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ON THE FREQUENCY OF TITCHMARSH'S PHENOMENON  
FOR  $\zeta(s)$ -IX

BY

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§ 1. INTRODUCTION. In a previous paper [R], I proved the following result on  $\zeta(1+it)$ .

**THEOREM 1.** Let  $0 \leq \theta < 2\pi$ ,  $z = e^{i\theta}$  and

$$f(H) = \min_{T \geq 1} \max_{T \leq t \leq T+H} |(\zeta(1+it))^z| \quad (1)$$

Then

$$f(H) \geq e^\gamma \lambda(\theta) (\log \log H - \log \log \log H) + O(1) \quad (2)$$

where  $H \geq \text{Exp}(e^e)$ ,

$$\lambda(\theta) = \prod_p \lambda_p(\theta),$$

$$\lambda_p(\theta) = \left\{ \left(1 - \frac{1}{p}\right) \left( \sqrt{1 - \frac{\text{Sin}^2 \theta}{p^2}} - \frac{\text{Cos } \theta}{p} \right)^{-\text{Cos } \theta} \text{Exp} \left( \text{Sin } \theta \text{Sin}^{-1} \left( \frac{\text{Sin } \theta}{p} \right) \right) \right\}. \quad (3)$$

In the present paper I prove that

$$f(H) \leq e^\gamma \lambda(\theta) (\log \log H + \log \log \log H) + O(1). \quad (4)$$

This result together with Theorem 1 gives the following Theorem.

**THEOREM 2.** We have

$$|f(H)e^{-\gamma(\lambda(\theta))^{-1}} - \log \log H| \leq \log \log \log H + O(1), \quad (5)$$

where  $H \geq \text{Exp}(e^e)$ .

**REMARK.** It is interesting to prove (or disprove!)  $f(H)e^{-\gamma(\lambda(\theta))^{-1}} =$

$\log \log H + O(1)$ .

§ 2. PROOF OF (4). We begin by

**LEMMA 1.** *Let  $T = \text{Exp}((\log H)^2)$  where  $H$  exceeds an absolute constant. Then there exists a sub-interval  $I$  of  $[T, 2T]$  of length  $H + 2(\log H)^{10}$ , such that the rectangle  $(\sigma \geq \frac{3}{4}, t \in I)$  does not contain any zero of  $\zeta(s)$  and moreover*

$$\max |\log \zeta(\sigma + it)| = O((\log H)^{\frac{1}{4}} (\log \log H)^{-\frac{3}{4}}) \quad (6)$$

the maximum being taken over the rectangle referred to.

**PROOF.** Follows from [BR] and the result (due to A.E. Ingham [I], see also [T] page 236 and p. 293-295 [AI]) that the number of zeros of  $\zeta(s)$  in  $(\sigma \geq \frac{3}{4}, T \leq t \leq 2T)$  is  $O(T^{\frac{1}{4}})$ .

**LEMMA 2.** *Let  $J$  be the interval obtained by removing from  $I$  intervals of length  $(\log H)^{10}$  from both ends. Then for  $t \in J$ , we have,*

$$\log \zeta(1 + it) = \sum_{m \geq 1} \sum_p (mp^{ms})^{-1} \text{Exp} \left( -\frac{p^m}{X} \right) + O((\log \log H)^{-1}) \quad (7)$$

where  $X = \log H \log \log H$  and  $s = 1 + it$ .

**PROOF.** The lemma follows from the fact that the double sum on the right is

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \log \zeta(s+w) X^w \Gamma(w) dw \quad (8)$$

where  $w = u + iv$  is a complex variable. Here we break off the portion  $|v| \geq (\log H)^9$  with an error  $O((\log \log H)^{-1})$  and move the line of integration to  $u = -\frac{1}{4}$ . Using Lemma 1 it is easily seen that the horizontal portions and the main integral contribute together  $O((\log \log H)^{-1})$ .

**LEMMA 3.** *Denote the double sum in (7) by  $S$ . Then*

$$S = \log \prod_{p \leq X} (1 - p^{-s})^{-1} + O((\log \log H)^{-1}). \quad (9)$$

**PROOF.** We use the fact that  $\text{Exp}(-p^m X^{-1}) = 1 + O(p^m X^{-1})$  if  $p^m \leq X$  and  $= O(X p^{-m})$  if  $p^m \geq X$ . Using this it is easy to see that

$$\begin{aligned} S &= \sum_{p^m \leq X} \sum_{p^m \leq X} (m p^{ms})^{-1} + O\left(\sum_{p^m \leq X} \sum_{p^m \leq X} X^{-1}\right) + O\left(\sum_{p^m \geq X} \sum_{p^m \geq X} X(m p^{2m})^{-1}\right) \\ &= \sum_{p^m \leq X} \sum_{p^m \leq X} (m p^{ms})^{-1} + O((\log \log H)^{-1}). \end{aligned}$$

Denoting the last double sum by  $S_0$ , we have,

$$S_0 - \sum_{p \leq X} \log(1 - p^{-s})^{-1} = O\left(\sum_{p^m \geq X, m \geq 2} \sum_{p^m \geq X, m \geq 2} (m p^m)^{-1}\right) = O((\log \log H)^{-1}).$$

**LEMMA 4.** We have, for  $t \in J$ ,

$$\log \zeta(1 + it) = \sum_{p \leq X} \log(1 - p^{-s})^{-1} + O((\log \log H)^{-1}), \quad (10)$$

where  $s = 1 + it$ .

**PROOF.** Follows from Lemmas 1, 2 and 3.

**LEMMA 5.** Let  $0 \leq r < 1, 0 \leq \phi < 2\pi$ . Then, we have,

$$\log |(1 - r e^{i\phi})^{-z}| \leq -\text{Cos } \theta \log \left(\sqrt{1 - r^2 \text{Sin}^2 \theta} - r \text{Cos } \theta\right) + \text{Sin } \theta \text{Sin}^{-1}(r \text{Sin } \theta). \quad (11)$$

**REMARK.** Put

$$\lambda_p(\theta) = (1 - p^{-1}) \left(\sqrt{1 - p^{-2} \text{Sin}^2 \theta} - p^{-1} \text{Cos } \theta\right)^{-\text{Cos } \theta} \text{Exp} \left(\text{Sin } \theta \text{Sin}^{-1} \left(\frac{\text{Sin } \theta}{p}\right)\right). \quad (12)$$

In the lemma replace  $r e^{i\phi}$  by  $p^{-s}$ . Lemmas 4 and 5 complete the proof of (4) and hence that of Theorem 2 since  $\sum_{p \geq X} \log \lambda_p(\theta) = O(X^{-1})$  and

$$\prod_{p \leq X} (1 - p^{-1})^{-1} = e^\gamma \log X + O(1). \quad (\text{See [P] page 81}).$$

**PROOF OF LEMMA 5.** Denote the LHS of (11) by  $g(\phi)$ . Then

$$\begin{aligned} g(\phi) &= \sum_{n=1}^{\infty} n^{-1} r^n \text{Cos}(n\phi + \theta) \\ g'(\phi) &= -\sum_{n=1}^{\infty} r^n \text{Sin}(n\phi + \theta) \\ &= \text{Im} \left\{ \frac{-r e^{i(\phi + \theta)} (1 - r e^{-i\phi})}{(1 - r e^{i\phi})(1 - r e^{-i\phi})} \right\}. \end{aligned}$$

Hence  $g'(\phi) = 0$  if  $\text{Sin}(\phi + \theta) = r \text{Sin } \theta$ , i.e. if

$$\phi = -\theta + \text{Sin}^{-1}(r \text{Sin } \theta). \quad (13)$$

At this point  $g(\phi)$  attains the maximum as we shall show in the end. Now

$$\begin{aligned} g(\phi) &= \text{Re} \left\{ -e^{i\theta} \left( \log \sqrt{1 - 2r \text{Cos } \phi + r^2} - i \text{Sin}^{-1} \frac{r \text{Sin } \phi}{\sqrt{1 - 2r \text{Cos } \phi + r^2}} \right) \right\} \\ &= -\text{Cos } \theta \log \sqrt{1 - 2r \text{Cos } \phi + r^2} - \text{Sin } \theta \text{Sin}^{-1} \left( \frac{r \text{Sin } \phi}{\sqrt{1 - 2r \text{Cos } \phi + r^2}} \right) \end{aligned} \quad (14)$$

From (13) we have

$$\begin{aligned} \text{Sin } \phi &= r \text{Sin } \theta \text{Cos } \theta - \sqrt{1 - r^2 \text{Sin}^2 \theta} \text{Sin } \theta \\ &= -\text{Sin } \theta \left( \sqrt{1 - r^2 \text{Sin}^2 \theta} - r \text{Cos } \theta \right), \\ \text{Cos } \phi &= \sqrt{1 - r^2 \text{Sin}^2 \theta} \text{Cos } \theta + r \text{Sin}^2 \theta, \\ 1 - 2r \text{Cos } \phi + r^2 &= 1 - 2r \text{Cos } \theta \sqrt{1 - r^2 \text{Sin}^2 \theta} - 2r^2 \text{Sin}^2 \theta + r^2 \\ &= \left( \sqrt{1 - r^2 \text{Sin}^2 \theta} - r \text{Cos } \theta \right)^2, \end{aligned}$$

since  $-r^2 \text{Sin}^2 \theta + r^2 \text{Cos}^2 \theta = -2r^2 \text{Sin}^2 \theta + r^2$ . Hence

$$g(\phi) \leq h(\theta) \quad (15)$$

where  $h(\theta)$  is the RHS of (11), provided  $g(\phi)$  attains its maximum for the value  $\phi$  given by (13). We now show that

(a) If  $\text{Cos } \theta \geq 0$  then  $g(\pi) < h(\theta)$

and

(b) If  $\text{Cos } \theta < 0$  then  $g(0) < h(\theta)$ .

Note that  $\text{Sin } \theta \text{Sin}^{-1}(r \text{Sin } \theta) \geq 0$ . Hence it suffices to prove (in case (a))

$$\begin{aligned} g(\pi) &= \text{Re} \log \left\{ (1 - re^{i\phi})^{-z} \right\}_{\phi=\pi} \\ &= -\text{Cos } \theta \log(1 + r) < -\text{Cos } \theta \log(\sqrt{1 - r^2 \text{Sin}^2 \theta} - r \text{Cos } \theta) \end{aligned}$$

$$\text{i.e. } \log(1+r) > \log(\sqrt{1-r^2 \sin^2 \theta} - r \cos \theta)$$

$$\text{i.e. } (1+r+r \cos \theta)^2 > 1-r^2 \sin^2 \theta$$

$$\text{i.e. } (1+r)^2 + 2r(1+r) \cos \theta > 1-r^2$$

$$\text{i.e. } 1+r+2r \cos \theta > 1-r \text{ (true since } \cos \theta \geq 0)$$

In case (b) it suffices to prove

$$\begin{aligned} g(0) &= \operatorname{Re} \log\{(1-re^{i\phi})^{-z}\}_{\phi=0} \\ &= -\cos \theta \log(1-r) < -\cos \theta \log(\sqrt{1-r^2 \sin^2 \theta} - r \cos \theta) \end{aligned}$$

$$\text{i.e. } \log(1-r) < \log(\sqrt{1-r^2 \sin^2 \theta} - r \cos \theta)$$

$$\text{i.e. } 1-r < \sqrt{1-r^2 \sin^2 \theta} - r \cos \theta$$

$$\text{i.e. } (1-r+r \cos \theta)^2 < 1-r^2 \sin^2 \theta$$

$$\text{i.e. } (1-r)^2 + 2r(1-r) \cos \theta < 1-r^2$$

$$\text{i.e. } 1-r+2r \cos \theta < 1+r \text{ (which is true).}$$

Thus Lemma 5 is completely proved and hence (4) and Theorem 2 are completely proved.

**REMARK 1.** In the notation of [BR] our method gives  $\gg TK^{-E}$  disjoint sub-intervals  $I$  (of length  $K$ ) of  $[T, 2T]$  for which  $\max_{t \in I} |(\zeta(1+it))^z|$  lies between  $e^{\gamma\lambda(\theta)}(\log \log K - \log \log \log K) + O(1)$  and  $e^{\gamma\lambda(\theta)}(\log \log K + \log \log \log K) + O(1)$ .

**REMARK 2.** By our method we can show that if  $\zeta(s) \neq 0$  in the open half plane  $\sigma > \frac{1}{2}$  then for  $t \geq 100$  we have

$$|(\zeta(1+it))^z| \leq 2e^{\gamma\lambda(\theta)} \log \log t + O(1).$$

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