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R Balasubramanian, K Ramachandra

To cite this version:

R Balasubramanian, K Ramachandra. On the zeros of a class of generalised Dirichlet series-IX. Hardy-Ramanujan Journal, 1991, Volume 14 - 1991, pp.34 - 43. 10.46298/hrj.1991.123 . hal-01104799

HAL Id: hal-01104799

https://hal.science/hal-01104799v1

Submitted on 22 Jan 2015

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Hardy-Ramanujan Journal
Vol.14 (1991) 34-43

ON THE ZEROS OF A CLASS OF GENERALISED DIRICHLET SERIES-IX

BY

R. BALASUBRAMANIAN AND K. RAMACHANDRA

§ 1. **INTRODUCTION.** In paper VII[4] of this series K. Ramachandra raised the question "Let $s = \sigma + it, T \geq T_0$ a large positive constant. For what values $\alpha = \alpha(T)$ the rectangle $(\sigma \geq \alpha(T), T \leq t \leq 2T)$ contains infinity of zeros of a generalised Dirichlet series of a certain type?" (In the earlier papers of this series he and R. Balasubramanian, sometimes individually and sometimes jointly considered the problem where $\alpha = \alpha(T)$ does not depend on T). Since the series considered in that paper were too general the answer $\left(\alpha(T) = \frac{1}{2} - \frac{D}{loglog\ T}\right)$ was perhaps too weak. In paper VIII[2] of the series we considered somewhat restricted series and obtained the result that we can take $\alpha(T) = \frac{1}{2} - C_0(loglog\ T)^{\frac{3}{2}}(log\ T)^{-\frac{1}{2}}$. In the present paper we assume a serious restrictive condition namely an Euler product and show that we can take $\alpha(T) = \frac{1}{2} - C_0(loglog\ T)(log\ T)^{-1}$. (Actually we work with shorter rectangles and also obtain a quantative result on the number of zeros). Accordingly we prove the following

THEOREM 1. *Let p run over primes and $\omega(p)$ complex numbers whose absolute value is* 1 *except for a finite set of primes where we define $\omega(p)$ to*

be zero. Let $s = \sigma + it$ as usual, and let

$$F(s) = \sum_{n=1}^{\infty}(a_n n^{-s}) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}, (\sigma > 1), \tag{1.1}$$

be continuable analytically in

$$\left\{\sigma \geq \frac{1}{2} - \frac{C\ loglog\ T}{log\ H}, T \leq t \leq T + H\right\} ((log\ T)^{C'} \leq H \leq T, C' \text{ to be specified}) \tag{1.2}$$

and there max $\mid F(s) \mid \leq T^A$ *where* $A(\geq 1)$ *is a constant. Let* $C(\geq 1)$ *be a large constant depending on* A *and let* H *exceed* $(log\ T)^{C'}$ *where* $C'(\geq 1)$ *is a large constant. Then* $F(s)$ *has at least* $(C'')^{-1}H(logloglog\ T)^{-1}$ *zeros in the rectangle (1.2), where* $C''(\geq 1)$ *is a large constant.*

REMARK 1. The condition on the Euler product (1.1) covers the Riemann zeta-function and the ordinary L-series. But it can be stated in such a way that it covers zeta and L-functions of algebraic number fields. The restriction (1.1) and the restriction $\mid F(s) \mid \leq T^A$ *practically force us to give these (and perhaps only these,* with trivial changes) *as examples* and in these cases the series have a functional equation and so in fact we can take $\alpha(T) = \frac{1}{2}$. Hence the main content of the present paper is the emphasis on results which can be proved without the assumption of a functional equation for $F(s)$. Thus we stress that *Theorem 1 does not depend on the assumption of a functional equation for* $F(s)$.

REMARK 2. In the case where $F(s)$ is the Riemann zeta-function or ordinary L-series we can prove (*without using the functional equation*) that

$$\frac{1}{T}\int_T^{2T} \mid F(\frac{1}{2} + it) \mid^2 dt \leq (log\ T)^2 \tag{1.3}$$

and from this we can deduce that the number of zeros of $F(s)$ in $(\sigma \geq \beta, T \leq t \leq 2T)$ is $\leq T^{\theta}(log\ T)^{A_1}$ where $\frac{1}{2} \leq \beta \leq 1, \theta = 4(1-\beta)(3-2\beta)^{-1}$ and $A_1 \geq 1$ is a certain numerical constant. (For a proof of this fact see [5]). Thus in these cases and with the extra condition $H = T$, the zeros of the theorem "already belong" to the rectangle

$$\left(\sigma \leq \frac{1}{2} + C\ \frac{loglog\ T}{log\ T}, T \leq t \leq 2T\right) \tag{1.4}$$

§ 2. **PROOF OF THEOREM 1.** In the proof we use the Ramachandra kernel $Exp((Sin\ z)^2)$, a well-known theorem due to Montgomery and Vaughan and the well-known

THEOREM 2 (Borel-Caratheodory Theorem). *Suppose $G(z)$ is analytic in $|\ z - z_0\ | \leq R$ and on $|\ z - z_0\ | = R$ we have $Re\ G(z) \leq U$. Then in $|\ z - z_0\ | \leq r < R$, we have,*

$$|\ G(z)\ | \leq \frac{2rU}{R-r} + \frac{R+r}{R-r}\ |\ G(z_0)\ |\ . \tag{2.1}$$

PROOF. See [6], page 174.

We break up the proof into a few lemmas. The rough idea of the proof is as follows. We first prove that on the line $\sigma = \frac{1}{2} - 2\delta$, where $\delta = \frac{1}{4}C(loglog\ T)(log\ H)^{-1}$, we have (between T and $T + H$) $\gg H$ well-spaced points which are all heavy for $log\ F(s)$ in a certain sense. Out of these we select $\gg H(logloglog\ T)^{-1}$ points with the property that any two (distinct) points are at a distance $\geq L = D\ logloglog\ T$, where $D(\geq 1)$ is a large constant. We then prove that if $\frac{1}{2} - 2\delta + it_j$ is any such point the rectangle $(\sigma \geq \frac{1}{2} - 4\delta, |\ t_j - t\ | \leq \frac{1}{3}L)$ contains a zero of $F(s)$. This will prove the theorem completely. We begin with

LEMMA 1. *For $u > 0$ let*

$$\Delta(u) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} u^w Exp\left(\left(Sin\ \frac{w}{1000}\right)^2\right)\frac{dw}{w}. \tag{2.2}$$

Then

$$\Delta(u) = O(u^2)\ \textit{and also}\ \Delta(u) = 1 + O(u^{-2})$$

where the implied constants are absolute.

PROOF. By moving the line of integration to $Re\ w = 2$ (resp. $Re\ w = -2$) the lemma follows by easy estimations.

LEMMA 2. *Let $\sigma = \frac{1}{2} - 2\delta, X = H^{\frac{1}{2}}$ and*

$$F_1(s) = \sum_p \omega(p)p^{-s}\Delta\left(\frac{X}{p}\right). \tag{2.3}$$

Then

$$\frac{1}{H}\int_T^{T+H} \mid F_1(s) \mid^2 dt \gg H^{2\delta}(\delta \log H)^{-1} \tag{2.4}$$

and

$$\frac{1}{H}\int_T^{T+H} \mid F_1(s) \mid^4 dt \ll H^{4\delta}(\delta \log H)^{-2}. \tag{2.5}$$

PROOF. By a well-known theorem of H.L. Montgomery and R.C. Vaughan (see [3] for a simpler proof of a special case of their result which we need here) LHS of (2.4) is

$$\sum_p \frac{\mid \omega(p) \mid^2}{p^{2\sigma}} \mid \Delta\left(\frac{X}{p}\right) \mid^2 \left(1 + O\left(\frac{p}{H}\right)\right) \tag{2.6}$$

which is $\gg H^{2\delta}(\delta \log H)^{-1}$ by the use of Lemma 1. (Since the proof of this is routine we postpone its proof to § 3). This proves (2.4).

Also LHS of (2.5) is

$$\leq \sum_{p_1,p_2} (p_1 p_2)^{-2\sigma} \mid \Delta\left(\frac{X}{p_1}\right) \Delta\left(\frac{X}{p_2}\right) \mid^2 \left(1 + O\left(\frac{p_1 p_2}{H}\right)\right) \tag{2.7}$$

which is $\ll H^{4\delta}(\delta \log H)^{-2}$ by the use of Lemma 1. (Since the proof of this is routine we postpone its proof to § 3). This proves (2.5).

Thus Lemma 2 is completely proved.

LEMMA 3. *Let T and H be positive integers (this can be assumed without loss of generality in Theorem 1). Then there exist $\gg H$ integers M satisfying $T \leq M \leq T + H - 1$ for which*

$$\int_M^{M+1} \mid F_1(s) \mid^2 dt \gg H^{2\delta}(\delta \log H)^{-1}. \tag{2.8}$$

PROOF. By (2.4) we have

$$\sum_M \int_M^{M+1} \mid F(s) \mid^2 dt \geq C_1 H \ \psi$$

where C_1 is a positive constant and $\psi = H^{2\delta}(\delta \log H)^{-1}$. Here in the sum we drop those M for which the integral from M to $M+1$ does not exceed $\frac{1}{2}C_1\psi$ and obtain

$$\sum_M{}' \int_M^{M+1} \mid F_1(s) \mid^2 dt \geq \frac{1}{2}C_1 H \psi$$

where the slash indicates that we restrict only to those integrals which exceed $\frac{1}{2}C_1\psi$. By applying Hölder's inequality we obtain

$$\left(\sum_M{}'1\right)^{\frac{1}{2}}\left(\sum_M{}'\left(\int_M^{M+1} \mid F_1(s) \mid^2 dt\right)^2\right)^{\frac{1}{2}} \geq \frac{1}{2}C_1 H\psi.$$

The inequality

$$\int_M^{M+1} \mid F_1(s) \mid^2 dt \leq \left(\int_M^{M+1} \mid F_1(s) \mid^4 dt\right)^{\frac{1}{2}}$$

completes the proof of the lemma on using (2.5).

REMARK. Actually the deduction of Lemma 3 from Lemma 2 involves a general principle which was first observed and applied for discussing the zeros of Dirichlet series by R. Balasubramanian and K. Ramachandra in paper III[1] of this series.

LEMMA 4. *In the interval* $[T, T+H]$ *there are* $R \gg H(logloglog\ T)^{-1}$ *points* $t_1, t_2, \cdots, t_R$ *with the properties* $\mid t_j - t_{j'} \mid \geq L$ *for all* $j \neq j'$ *where* $L = D\ logloglog\ T$ *and further*

$$\mid F_1\left(\frac{1}{2} - 2\delta + t_j\right) \mid \gg H^{\delta}(\delta\ log\ H)^{-\frac{1}{2}}. \tag{2.9}$$

As has been said before D *is a large positive constant.*

PROOF. The proof follows from Lemma 3.

LEMMA 5. *Consider any of the points* $t_j (= t_0$ *say) given by Lemma 4 excluding the upper and the lower extreme points. Assume that the region* $(\sigma \geq \frac{1}{2} - 4\delta, \mid t - t_0 \mid \leq \frac{1}{3}L)$ *is zero-free for* $F(s)$. *Then in* $(\sigma \geq \frac{1}{2} - 3\delta,$ $\mid t - t_0 \mid \leq \frac{1}{4}L)$ *we have,*

$$\mid log\ F(s) \mid \leq (log\ T)^3. \tag{2.10}$$

PROOF. Apply Borel-Caratheodory theorem as follows. Take $z_0 = 2 + it$ at which definitely $\mid log\ F(z_0) \mid \leq 10$, whatever be t. Choose R such that the circle $\mid z - z_0 \mid = R$ touches $\sigma = \frac{1}{2} - 4\delta$ (we mean the line $Re\ z = \frac{1}{2} - 4\delta$)

and r to be $R-\delta$. This proves the lemma.

LEMMA 6. *Put* $s_0 = \frac{1}{2} - 2\delta + it_0$. *Then*

$$F_1(s_0) = log\ F(s_0) + O(1). \tag{2.11}$$

PROOF. Since the powers of primes higher than the first contribute $O(1)$, LHS of (2.11) is

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} log\ F(s_0+w)X^w Exp\left(\left(Sin\ \frac{w}{1000}\right)^2\right)\frac{dw}{w} + O(1).$$

We now break off the portion $|\ Im\ w\ | \geq \frac{1}{4}L$ of the integral with an error $O(1)$ and in the rest of the integral move the line of integration to $Re\ w = -\delta$. This involves an error $O(X^{-\delta}(log\ T)^4) = O(1)$ if C is large. The pole at $w = 0$ contributes $log\ F(s_0)$. This proves the lemma.

LEMMA 7. *The equations (2.9), (2.10) and (2.11) are contradictory and hence the assumption of Lemma 5 is not true.*

PROOF. The equations referred to in the lemma give

$$H^{\delta}(\delta\ log\ H)^{-\frac{1}{2}} \ll (log\ T)^3.$$

This is false if $\delta = \frac{1}{4}C(loglog\ T)(log\ H)^{-1}$ (as specified already) and C is a large positive constant.

This completes the proof of Theorem 1.

§ 3. EXPLANATIONS REGARDING (2.6) AND (2.7).

(A) In (2.6) the main term is

$$= \sum_{p\leq X} p^{-2\sigma}\left(1 + O\left(\frac{p^2}{X^2}\right)\right)\left(1 + O\left(\frac{p}{X^2}\right)\right) + O(1)$$

$$= \sum_{p\leq X} p^{-2\sigma}\left(1 + O\left(\frac{p^2}{X^2}\right)\right) + O(1)$$

$$= \sum_{p\leq X} p^{-2\sigma} + O(X^{1-2\sigma}(log\ H)^{-1})$$

$$= \sum_{p \le X} p^{-2\sigma} + O(H^{2\delta}(log\ H)^{-1}).$$

The error term is

$$\ll \sum_{X \le p \le H} p^{-2\sigma} \left(\frac{X}{p}\right)^2 (1 + \frac{p}{H}) + \sum_{p > H} p^{-2\sigma} \left(\frac{X}{p}\right)^2 \left(\frac{p}{X^2}\right)$$

$$\ll \sum_{X \le p \le H} p^{-2\sigma} \left(\frac{X}{p}\right)^2 + \sum_{p > H} p^{-1-2\sigma}$$

$$\ll X^2 (log\ H)^{-1} X^{-1-2\sigma} + (log\ H)^{-1} H^{-2\sigma}$$

$$\ll (log\ H)^{-1}(XH^{-\sigma} + H^{-2\sigma}) \ll H^{2\delta}(log\ H)^{-1}.$$

Thus (2.6) is $\sum_{p \le X} p^{-2\sigma} + O(H^{2\delta}(log\ H)^{-1})$. Now the first term here is $\asymp$ $\sum_{U \le X} U^{4\delta}\ (log\ U)^{-1}$ (U runs over powers of 2) which is

$$\asymp \sum_{X^{\frac{1}{2}} \le U \le X} U^{4\delta}\ (log\ H)^{-1} + O(H^{\delta} loglog\ H)$$

$$\asymp H^{2\delta}(log\ H)^{-1} \left(\sum_{\nu=0}^{Q} 2^{-4\nu\delta} + O\left(2^{-4Q\delta}\delta^{-1}\right) \right) + O(H^{\delta} loglog\ H) \text{ (where } Q$$

$$\left[\frac{1}{4}\ \frac{log\ H}{log\ 2}\right]),$$

$$\asymp H^{2\delta}(log\ H)^{-1} \left(\frac{1}{\delta} + O\left(H^{-\delta}\delta^{-1}\right)\right) + O(H^{\delta}\ loglog\ H)$$

$$= H^{2\delta}(\delta\ log\ H)^{-1}(1 + O(H^{-\delta} + H^{-\delta}(\delta\ log\ H) loglog\ H))$$

$$= H^{2\delta}(\delta\ log\ H)^{-1}(1 + O(H^{-\delta} + H^{-\delta/2} loglog\ H)), \text{(since } H^{\delta/2} \ge \frac{\delta}{2}\ log\ H),$$

$\asymp H^{2\delta}(\delta\ log\ H)^{-1}$ provided $H^{\delta} \ge (loglog\ H)^3$ (which is certainly satisfied since $\delta = \frac{1}{4}C(loglog\ T)(log\ H)^{-1}$ and C is assumed to be large).

(B) The quantity in (2.7) is

$$\ll \left(\sum_p |\Delta\left(\frac{X}{p}\right)|^2 p^{-2\sigma}\right)^2 + \frac{1}{H}\left(\sum_p |\Delta\left(\frac{X}{p}\right)|^2 p^{1-2\sigma}\right)^2$$

$$\ll \left(\sum_{p\le X} p^{-2\sigma}\right)^2 + \left(\sum_{p>X} \frac{X^4}{p^{4+2\sigma}}\right)^2 + \frac{1}{X^2}\left(\sum_{p\le X} p^{1-2\sigma}\right)^2 + \frac{1}{X^2}\left(\sum_{p>X} \frac{X^4}{p^{3+2\sigma}}\right)^2$$

$$\ll H^{4\delta}(\delta \log H)^{-2} + \left(\frac{X^4}{X^{3+2\sigma}\log H}\right)^2 + \frac{1}{X^2}\left(\frac{X^{2-2\sigma}}{\log H}\right)^2$$

$$+\frac{1}{X^2}\left(\frac{X^4}{X^{2+2\sigma}\log H}\right)^2 \text{ (if } H^\delta \ge (\log\log H)^3 \text{ which is satisfied by our assumption on } H\text{)}$$

$$\ll H^{4\delta}(\delta \log H)^{-2} + H^{4\delta}(\log H)^{-2}$$

$$\ll H^{4\delta}(\delta \log H)^{-2} \text{ if } \delta \text{ is small.}$$

This completes explanations regarding (2.6) and (2.7).

Thus Theorem 1 is completely proved.

ACKNOWLEDGEMENT. The authors are grateful to Professor M. JUTILA for encouragement.

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ADDRESS OF THE AUTHORS

(1) PROFESSOR R. BALASUBRAMANIAN
MATSCIENCE
THARAMANI P.O.
MADRAS 600 113
TAMIL NADU
INDIA

(2) PROFESSOR K. RAMACHANDRA
SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
HOMI BHABHA ROAD
COLABA
BOMBAY 400 005
INDIA

MANUSCRIPT COMPLETED ON 23 FEBRUARY 1991