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K Ramachandra

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ON THE FREQUENCY OF TITCHMARSH'S PHENOMENON
FOR $\zeta(s)$ -IX

BY

K. RAMACHANDRA

§ 1. INTRODUCTION. In a previous paper [R], I proved the following result on $\zeta(1+it)$.

THEOREM 1. Let $0 \leq \theta < 2\pi$, $z = e^{i\theta}$ and

$$f(H) = \min_{T \geq 1} \max_{T \leq t \leq T+H} |(\zeta(1+it))^z| \quad (1)$$

Then

$$f(H) \geq e^\gamma \lambda(\theta) (\log \log H - \log \log \log H) + O(1) \quad (2)$$

where $H \geq \text{Exp}(e^e)$,

$$\lambda(\theta) = \prod_p \lambda_p(\theta),$$

$$\lambda_p(\theta) = \left\{ \left(1 - \frac{1}{p}\right) \left(\sqrt{1 - \frac{\text{Sin}^2 \theta}{p^2}} - \frac{\text{Cos } \theta}{p} \right)^{-\text{Cos } \theta} \text{Exp} \left(\text{Sin } \theta \text{Sin}^{-1} \left(\frac{\text{Sin } \theta}{p} \right) \right) \right\}. \quad (3)$$

In the present paper I prove that

$$f(H) \leq e^\gamma \lambda(\theta) (\log \log H + \log \log \log H) + O(1). \quad (4)$$

This result together with Theorem 1 gives the following Theorem.

THEOREM 2. We have

$$|f(H)e^{-\gamma(\lambda(\theta))^{-1}} - \log \log H| \leq \log \log \log H + O(1), \quad (5)$$

where $H \geq \text{Exp}(e^e)$.

REMARK. It is interesting to prove (or disprove!) $f(H)e^{-\gamma(\lambda(\theta))^{-1}} =$

$\log \log H + O(1)$.

§ 2. PROOF OF (4). We begin by

LEMMA 1. *Let $T = \text{Exp}((\log H)^2)$ where H exceeds an absolute constant. Then there exists a sub-interval I of $[T, 2T]$ of length $H + 2(\log H)^{10}$, such that the rectangle $(\sigma \geq \frac{3}{4}, t \in I)$ does not contain any zero of $\zeta(s)$ and moreover*

$$\max |\log \zeta(\sigma + it)| = O((\log H)^{\frac{1}{4}} (\log \log H)^{-\frac{3}{4}}) \quad (6)$$

the maximum being taken over the rectangle referred to.

PROOF. Follows from [BR] and the result (due to A.E. Ingham [I], see also [T] page 236 and p. 293-295 [AI]) that the number of zeros of $\zeta(s)$ in $(\sigma \geq \frac{3}{4}, T \leq t \leq 2T)$ is $O(T^{\frac{1}{2}})$.

LEMMA 2. *Let J be the interval obtained by removing from I intervals of length $(\log H)^{10}$ from both ends. Then for $t \in J$, we have,*

$$\log \zeta(1 + it) = \sum_{m \geq 1} \sum_p (mp^{ms})^{-1} \text{Exp} \left(-\frac{p^m}{X} \right) + O((\log \log H)^{-1}) \quad (7)$$

where $X = \log H \log \log H$ and $s = 1 + it$.

PROOF. The lemma follows from the fact that the double sum on the right is

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \log \zeta(s+w) X^w \Gamma(w) dw \quad (8)$$

where $w = u + iv$ is a complex variable. Here we break off the portion $|v| \geq (\log H)^9$ with an error $O((\log \log H)^{-1})$ and move the line of integration to $u = -\frac{1}{4}$. Using Lemma 1 it is easily seen that the horizontal portions and the main integral contribute together $O((\log \log H)^{-1})$.

LEMMA 3. *Denote the double sum in (7) by S . Then*

$$S = \log \prod_{p \leq X} (1 - p^{-s})^{-1} + O((\log \log H)^{-1}). \quad (9)$$

PROOF. We use the fact that $\text{Exp}(-p^m X^{-1}) = 1 + O(p^m X^{-1})$ if $p^m \leq X$ and $= O(X p^{-m})$ if $p^m \geq X$. Using this it is easy to see that

$$\begin{aligned} S &= \sum_{p^m \leq X} \sum_{p^m \leq X} (m p^{ms})^{-1} + O\left(\sum_{p^m \leq X} \sum_{p^m \leq X} X^{-1}\right) + O\left(\sum_{p^m \geq X} \sum_{p^m \geq X} X(m p^{2m})^{-1}\right) \\ &= \sum_{p^m \leq X} \sum_{p^m \leq X} (m p^{ms})^{-1} + O((\log \log H)^{-1}). \end{aligned}$$

Denoting the last double sum by S_0 , we have,

$$S_0 - \sum_{p \leq X} \log(1 - p^{-s})^{-1} = O\left(\sum_{p^m \geq X, m \geq 2} \sum_{p^m \geq X, m \geq 2} (m p^m)^{-1}\right) = O((\log \log H)^{-1}).$$

LEMMA 4. We have, for $t \in J$,

$$\log \zeta(1 + it) = \sum_{p \leq X} \log(1 - p^{-s})^{-1} + O((\log \log H)^{-1}), \quad (10)$$

where $s = 1 + it$.

PROOF. Follows from Lemmas 1, 2 and 3.

LEMMA 5. Let $0 \leq r < 1, 0 \leq \phi < 2\pi$. Then, we have,

$$\log |(1 - r e^{i\phi})^{-z}| \leq -\text{Cos } \theta \log(\sqrt{1 - r^2 \text{Sin}^2 \theta} - r \text{Cos } \theta) + \text{Sin } \theta \text{Sin}^{-1}(r \text{Sin } \theta). \quad (11)$$

REMARK. Put

$$\lambda_p(\theta) = (1 - p^{-1}) \left(\sqrt{1 - p^{-2} \text{Sin}^2 \theta} - p^{-1} \text{Cos } \theta \right)^{-\text{Cos } \theta} \text{Exp} \left(\text{Sin } \theta \text{Sin}^{-1} \left(\frac{\text{Sin } \theta}{p} \right) \right). \quad (12)$$

In the lemma replace $r e^{i\phi}$ by p^{-s} . Lemmas 4 and 5 complete the proof of (4) and hence that of Theorem 2 since $\sum_{p \geq X} \log \lambda_p(\theta) = O(X^{-1})$ and

$$\prod_{p \leq X} (1 - p^{-1})^{-1} = e^\gamma \log X + O(1). \quad (\text{See [P] page 81}).$$

PROOF OF LEMMA 5. Denote the LHS of (11) by $g(\phi)$. Then

$$\begin{aligned} g(\phi) &= \sum_{n=1}^{\infty} n^{-1} r^n \text{Cos}(n\phi + \theta) \\ g'(\phi) &= -\sum_{n=1}^{\infty} r^n \text{Sin}(n\phi + \theta) \\ &= \text{Im} \left\{ \frac{-r e^{i(\phi + \theta)} (1 - r e^{-i\phi})}{(1 - r e^{i\phi})(1 - r e^{-i\phi})} \right\}. \end{aligned}$$

Hence $g'(\phi) = 0$ if $\text{Sin}(\phi + \theta) = r \text{Sin } \theta$, i.e. if

$$\phi = -\theta + \text{Sin}^{-1}(r \text{Sin } \theta). \quad (13)$$

At this point $g(\phi)$ attains the maximum as we shall show in the end. Now

$$\begin{aligned} g(\phi) &= \text{Re} \left\{ -e^{i\theta} \left(\log \sqrt{1 - 2r \text{Cos } \phi + r^2} - i \text{Sin}^{-1} \frac{r \text{Sin } \phi}{\sqrt{1 - 2r \text{Cos } \phi + r^2}} \right) \right\} \\ &= -\text{Cos } \theta \log \sqrt{1 - 2r \text{Cos } \phi + r^2} - \text{Sin } \theta \text{Sin}^{-1} \left(\frac{r \text{Sin } \phi}{\sqrt{1 - 2r \text{Cos } \phi + r^2}} \right) \end{aligned} \quad (14)$$

From (13) we have

$$\begin{aligned} \text{Sin } \phi &= r \text{Sin } \theta \text{Cos } \theta - \sqrt{1 - r^2 \text{Sin}^2 \theta} \text{Sin } \theta \\ &= -\text{Sin } \theta \left(\sqrt{1 - r^2 \text{Sin}^2 \theta} - r \text{Cos } \theta \right), \\ \text{Cos } \phi &= \sqrt{1 - r^2 \text{Sin}^2 \theta} \text{Cos } \theta + r \text{Sin}^2 \theta, \\ 1 - 2r \text{Cos } \phi + r^2 &= 1 - 2r \text{Cos } \theta \sqrt{1 - r^2 \text{Sin}^2 \theta} - 2r^2 \text{Sin}^2 \theta + r^2 \\ &= \left(\sqrt{1 - r^2 \text{Sin}^2 \theta} - r \text{Cos } \theta \right)^2, \end{aligned}$$

since $-r^2 \text{Sin}^2 \theta + r^2 \text{Cos}^2 \theta = -2r^2 \text{Sin}^2 \theta + r^2$. Hence

$$g(\phi) \leq h(\theta) \quad (15)$$

where $h(\theta)$ is the RHS of (11), provided $g(\phi)$ attains its maximum for the value ϕ given by (13). We now show that

(a) If $\text{Cos } \theta \geq 0$ then $g(\pi) < h(\theta)$

and

(b) If $\text{Cos } \theta < 0$ then $g(0) < h(\theta)$.

Note that $\text{Sin } \theta \text{Sin}^{-1}(r \text{Sin } \theta) \geq 0$. Hence it suffices to prove (in case (a))

$$\begin{aligned} g(\pi) &= \text{Re} \log \left\{ (1 - re^{i\phi})^{-z} \right\}_{\phi=\pi} \\ &= -\text{Cos } \theta \log(1 + r) < -\text{Cos } \theta \log(\sqrt{1 - r^2 \text{Sin}^2 \theta} - r \text{Cos } \theta) \end{aligned}$$

$$\text{i.e. } \log(1+r) > \log(\sqrt{1-r^2 \sin^2 \theta} - r \cos \theta)$$

$$\text{i.e. } (1+r+r \cos \theta)^2 > 1-r^2 \sin^2 \theta$$

$$\text{i.e. } (1+r)^2 + 2r(1+r) \cos \theta > 1-r^2$$

$$\text{i.e. } 1+r+2r \cos \theta > 1-r \text{ (true since } \cos \theta \geq 0)$$

In case (b) it suffices to prove

$$\begin{aligned} g(0) &= \operatorname{Re} \log\{(1-re^{i\phi})^{-z}\}_{\phi=0} \\ &= -\cos \theta \log(1-r) < -\cos \theta \log(\sqrt{1-r^2 \sin^2 \theta} - r \cos \theta) \end{aligned}$$

$$\text{i.e. } \log(1-r) < \log(\sqrt{1-r^2 \sin^2 \theta} - r \cos \theta)$$

$$\text{i.e. } 1-r < \sqrt{1-r^2 \sin^2 \theta} - r \cos \theta$$

$$\text{i.e. } (1-r+r \cos \theta)^2 < 1-r^2 \sin^2 \theta$$

$$\text{i.e. } (1-r)^2 + 2r(1-r) \cos \theta < 1-r^2$$

$$\text{i.e. } 1-r+2r \cos \theta < 1+r \text{ (which is true).}$$

Thus Lemma 5 is completely proved and hence (4) and Theorem 2 are completely proved.

REMARK 1. In the notation of [BR] our method gives $\gg TK^{-E}$ disjoint sub-intervals I (of length K) of $[T, 2T]$ for which $\max_{t \in I} |(\zeta(1+it))^z|$ lies between $e^{\gamma\lambda(\theta)}(\log \log K - \log \log \log K) + O(1)$ and $e^{\gamma\lambda(\theta)}(\log \log K + \log \log \log K) + O(1)$.

REMARK 2. By our method we can show that if $\zeta(s) \neq 0$ in the open half plane $\sigma > \frac{1}{2}$ then for $t \geq 100$ we have

$$|(\zeta(1+it))^z| \leq 2e^{\gamma\lambda(\theta)} \log \log t + O(1).$$

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ADDRESS OF THE AUTHOR

PROFESSOR K. RAMACHANDRA
SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
HOMI BHABHA ROAD
BOMBAY 400 005
INDIA

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