, x) \cdot \nabla \phi(x) \, dx = \int_{\Omega} f(t, x) \phi(x) \, dx
\end{equation}

a.e. $t \in (0, T)$. Hence, from the density of $C^\infty_c(\Omega)$ in $H^1_0(\Omega)$, we have (2.5) for all $\phi \in H^1_0(\Omega)$. Now, we recall the definition of a weak solution for (1.1) (see [9])

**Definition 2.3.** We say a function

$$u \in L^2(0, T; H^1_0(\Omega)) \text{ with } \frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega)) \text{ and } \frac{\partial^2 u}{\partial t^2} \in L^2(0, T; H^{-1}(\Omega))$$

is a weak solution of Problem (1.1) provided (2.5) holds true for all $\phi \in H^1_0(\Omega)$ and $0 \leq t \leq T$ a.e., and $u(0, x) = u_0(x)$ and $\frac{\partial u}{\partial t}(0, x) = u_1(x)$.

We remark that the initial conditions $u(0, x) = u_0(x)$ and $\frac{\partial u}{\partial t}(0, x) = u_1(x)$ make sense because of regularity of a weak solution; indeed we have $u \in C(0, T; L^2(\Omega))$ and $\frac{\partial u}{\partial t} \in C(0, T; H^{-1}(\Omega))$. 

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3. The main result

Now, we may give the main result:

**Theorem 3.1.** Suppose \( a_{ij} \in VMO \cap L^\infty(\Omega), \ \nabla a_{ij} \in [L^p(\Omega)]^d \) with \( p > d \) such that conditions (2.3) are ensured. We also suppose \( f \in H^1(L^2(\Omega)), \ u_0 \in H^2(\Omega) \) and \( u_1 \in H^1_0(\Omega) \). Then there exists a unique solution \( u \) of (1.1) such that

\[
\begin{align*}
&u \in L^\infty(\Omega), \quad \frac{\partial u}{\partial t} \in L^\infty(H^1_0(\Omega)), \quad \frac{\partial^2 u}{\partial t^2} \in L^\infty(L^2(\Omega)), \quad \frac{\partial^3 u}{\partial t^3} \in L^2(H^{-1}(\Omega)) \\
&\text{with the estimate}
\end{align*}
\]

\[
\max_{0 \leq t \leq T} \left( \|u(t)\|_{H^2(\Omega)} + \left\| \frac{\partial u}{\partial t}(t) \right\|_{H^1_0(\Omega)} + \left\| \frac{\partial^2 u}{\partial t^2}(t) \right\|_{L^2(\Omega)} + \left\| \frac{\partial^3 u}{\partial t^3} \right\|_{L^2(H^{-1}(\Omega))} \right) \leq C \left( \|f\|_{H^1(L^2(\Omega))} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} \right)
\]

with the constant \( C \) depending on \( \Omega, T \) and the coefficients \( a_{ij} \).

**Proof.** We split the proof in several steps.

**Step 1:** Finite-dimensional approximate solutions.

For sake of simplicity, we denote \( u' = \frac{\partial u}{\partial t}, \ u'' = \frac{\partial^2 u}{\partial t^2}, \ u''' = \frac{\partial^3 u}{\partial t^3} \) and \( f' = \frac{\partial f}{\partial t} \) in the proof. We construct finite-dimensional approximate solutions of (2.5) by the method of Faedo–Galerkin.

As \( H^1_0(\Omega) \) is a separable Hilbert space, there exist a family of functions \( \{w_m\}_{m \geq 1} \) in \( H^1_0(\Omega) \) such that

\[
\{w_m\}_{m \geq 1} \text{ is an orthogonal basis of } H^1_0(\Omega)
\]

and

\[
\{w_m\}_{m \geq 1} \text{ is an orthonormal basis of } L^2(\Omega).
\]

Fix now a positive \( m \), we look for approximate solutions of (2.5) \( u_m : [0, T] \to H^1_0(\Omega) \), as

\[
(3.6) \quad u_m(t) = \sum_{i=1}^{m} g_{im}(t)w_i
\]

with \( g_m := (g_{1m}, g_{2m}, \ldots, g_{mm}) \) satisfying

\[
(3.7) \quad \left\{ \begin{array}{ll}
(u''_m(t), w_j) + (a_{ij} \nabla u_m(t), \nabla w_j) = (f(t), w_j) \\
g_{im}(0) = (u_0, w_i), \quad g'_{im}(0) = (u_1, w_i) \quad (i = 1, 2, \ldots, m)
\end{array} \right.
\]

where \((., .)\) denotes the scalar product in \( L^2(\Omega) \). The initial conditions in system (3.7) mean that \( u_m(0) \) and \( u'_m(0) \) are the respective projections of \( u_0 \) and \( u_1 \) onto the subspace spanned by \( \{w_1, w_2, \ldots, w_m\} \); thus we have \( \lim_{m \to +\infty} u_m(0) = u_0 \) and \( \lim_{m \to +\infty} u'_m(0) = u_1 \) (see, for example [3, Chapter 5]). From the classical theory of ordinary differential equations and assumptions of \( w_i \), system (3.7) admits a unique local solution \( g_m \) such that \( g_{jm} \in C^2(0, T_m) \) for \( j = 1, 2, \ldots, m \). Then, for each fixed \( m \), \( u_m \) defined by (3.6) is solution of (3.7).

**Step 2:** A priori estimates.

Multiplying (3.7) by \( g'_{jm} \), summing for \( j = 1, \ldots, m \) and taking relation (3.6) into account, we get

\[
(3.8) \quad (u''_m(t), u'_m(t)) + (a_{ij} \nabla u_m(t), \nabla u'_m(t)) = (f(t), u'_m(t)), \quad \text{a.e.} \ t \in [0, T]
\]
or equivalently

\[ \frac{d}{dt} \left\| u_m(t) \right\|_{L^2(\Omega)}^2 + \frac{\partial}{\partial t} \left\| a_{ij} \nabla u_m(t) \right\|_{L^2(\Omega)}^2 \geq 0 \]

Integrating on \((0, s)\), we deduce

\[ \left\| u_m(s) \right\|_{L^2(\Omega)}^2 \leq \left\| f \right\|_{L^2(\Omega)}^2 + \int_0^s \left\| u'_m(t) \right\|_{L^2(\Omega)}^2 \ dt + \left\| u_m(0) \right\|_{L^2(\Omega)}^2 \]

and with Gronwall’s inequality

\[ \left\| u'_m(t) \right\|_{L^2(\Omega)}^2 \leq e^{\exp(T)} \left( \left\| f \right\|_{L^2(\Omega)}^2 + \left\| u_m(0) \right\|_{L^2(\Omega)}^2 \right) \]

Therefore,

\[ \max_{0 \leq t \leq T} \left\| u'_m(t) \right\|_{L^2(\Omega)}^2 \leq C \left( \left\| f \right\|_{L^2(\Omega)}^2 + \left\| u_1 \right\|_{L^2(\Omega)}^2 \right) \]

with \( C \) depending on \( T \) and \( \Omega \). Here, we have used that \( u'_m(0) \) is the projection of \( u_1 \) onto the subspace spanned by \( \{u_1, \ldots, u_m\} \).

Using (3.9) again, integrating on \((0, s)\) and using (3.10), we obtain

\[ \left\| a_{ij} \nabla u_m(t) \right\|_{L^2(\Omega)}^2 \leq \left\| f \right\|_{L^2(\Omega)}^2 + \left\| a_{ij} \nabla u_m(0) \right\|_{L^2(\Omega)}^2 + \int_0^t \left\| u_m(s) \right\|_{L^2(\Omega)}^2 \ ds \]

\[ \leq \left\| f \right\|_{L^2(\Omega)}^2 + \left( \alpha_{\text{max}} \right)^2 \left\| \nabla u_0 \right\|_{L^2(\Omega)}^2 + TC \left( \left\| f \right\|_{L^2(\Omega)}^2 + \left\| h \right\|_{L^2(\Omega)}^2 \right) \]

\[ \leq C \left( \left\| f \right\|_{L^2(\Omega)}^2 + \left\| \nabla u_0 \right\|_{L^2(\Omega)}^2 + \left\| u_1 \right\|_{L^2(\Omega)}^2 \right) \]

with \( C \) depending on \( T \), \( \Omega \) and \( \alpha_{\text{max}} \). From hypothesis on \( a_{ij} \) and Poincaré inequality we get

\[ \left\| a_{ij} \nabla u_m(t) \right\|_{L^2(\Omega)}^2 \geq C a_{\text{min}}^2 \left\| u_m(t) \right\|_{H_0^1(\Omega)}^2 \]

So,

\[ \max_{0 \leq t \leq T} \left\| u_m(t) \right\|_{H_0^1(\Omega)}^2 \leq C \left( \left\| f \right\|_{L^2(\Omega)}^2 + \left\| u_0 \right\|_{H^1(\Omega)}^2 + \left\| u_1 \right\|_{L^2(\Omega)}^2 \right) \]

with \( C \) depending on \( \Omega \), \( T \), \( a_{\text{min}} \) and \( \alpha_{\text{max}} \).

Now, we estimate \( \left\| u''_m \right\|_{L^2(H^{-1}(\Omega))} \):

\[ \left\| u''_m(t) \right\|_{H^{-1}(\Omega)} = \sup_{\phi \in H_0^1(\Omega) \atop \left\| \phi \right\|_{H_0^1(\Omega)} = 1} \left\langle u''_m(t), \phi \right\rangle_{H^{-1}, H_0^1} = \sup_{\phi \in H_0^1(\Omega) \atop \left\| \phi \right\|_{H_0^1(\Omega)} = 1} \left[ (f(t), \phi) - \left( a_{ij} \nabla u_m(t), \nabla \phi \right) \right] \]

\[ \leq \left\| f(t) \right\|_{L^2(\Omega)} + \left\| a_{ij} \nabla u_m(t) \right\|_{L^2(\Omega)} \]

Thus

\[ \left\| u''_m \right\|_{L^2(H^{-1}(\Omega))} \leq C \left( \left\| f \right\|_{L^2(\Omega)}^2 + \left\| u_0 \right\|_{H^1(\Omega)}^2 + \left\| u_1 \right\|_{L^2(\Omega)}^2 \right) \]

where \( C \) depends on \( \Omega \), \( T \), \( a_{\text{min}} \) and \( \alpha_{\text{max}} \).

From these estimates we can conclude that \( T_m = T \), that is \( g_m = (g_{1m}, g_{2m}, \ldots, g_{mm}) \) is a global solution of system (3.7) and consequently a global solution \( u_m \).
Step 3: passage to the limit.
The estimates of step 2 allow us to conclude there exists a subsequence \( u_{mk} \) still denoted \( u_m \) and a function \( u \) such that
\[
\begin{align*}
u_m &\rightharpoonup u & L^2(0, T; H^1_0(\Omega)) \\
u_m' &\rightharpoonup u' & L^2(0, T; L^2(\Omega)) \\
u_m'' &\rightharpoonup u'' & L^2(0, T; H^{-1}(\Omega))
\end{align*}
\]
where \( \rightharpoonup \) stands for the weak convergence. This yields
\[
\int_\Omega \frac{\partial^2 u_m}{\partial t^2} w_j \, dx \rightarrow \int_\Omega \frac{\partial^2 u}{\partial t^2} w_j \, dx \quad \text{as} \quad m \rightarrow +\infty
\]
\[
\int_\Omega a_{ij}(x) \nabla u_m(t) \cdot \nabla w_j \, dx \rightarrow \int_\Omega a_{ij}(x) \nabla u(t) \cdot \nabla w_j \, dx \quad \text{as} \quad m \rightarrow +\infty
\]
for every \( w_j \), by a density argument, for every \( H^1_0 \) function so equation (2.5) is satisfied. Furthermore, by standard arguments is possible to show that \( u(0) = u_0 \) and \( u'(0) = u_1 \). This proves that \( u \) is a weak solution of (1.1). Moreover from (3.10)-(3.12) we have \( u \in L^\infty(0, T; H^1_0(\Omega)) \), \( u' \in L^\infty(0, T; L^2(\Omega)) \) and \( u'' \in L^2(0, T; H^{-1}(\Omega)) \).

Step 4: The uniqueness solution of (1.1) follows similarly to the classical results for hyperbolic equations (for example [9, §7.2]) and from the conditions (2.3) for \( a_{ij} \).

Step 5: Regularity improvement.
Let us differentiate the main equation of (3.7) with respect to \( t \) and multiply by \( g_m \)
\[
\left( u_m''(t), u_m''(t) \right) + \left( a_{ij} \nabla u_m(t), \nabla u_m''(t) \right) = \left( f'(t), u_m''(t) \right),
\]
that is,
\[
(3.13) \quad \frac{\partial}{\partial t} \left\| u_m''(t) \right\|_{L^2(\Omega)}^2 + \frac{\partial}{\partial t} \left\| a_{ij} \nabla u_m(t) \right\|_{L^2(\Omega)}^2 \leq \left( \left\| f'(t) \right\|_{L^2(\Omega)}^2 + \left\| u_m(t) \right\|_{L^2(\Omega)}^2 \right).
\]
Integrating on \( (0, s) \) gives
\[
\left\| u_m''(0) \right\|_{L^2(\Omega)}^2 \leq \left\| f' \right\|_{L^2(\Omega)}^2 + s \left\| u_m(t) \right\|_{L^2(\Omega)}^2 + \left\| u_m'(0) \right\|_{L^2(\Omega)}^2
\]
and with (3.8) we deduce
\[
\left\| u_m''(0) \right\|_{L^2(\Omega)}^2 \leq C \left( \left\| f \right\|_{H^1(\Omega)}^2 + \left\| u_m(0) \right\|_{H^2(\Omega)}^2 \right) \leq C \left( \left\| f \right\|_{H^1(\Omega)}^2 + \left\| u_0 \right\|_{H^2(\Omega)}^2 \right).
\]
Then Gronwall’s inequality gives
\[
\max_{0 \leq t \leq T} \left\| u_m''(t) \right\|_{L^2(\Omega)}^2 \leq C \left( \left\| f \right\|_{H^1(\Omega)}^2 + \left\| u_0 \right\|_{H^2(\Omega)}^2 \right)
\]
where \( C \) depends on \( T, a_{\min} \) and \( \Omega \).
On the other hand, by integrating on \((0, s)\) in (3.13) and using the last inequality, we obtain
\[
\left\| a_{ij} \nabla u_m'(t) \right\|_{L^2(\Omega)}^2 \leq \left\| f' \right\|_{L^2(\Omega)}^2 + \left\| u_m'' \right\|_{L^2(\Omega)}^2 + \left\| a_{ij} \nabla u_m'(0) \right\|_{L^2(\Omega)}^2 \leq C \left( \left\| f \right\|_{H^1(\Omega)}^2 + \left\| u_0 \right\|_{H^2(\Omega)}^2 + \left\| u_m'(0) \right\|_{H^1(\Omega)}^2 \right)
\]
\[
\leq C \left( \left\| f \right\|_{H^1(\Omega)}^2 + \left\| u_0 \right\|_{H^2(\Omega)}^2 + \left\| u_1 \right\|_{H^1(\Omega)}^2 \right)
\]
with \(C\) depending on \(T, a_{\text{min}}, a_{\text{max}}\) and \(\Omega\). Therefore,
\[
(3.14)
\]
\[
\max_{0 \leq t \leq T} \left( \left\| u_m''(t) \right\|_{L^2(\Omega)}^2 + \left\| u_m'(t) \right\|_{H^1(\Omega)}^2 \right) \leq C \left( \left\| f \right\|_{H^1(\Omega)}^2 + \left\| u_0 \right\|_{H^2(\Omega)}^2 + \left\| u_1 \right\|_{H^1(\Omega)}^2 \right)
\]
where \(C\) depends on \(T, a_{\text{min}}, a_{\text{max}}\) and \(\Omega\).

In order to establish the higher regularity for \(u\), we remark that, from (3.8), for \(a.e. t \in [0, T]\) we have
\[
- \text{div}(a_{ij} \nabla u_m(t), \phi) = \left( f(t) - u_m'(t), \phi \right)
\]
for every \(\phi \in H^1_0(\Omega)\). We taking \(q = 2\), \(a_{ij} = a_{ij}, \bar{b}_i = \frac{\partial a_{11}}{\partial x_1} + \ldots + \frac{\partial a_{ij}}{\partial x_d}\) and \(\bar{c} = 0\) in Theorem 2.1 and from hypothesis for \(a_{ij}\) and \(\nabla a_{ij}\), we get \(u_m(t) \in H^2(\Omega)\) and
\[
\left\| u_m(t) \right\|_{H^2(\Omega)} \leq C \left\| f(t) - u_m'(t) \right\|_{L^2(\Omega)}
\]
where \(C\) depends on \(\Omega\) and the coefficients \(a_{ij}\) (via \(\left\| \nabla a_{ij} \right\|_{L^p}\) and the VMO modulus of \(a_{ij}\)).

Hence, by using (3.14) we deduce
\[
\max_{0 \leq t \leq T} \left\| u_m(t) \right\|_{H^2(\Omega)} \leq C \max_{0 \leq t \leq T} \left\| f(t) - u_m'(t) \right\|_{L^2(\Omega)} \leq C \left( \left\| f \right\|_{H^1(\Omega)}^2 + \left\| u_0 \right\|_{H^2(\Omega)}^2 + \left\| u_1 \right\|_{L^2(\Omega)}^2 \right)
\]
with \(C\) depending on \(\Omega, T\) and the coefficients \(a_{ij}\).

Last, we estimate \(\left\| u_m'' \right\|_{L^2(H^{-1}(\Omega))}\)
\[
\left\| u_m''(t) \right\|_{H^{-1}(\Omega)} = \sup_{\phi \in H^1_0(\Omega) \backslash \{0\}} \left( \left\langle u_m''(t), \phi \right\rangle_{H^{-1}, H^1_0} \left\| \phi \right\|_{H^1_0(\Omega)}^{-1} \right) \sup_{\phi \neq 0} \left[ \left\langle f'(t), \phi \right\rangle - \left( a_{ij} \nabla u_m'(t), \nabla \phi \right) \right] \left\| \phi \right\|_{H^1_0(\Omega)}^{-1}
\]
\[
\leq \left\| f'(t) \right\|_{L^2(\Omega)} + \left| a_{ij} \nabla u_m'(t) \right|_{L^2(\Omega)}.
\]
Thus, from (3.14)
\[
\left\| u_m'' \right\|_{L^2(H^{-1}(\Omega))} \leq C \left( \left\| f \right\|_{H^1(\Omega)}^2 + \left\| u_0 \right\|_{H^2(\Omega)}^2 + \left\| u_1 \right\|_{H^1(\Omega)}^2 \right)
\]
where \(C\) depends on \(\Omega, T\) and the coefficients \(a_{ij}\). Passing to limit as \(m \to +\infty\), we obtain the same regularity and bounds for \(u\). This concludes the proof of the theorem.

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