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ON THE ZEROS OF A CLASS OF GENERALISED  
DIRICHLET SERIES-VIII

BY

R. BALASUBRAMANIAN AND K. RAMACHANDRA

§ 1. INTRODUCTION AND NOTATION. In the paper VII<sup>[4]</sup> of this series (for the earlier papers of the series see the list of references in the paper VII<sup>[4]</sup>) K. Ramachandra started a new problem "Let  $s = \sigma + it, T \geq T_0$ . For what values  $\alpha = \alpha(T)$  the rectangle  $(\sigma \geq \alpha(T), T \leq t \leq 2T)$  contains infinity of zeros of a generalised Dirichlet series of a certain type?" (In the earlier papers of this series he and R. Baslasubramanian, sometimes individually and sometimes jointly, considered the problem where  $\alpha = \alpha(T)$  is independent of  $T$ ). Since the series considered in that paper were too general the answer  $(\alpha(T) = \frac{1}{2} - \frac{D}{\log \log T})$  was perhaps too weak. In the present paper we consider some of the Dirichlet series of the form  $F(s) = \sum_{n=1}^{\infty} (a_n b_n \lambda_n^{-s})$  which were considered in the paper V<sup>[3]</sup> of this series. (The method of the present paper does not succeed for *all* the series considered in V<sup>[3]</sup> let alone those considered in VI<sup>[2]</sup>). Before we recall the general series of V<sup>[3]</sup>, we record two neat results (the second being deeper than the first) as two theorems. In what follows  $T$  is the only variable and we assume that  $T$  exceeds a large positive constant.

**THEOREM 1.** *Let  $\{\chi(n)\}(n = 1, 2, 3, \dots)$  be any sequence of complex*

numbers with  $\sum_{n \leq x} \chi(n) = O(1)$ . Let, as usual,  $s = \sigma + it$ . Then the number of zeros of  $\zeta(s) + \sum_{n=1}^{\infty} (\chi(n)n^{-s})$  in the rectangle

$$\left\{ \sigma \geq \frac{1}{2} - C_0(\log \log T)^{\frac{3}{2}}(\log T)^{-\frac{1}{2}}, T \leq t \leq 2T \right\}$$

is  $\gg T(\log \log T)^{-1}$  for a suitable positive constant  $C_0$ .

**THEOREM 2.** Let  $1 = \lambda_1 < \lambda_2 < \lambda_3 < \dots$  be an infinite sequence of real numbers such that for  $n \geq n_0$  ( $n_0$ , a constant),  $\lambda_n$  is the restriction to integers of a twice continuously differentiable function  $g(x)$  of a real variable  $x$  with the following properties.

- (1) As  $x \rightarrow \infty$ ,  $x^{-1}g(x)$  tends to a positive limit.
- (2) There exist positive constants  $a$  and  $b$  such that for all  $x \geq n_0$ , we have,

$$a \leq g'(x) \leq b$$

and

$$a \leq (g'(x))^2 - g(x)g''(x) \leq b.$$

Then the number of zeros of  $F(s) = \sum_{n=1}^{\infty} ((-1)^n \lambda_n^{-s})$  in the rectangle

$$\left\{ \sigma \geq \frac{1}{2} - C_0(\log \log T)^{\frac{3}{2}}(\log T)^{-\frac{1}{2}}, T \leq t \leq 2T \right\}$$

is  $\gg T(\log \log T)^{-1}$  for a suitable positive constant  $C_0$ .

**REMARK.** For  $n = 1, 2, 3, \dots$ , let  $\beta_n = \beta_n^{(1)} + \beta_n^{(2)}$  where  $\beta_n^{(1)}$  and  $\beta_n^{(2)}$  are two bounded monotonic sequences of real numbers. Then for  $n \geq n_0$  we can replace  $\lambda_n$  by  $\lambda_n + \beta_n$  and the result is practically unchanged (i.e. except for a change of  $C_0$ ).

The general theorem is too lengthy to state. We now proceed to state it. We consider series of the form  $F(s) = \sum_{n=1}^{\infty} (a_n b_n \lambda_n^{-s})$  where  $\lambda_n$  has been introduced already (the change of  $\lambda_n$  to  $\lambda_n + \beta_n$  mentioned in the remark

below Theorem 2 is certainly permissible in what follows). Let  $f(x)$  be a positive real valued function with the following properties.

(1)  $f(x)x^\eta$  is increasing and  $f(x)x^{-\eta}$  is decreasing for every  $\eta > 0$  and all  $x \geq x_0(\eta)$ .

(2) For  $n \geq n_0$ ,  $a \leq |b_n| (f(n))^{-1} \leq b$ .

(3) For all  $x \geq 1$ ,  $\sum_{x \leq n \leq 2x} |b_{n+1} - b_n| \leq bf(x)$ . We next assume that  $\{a_n\}$  and  $\{b_n\}$  satisfy one at least of the following two conditions.

(4) **Monotonicity condition.** Let  $a_n (n = 1, 2, 3, \dots)$  be a bounded sequence of complex numbers such that  $x^{-1} \sum_{n \leq x} a_n$  tends to a non-zero limit (which may be complex) and further  $|b_n| \lambda_n^{-\eta}$  is monotonic decreasing for every  $\eta > 0$  and all  $n \geq n_0(\eta)$ .

(5) **Real part condition.** There exists an infinite arithmetic progression  $J$  of positive integers such that

$$\liminf_{x \rightarrow \infty} \left( \frac{1}{x} \sum_{\substack{x \leq \lambda_n \leq 2x, \\ n \in J}} \operatorname{Re} a_n \right) > 0$$

and

$$\lim_{x \rightarrow \infty} \left( \frac{1}{x} \sum_{\substack{x \leq \lambda_n \leq 2x, \\ n \in J}} a_n \right) = 0.$$

We are now in a position to state our general theorem.

**THEOREM 3.** Let  $F(s) = \sum_{n=1}^{\infty} (a_n b_n \lambda_n^{-s})$  be as described above. Let  $\operatorname{Exp}(-\sqrt{\log x}) \leq f(x)$  for  $x \geq x_0$ . Let  $\beta$  be a positive constant  $< \frac{1}{2}$  and that  $F(s)$  can be continued analytically in  $(\sigma \geq \beta, \frac{1}{2}T \leq t \leq \frac{5}{2}T)$  and here  $\max |F(s)| \leq T^{A_1}$  where  $A_1 \geq 2$  is a positive constant. Finally let

$$\frac{1}{T} \int_{\frac{1}{2}T}^{\frac{5}{2}T} |F(\frac{1}{2} + it)|^2 dt \leq (\log T)^{A_2}$$

where  $A_2 \geq 2$  is a constant. Then the number of zeros of  $F(s)$  in the rectangle

$$\left\{ \sigma \geq \frac{1}{2} - C_0(\log \log T)^{\frac{3}{2}}(\log T)^{-\frac{1}{2}}, T \leq t \leq 2T \right\}$$

is  $\gg T(\log \log T)^{-1}$  where  $C_0 \geq 0$  is a certain constant.

**REMARK 1.** The restriction of the theorem regarding the upper bound for the mean square of  $|F(\frac{1}{2} + it)|$  is very strong. Practically (since the mean square can be proved to be  $\gg (f(T))^2$ ) it forces us to consider the series of  $V^{[3]}$ , with the extra restriction  $f(x) \leq (\log x)^A$  for some constant  $A \geq 2$  and all  $x \geq x_0(A)$ . Further the restriction  $f(x) \geq \text{Exp}(-\sqrt{\log x})$  forces us to consider only a sub-class of functions considered in  $V^{[3]}$ . It may be remarked that the mean square hypothesis is satisfied for all functions considered in  $V^{[3]}$  by imposing  $f(x) \leq (\log x)^A$ .

**REMARK 2.** A nice example of the functions covered by Theorem 3 is  $\sum_{n=1}^{\infty} ((-1)^n \text{Exp}(-\sqrt{\log n}) n^{-s})$ . It may be noted (as a special case of a very general Theorem [1]) that this is an entire function.

**REMARK 3.** In the theorem it is not difficult to relax the rectangle of analytic continuation to  $(\sigma \geq \beta, T \leq t \leq 2T)$  and replace the mean-value condition by

$$\frac{1}{T} \int_T^{2T} |F(\frac{1}{2} + it)|^2 \leq (\log T)^{A_2}$$

where  $A_2 \geq 2$  is a constant.

**REMARK 4.** It is possible to generalise our results further. As a simple example we can in Theorem 1 replace  $\zeta(s) + \sum_{n=1}^{\infty} (\chi(n)n^{-s})$  by

$$K^{-s}(\zeta(s) + \sum_{n=1}^{\infty} (\chi(n)n^{-s})) + \sum_{n=1}^{\infty} d_n \lambda_n^{-s}$$

where  $\sum_{n \leq x} d_n = O(1)$ ,  $K$  is a positive constant,  $|\lambda_m - Kn| \geq (100)^{-1}$  for all  $m, n$ ,  $1 \ll \lambda_{n+1} - \lambda_n$  and finally  $\lambda_n = O(n)$ .

**REMARK 5.** We have imposed the restriction  $f(x) \geq \text{Exp}(-\sqrt{\log x})$  for

$x \geq x_0$  to obtain some worthwhile results, but it is possible to obtain weaker results by relaxing this condition.

**NOTATION.** The letter  $A$  with or without subscripts will denote constants  $\geq 2$ . The letter  $C$  with or without subscripts will denote positive constants.

§ 2. **A GENERAL LEMMA.** Let  $1 = \lambda_1 < \lambda_2 < \lambda_3 < \dots$  be an infinite sequence of real numbers with  $1 \gg \lambda_{n+1} - \lambda_n \gg 1$  and  $\{k_n\} (n = 1, 2, 3, \dots)$  be any sequence of complex numbers such that  $k_1 = 1$  and the series  $\phi(s) = \sum_{n=1}^{\infty} (k_n \lambda_n^{-s})$  is convergent in  $\sigma \geq A_1$  and is continuable analytically in  $(\sigma \geq \beta, T - (\log T)^2 \leq t \leq T + (\log T)^2)$  and there  $\max |\phi(s)| \leq T^{A_2}$ , where  $\beta < \frac{1}{2}$  is a positive constant. Let

$$\frac{1}{T} \int_{T - (\log T)^2}^{2T + (\log T)^2} \left| \phi\left(\frac{1}{2} + it\right) \right|^2 dt \leq (\log T)^{A_3}.$$

Then, we have,

$$\frac{1}{T} \int_{\frac{1}{2} - (\log T)^{-1}}^{A_1 + 2} \int_{T-1}^{2T+1} \left| \phi(\sigma + it) \right|^2 dt d\sigma \leq (\log T)^{A_4}.$$

**REMARK.** This lemma is well-known to experts in the subject and so its proof will be postponed to the last section. Also it is possible to replace  $(\log T)^2$  by a constant multiple of  $\log \log T$ .

§ 3. **THE FUNCTION  $F_2(s)$ .** As in VI<sup>[2]</sup> we introduce the function (in VI<sup>[2]</sup> we have used the kernel  $\text{Exp}(W^{4a+2})$  but we now use the kernel  $\text{Exp}((\text{Sin } W)^2)$ )

$$F_2(s) = \sum_{n=1}^{\infty} a_n b_n (\Delta(T) - \Delta(TD^{-1})) \lambda_n^{-s}$$

where  $D$  is a large positive constant and  $\Delta(x)$  for  $x > 0$  is defined by

$$\Delta(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(W) x^W \text{Exp}\left(\left(\text{Sin } \frac{W}{1000}\right)^2\right) \frac{dW}{W}.$$

As in VI<sup>[2]</sup> we have

**LEMMA 1.** *Let  $q$  be any real constant satisfying  $\beta < q < \frac{1}{2}$ . Then we have the inequalities*

$$(1) \quad \frac{1}{T} \int_T^{2T} |F_2(q+it)|^2 dt \ll T^{1-2q} (f(T))^2,$$

and

$$(2) \quad \frac{1}{T} \int_T^{2T} |F_2(q+it)| \gg T^{\frac{1}{2}-q} f(T).$$

**PROOF.** Similar to the proof of Lemma 10 of VI<sup>[2]</sup>.

**LEMMA 2.** *Let  $T$  be an integer. Then the number of integers  $M$  in the range  $T \leq M \leq 2T - 1$  for which*

$$\int_M^{M+1} |F_2(q+it)| dt > C_1 T^{\frac{1}{2}-q} f(T)$$

*exceeds  $C_2 T$ .*

**PROOF.** Similar to that of Lemma 4 of VI<sup>[2]</sup>.

**LEMMA 3.** *There exist at least  $C_3 T (\log \log T)^{-1}$  points  $t_j$  with*

$$|F_2(q+it_j)| > c_1 T^{\frac{1}{2}-q} f(T)$$

*and such that any two points  $t_j$  and  $t_{j'}$  with  $j \neq j'$  differ by at least  $C_4 \log \log T$*

**REMARK.** Here  $C_4$  is arbitrary and  $C_3$  depends on it.

**PROOF.** Follows from Lemma 2.

**LEMMA 4.** *Let  $r$  be a constant satisfying  $\beta < r < q < \frac{1}{2}$ . Put  $C_5 = \frac{1}{100} C_4$  and  $H = C_5 \log \log T$ . Then*

$$\int_{t_j-H}^{t_j+H} |F_2(r+it)| \geq C_6 V \log \log T$$

*where  $V = T^{\frac{1}{2}-r} f(T)$  for at most  $C_7 C_6^{-1} T (\log \log T)^{-1}$  points  $t_j$ .*

**REMARK.** Here  $C_6$  is arbitrary and  $C_7$  is independent of  $C_6$ .

**PROOF.** By (1) of Lemma 1, the sum over  $j$  of the quantity on the LHS does not exceed  $C_7VT$  and this gives Lemma 4.

**LEMMA 5.** *There are at least  $\frac{1}{2}C_3T(\log\log T)^{-1}$  points  $t_j$  separated by (distances) at least  $C_4\log\log T$  such that if  $H = \frac{1}{100}C_4\log\log T$  then with  $V = T^{\frac{1}{2}-r}f(T)$ , we have,*

$$\int_{t_j-H}^{t_j+H} |F_2(r+it)| dt \leq C_6V \log\log T.$$

**REMARK.** Here  $C_4$  is arbitrary and  $C_3$  depends on it.

**PROOF.** The lemma follows by choosing a large  $C_6$  in Lemma 4.

**LEMMA 6.** *Uniformly in  $\sigma$  with  $q < \sigma_0 \leq \sigma < \frac{1}{2}$ , we have, for the points  $t_j$  of Lemma 5,*

$$\int_{t_j-2H}^{t_j+2H} |F_2(\sigma+iv) \text{Exp}((\text{Sin} \frac{W}{1000})^2) \frac{dW}{W}| > C_8T^{\frac{1}{2}-\sigma} f(T)(\log\log T)^{-\theta}$$

where  $\sigma_0$  is a constant  $W = \sigma - q + iv$ , and  $\theta = \frac{1}{2(q-r)}$ .

**PROOF.** Put  $s_0 = q + it_j$ , we have

$$F_2(s_0) = \frac{1}{2\pi i} \int F_2(s_0+W) X^W \text{Exp}((\text{Sin} \frac{W}{1000})^2) \frac{dW}{W}$$

where the integral is taken over the (anticlockwise) boundary of the rectangle bounded by the lines  $\text{Re } W = r - q$ ,  $\text{Re } W = \sigma - q$ ,  $\text{Im } W = \pm H$ . We take the absolute values (using Lemma 3) of the integrand on the RHS and choose  $X = C_8T(\log\log T)^{(q-r)^{-1}}$ , where  $C_8$  is a large positive constant. This leads to Lemma 6.

**LEMMA 7.** *Given any  $\sigma$  in  $\sigma_0 \leq \sigma < \frac{1}{2}$ , there exist points  $v_j$  satisfying  $t_j - 2H \leq v_j \leq t_j + 2H$ , such that uniformly in  $\sigma$  there holds*

$$|F_2(\sigma+iv_j)| > C_9T^{\frac{1}{2}-\sigma} f(T)(\log\log T)^{-\theta}$$

where  $\theta = (2(q-r))^{-1}$ .

**REMARK.** Note that  $v_j$  are separated by (distances) at least  $\frac{24}{25}C_4\log\log T$



where  $C_4$  is at our disposal.

**PROOF.** Follows from Lemma 6.

**LEMMA 8.** *Given any  $\sigma$  in  $\sigma_0 \leq \sigma < \frac{1}{2}$  there exist points  $p_j$  satisfying  $v_j - H \leq p_j \leq v_j + H$  such that uniformly in  $\sigma$ , there holds,*

$$|F(\sigma + ip_j)| > C_{10} T^{\frac{1}{2} - \sigma} f(T) (\log \log T)^{-\theta}$$

where  $\theta$  is the constant defined before.

**REMARK 1.** Note that  $p_j$  are separated by (distances) at least  $\frac{1}{2} C_4 \log \log T$ . Also the number of points  $p_j$  is at least  $\frac{1}{2} C_3 T (\log \log T)^{-1}$ . Here  $C_4$  is arbitrary and  $C_3$  depends on it. (Both are independent of  $\sigma$ ).

**REMARK 2.** We can refine the lower bound for  $|F(\sigma + ip_j)|$  but we do not do it since it does not have an application.

**PROOF.** We start with

$$F_2(\sigma + iv_j) = \frac{1}{2\pi i} \int F(\sigma + iv_j + W) T^W (1 - D^{-W}) \text{Exp}((\text{Sin } \frac{W}{1000})^2) \frac{dW}{W}$$

where the integration is over  $\text{Re } W = 2$ . We break off the portion  $|v| \geq C_{11} \log \log T$  with a small error and move the line of integration in the rest to  $\text{Re } W = 0$ . Here  $C_{11}$  is a specific constant and not arbitrary. We now use Lemma 7 and majorise the integrand. This leads to the lemma.

The rest of the proof consists in proving that at least  $\frac{1}{3} C_3 T (\log \log T)^{-1}$  of the rectangles

$$\left\{ \sigma \geq \frac{1}{2} - C_0 (\log \log T)^{\frac{3}{2}} (\log T)^{-\frac{1}{2}}, p_j - H \leq t \leq p_j + H \right\}$$

contain a zero of  $F(s)$  if  $C_0$  is a large positive constant. This would complete the proof of Theorem 3.

**§ 4. TWO APPLICATIONS OF BOREL-CARATHÉODORY THEOREM.** Suppose that the rectangle

$$\left\{ \sigma \geq \frac{1}{2} - K\delta, p_j - H \leq t \leq p_j + H \right\}$$

is zero free for  $F(s)$ , where  $\delta$  and  $K$  are positive quantities to be chosen in the next section. (The quantity  $\delta$  will be chosen to be small and  $K$  to be large).

**LEMMA 1.** (Borel-Carathéodory Theorem. See [5] page 174). *Suppose  $G(z)$  is analytic in  $|z - z_0| \leq R$  and on  $|z - z_0| = R$  we have  $\operatorname{Re} G(z) \leq U$ . Then in  $|z - z_0| \leq r < R$ , we have,*

$$|G(z)| \leq \frac{2rU}{R-r} + \frac{R+r}{R-r} |G(z_0)|.$$

**REMARK.** The  $r$  of this lemma is not to be confused with that of the preceding section.

**LEMMA 2.** *In the rectangle*

$$\left\{ \sigma \geq \frac{1}{2} - (K-1)\delta, p_j - H + C_{12} \leq t \leq p_j + H - C_{12} \right\}$$

*we have,*

$$|\log F(s)| \leq C_{13} \delta^{-1} \log T.$$

**PROOF.** Choose  $z_0$  to be a point in

$$\left\{ \sigma \geq 2, p_j - H + C_{12} \leq t \leq p_j + H - C_{12} \right\}$$

where  $\log F(s)$  is bounded and then take  $R$  to be such that the circle with centre  $z_0$  and radius  $R$  touches  $\sigma = \frac{1}{2} - K\delta$  and lies within the rectangle  $\left\{ \sigma \geq \frac{1}{2} - K\delta, p_j - H \leq t \leq p_j + H \right\}$ . Next choose  $r = R - \delta$ . This proves Lemma 2.

**LEMMA 3.** *Let  $M_j$  denote the maximum of  $|F(s)|$  in  $\left\{ \sigma \geq \frac{1}{2}, p_j - H \leq t \leq p_j + H \right\}$ . Then, we have,*

$$\sum_j M_j^2 \leq T(\log T)^{A_5}.$$

**PROOF.** Let  $M_j$  be attained at  $s_j$  say. Then  $M_j^2$  is majorised by the mean of  $|F(s)|^2$  over a disc of radius  $(\log T)^{-1}$  with centre  $s_j$ . The lemma now follows from the general result of § 2.

**LEMMA 4.** *We have,*

$$M_j^2 \geq (\log T)^{11A_5}$$

for at most  $T(\log T)^{-10}$  values of  $j$ . Hence we are still left with at least  $\frac{1}{3} C_3 T (\log \log T)^{-1}$  values of  $j$  for which

$$M_j^2 \leq (\log T)^{11A_5}.$$

**REMARK.** From now on we restrict  $j$  only to these values.

**PROOF.** Follows from Lemma 3.

**LEMMA 5.** *In the rectangle*

$$\left\{ \sigma \geq \frac{1}{2} - \delta, p_j - H + C_{12} \leq t \leq p_j + H - C_{12} \right\}$$

we have,

$$|\log F(s)| \leq C_{14} \delta^{-1} \log \log T.$$

**PROOF.** Choose  $z_0$  to be a point in

$$\left\{ \sigma \geq 2, p_j - H + C_{12} \leq t \leq p_j + H - C_{12} \right\}$$

and then take  $R$  to be such that the circle with centre  $z_0$  and radius  $R$  touches  $\sigma = \frac{1}{2}$  and lies within the rectangle  $\left\{ \sigma \geq \frac{1}{2}, p_j - H \leq t \leq p_j + H \right\}$ . Next choose  $r = R - \delta$ . The lemma now follows from Lemma 4.

**§ 5. COMPLETION OF THE PROOF.** Suppose that for a certain  $j$ , the rectangle  $\left\{ \sigma \geq \frac{1}{2} - K\delta, |p_j - t| \leq H \right\}$  does not contain a zero of  $F(s)$ . We obtain a contradiction in the following way. Put  $s_0 = \sigma + ip_j$  where  $\sigma = \frac{1}{2} - \delta$ , and also let  $\sigma_1 = \frac{1}{2} - (K-1)\delta$ ,  $\sigma_2 = \sigma$ , and  $\sigma_3 = \frac{1}{2} + \delta$ . We apply maximum modulus principle to

$$\psi(W) = \log F(s_0 + W) X^W \text{Exp}\left(\left(\text{Sin} \frac{W}{1000}\right)^2\right)$$

according to which

$$|\psi(0)| \leq \max |\psi(W)|$$

maximum being taken over the boundary of the rectangle bounded by  $Re W = -(K - 2)\delta, Re W = 2\delta, Im W = \pm \frac{1}{2}H$ . If  $\delta \geq 6(\log T)^{-\frac{1}{2}}$  we have (by a suitable choice of  $X$  and  $C_4$ )

$$\begin{aligned} \frac{\delta}{2} \log T &\leq \delta \log T - 3\sqrt{\log T} \leq |\psi(0)| \\ &\leq C_{15}(\delta^{-1} \log T)^{\frac{2}{K}} (\delta^{-1} \log \log T)^{\frac{K-2}{K}}. \end{aligned}$$

We now choose  $K = \log \log T$  and obtain

$$\frac{\delta}{2} \log T \leq C_{16} \delta^{-1} \log \log T.$$

This is a contradiction if we choose  $\delta = C_{17}(\log \log T)^{\frac{1}{2}}(\log T)^{-\frac{1}{2}}$  and  $C_{17}^2 > 2C_{16}$ . This proves Theorem 3 provided we prove the general lemma of § 2.

**§ 6. PROOF OF THE GENERAL LEMMA.** Let  $\varepsilon > 0$  be arbitrary but fixed. Then in  $\{\sigma \geq \beta + \varepsilon, T \leq t \leq 2T\}$ , we have, by Cauchy's theorem  $|\phi'(s)| \leq T^{A_2+1}$  and so in  $\{|\sigma - \frac{1}{2}| \leq T^{-4A_2}, T \leq t \leq 2T\}$  we have

$$|\phi^2(\frac{1}{2} + it) - \phi^2(\sigma + it)| \leq 1.$$

Hence it suffices to consider in this rectangle the portion  $|\sigma - \frac{1}{2}| \geq T^{-4A_2}$ . If now  $\frac{1}{2} - (\log T)^{-1} \leq \sigma \leq \frac{1}{2} - T^{-4A_2}$  we have

$$|\phi^2(s) = \frac{1}{2\pi i} \int \phi^2(s + W) X^W \text{Exp}(W^2) \frac{dW}{W}$$

the contour being the (anticlockwise) boundary of the rectangle bounded by  $Re W = \beta - \sigma, Re W = \frac{1}{2} - \sigma, Im W = \pm \log T$ . We choose  $X$  to be a large power of  $T$  so that the integral over the left boundary is negligible. Clearly the integrals over the horizontal boundaries are together negligible. We take absolute values and integrate with respect to  $t$  from  $t = T$  to  $t = 2T$ . This leads to the result since on the right boundary  $|X^W| \leq 1$  and  $\int | \frac{dW}{W} | \ll \log T$ .

If now  $\sigma \geq \frac{1}{2} + T^{-4A_2}$  we start with

$$\phi^2(s) = \frac{1}{2\pi i} \int \phi^2(s + W) \text{Exp}(W^2) \frac{dW}{W}$$

the contour being the (anticlockwise) boundary of the rectangle bounded by  $Re W = \frac{1}{2} - \sigma$ ,  $Re W = 3A_1 - \sigma$ ,  $Im W = \pm \log T$ . The proof proceeds as before using  $\phi(s + W) = O(1)$  on the right boundary and negligible on the horizontal boundaries and the fact  $\int \left| \frac{dW}{W} \right| \ll \log T$  on the left boundary. This completes the proof of the general lemma.

Theorem 3 is now completely proved.

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