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PROOF OF SOME CONJECTURES ON THE MEAN-VALUE OF TITCHMARSH SERIES WITH APPLICATIONS TO TITCHMARSH'S PHENOMENON

BY

K. RAMACHANDRA

§ 1. INTRODUCTION. This is a continuation of [R]₁ but no previous knowledge of this paper is necessary. In fact we improve these results a good deal. However for some applications of the main results of the present paper, a knowledge of the results of [BRS] is assumed. The main results of the present paper are the following two theorems on what I call weak Titchmarsh series. We begin with a definition.

WEAK TITCHMARSH SERIES. Let $0 \leq \epsilon < 1, D \geq 1, C \geq 1$ and $H \geq 10$. Put $R = H^\epsilon$. Let $a_1 = \lambda_1 = 1$ and $\{\lambda_n\}$ ($n = 1, 2, 3, \dots$) be any sequence of real numbers with $\frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C$ ($n = 1, 2, 3, \dots$) and $\{a_n\}$ ($n = 1, 2, 3, \dots$) any sequence of complex numbers satisfying

$$\sum_{\lambda_n \leq X} |a_n| \leq D(\log X)^R$$

for all $X \geq 3C$. Then for complex $s = \sigma + it$ ($\sigma > 0$) we define the analytic function $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ as a weak Titchmarsh series associated with the parameters occurring in the definition.

THEOREM 1 (FOURTH MAIN THEOREM). For a weak Titchmarsh series $F(s)$ with $H \geq 36C^2 H^\epsilon$, we have

$$\liminf_{\sigma \rightarrow +0} \int_0^H |F(\sigma + it)| dt \geq H - 36C^2 H^\epsilon - 12CD.$$

THEOREM 2 (FIFTH MAIN THEOREM). *For a weak Titchmarsh series $F(s)$ with $\log H \geq 4320 C^2(1 - \varepsilon)^{-5}$, we have,*

$$\liminf_{\sigma \rightarrow +0} \int_0^H |F(\sigma + it)|^2 dt \geq \sum_{n \leq M} \left(H - \frac{H}{\log H} - 100C^2n \right) |a_n|^2 - 2D^2$$

where $M = (36C^2)^{-1} H^{1-\varepsilon} (\log H)^{-4}$.

REMARKS. Theorems 1 and 2 have been referred to as the fourth and the fifth main theorems in $[R]_2$. Also we remark that it is not difficult to improve the conditions in the theorems slightly.

§ 2. **PROOF OF THEOREM 1.** We can argue with $\sigma > 0$ and then pass to the limit as $\sigma \rightarrow +0$. But formally the notation is simplified if we treat as though $F(s)$ is convergent absolutely if $\sigma = 0$ and there is no loss of generality. Let r be a positive integer and $0 < U \leq r^{-1}H$. Then since $|F(s)| \geq 1 + \operatorname{Re}(F(s))$, we have (with $\lambda = u_1 + \dots + u_r$),

$$\begin{aligned} \int_0^H |F(it)| dt &\geq U^{-r} \int_0^U du_r \dots \int_0^U du_1 \int_\lambda^{H-rU+\lambda} |F(it)| dt \\ &\geq U^{-r} \int_0^U du_r \dots \int_0^U du_1 \int_\lambda^{H-rU+\lambda} \{1 + \operatorname{Re}(F(it))\} dt \\ &\geq H - rU - 2^{r+1}U^{-r}J \end{aligned}$$

where $J = \sum_{n=2}^{\infty} |a_n| (\log \lambda_n)^{-r-1}$. Now $J = S_0 + \sum_{j=1}^{\infty} S_j$ where $S_0 = \sum_{\lambda_n \leq 3C} |a_n| (\log \lambda_n)^{-r-1}$ and $S_j = \sum_{3^j C \leq \lambda_n \leq 3^{j+1}C} |a_n| (\log \lambda_n)^{-r-1}$. In S_0 we use $\lambda_n \geq \lambda_2 \geq 1 + C^{-1}$ and so $(\log \lambda_n)^{-r-1} \leq (2C)^{r+1}$ and we obtain $S_0 \leq D(2C)^{r+1}(3C)^R$. Also, we have,

$$\begin{aligned} S_j &\leq D(\log(3^{j+1}C))^R (\log(3^jC))^{-r-1} \\ &\leq D2^R (\log(3^jC))^{R-r-1}, (\text{since } 3^{j+1}C \leq (3^jC)^2), \\ &\leq D2^R j^{-2} \text{ by fixing } r = [3R]. \end{aligned}$$

Thus for $r = [3R]$ we have

$$\begin{aligned} J &\leq D(2C)^{r+1}(3C)^R + 2D2^R, (\text{since } \sum_{j=1}^{\infty} j^{-2} < 2), \\ &\leq 3D(2C)^{r+1}(3C)^R. \end{aligned}$$

Collecting we have,

$$\begin{aligned} 2^{r+1}U^{-r}J &\leq 12CD(3C)^R\left(\frac{4C}{U}\right)^r \\ &\leq 12CD\left(\frac{12C^2}{U}\right)^R \text{ if } U \geq 4C \\ &\leq 12CD \text{ by fixing } U = 12C^2. \end{aligned}$$

The only condition which we have to satisfy is $rU \leq H$ which is secured by $H \geq 36C^2H^\epsilon$. This completes the proof of Theorem 1.

§ 3. PROOF OF THEOREM 2. We write $\lambda = u_1 + \dots + u_r$, where $0 \leq u_i \leq U$ and $0 < U \leq r^{-1}H$. We put $M_1 = [M]$, $A(s) = \sum_{m \leq M_1} a_m \lambda_m^{-s}$ and $B(s) = \sum_{n \geq M_1+1} a_n \lambda_n^{-s}$ so that $F(s) = A(s) + B(s)$. For the moment we suppose M to be a free parameter with the restriction $3 \leq M \leq H$. We use

$$|F(it)|^2 \geq |A(it)|^2 + 2 \operatorname{Re}(A(it)\overline{B(it)}).$$

Now by a well-known theorem of H.L. Montgomery and R.C. Vaughan we have

$$\int_{\lambda}^{H-rU+\lambda} |A(it)|^2 dt \geq \sum_{n \leq M} (H - rU - 100C^2n) |a_n|^2$$

Next the absolute value of

$$2U^{-r} \int_0^U du_r \dots \int_0^U du_1 \int_{\lambda}^{H-rU+\lambda} (A(it)\overline{B(it)}) dt \quad (3.1)$$

does not exceed

$$\begin{aligned} &2^{r+2}U^{-r} \sum_{m \leq M_1, n \geq M_1+1} |a_m \overline{a_n}| \left(\log \frac{\lambda_n}{\lambda_m}\right)^{-r-1} \\ &\leq 2^{r+2}U^{-r} \left(\sum_{m \leq M_1} |a_m| \right) \left(\sum_{n \geq M_1+1} |a_n| \left(\log \frac{\lambda_n}{\lambda_{M_1}}\right)^{r+1} \right) \end{aligned}$$

Here the m -sum is $\leq D(\log \lambda_{M_1})^R \leq D(\log(3MC))^R$, since $\lambda_{M_1} \leq M_1C \leq MC$. It is enough to choose $M \geq 1$ for the bound for the m -sum. The n -sum can be broken up into $\lambda_n \leq 3\lambda_{M_1}$ and $3^j\lambda_{M_1} < \lambda_n \leq 3^{j+1}\lambda_{M_1}$ ($j =$

1, 2, 3, ...). Let us denote these sums by S_0 and S_j . Now since $\left(\log \frac{\lambda_n}{\lambda_{M_1}}\right) \geq \left(\log \frac{\lambda_{M_1+1}}{\lambda_{M_1}}\right) \geq \log \left(1 + \frac{1}{C\lambda_{M_1}}\right) \geq (2C\lambda_{M_1})^{-1} \geq (2C^2M)^{-1}$, we obtain

$$S_0 \leq D(\log(3\lambda_{M_1}))^R (2C^2M)^{r+1} \leq D(\log(3MC))^R (2C^2M)^{r+1}.$$

Also

$$\begin{aligned} S_j &\leq D(\log(3^{j+1}\lambda_{M_1}))^R (j \log 3)^{-r-1} \\ &\leq D(j \log 3 + \log(3MC))^R j^{-r-1} \\ &\leq 2^R D(j \log 3)^R (\log(3MC))^R j^{-r-1} \\ &\leq 4^R D(\log(3MC))^R j^{-2}, \text{ if } r \geq R+1, \end{aligned}$$

and so (since $\sum_{j=1}^{\infty} j^{-2} < 2$),

$$\left(\sum_m \dots\right) \left(\sum_n \dots\right) \leq D^2 (\log(3MC))^R (\log(3MC))^R Y$$

(where $Y = (2C^2M)^{r+1} + 2(4^R)$)

$$\leq D^2 (\log(3MC))^{2R} ((2C^2M)^{r+1} + 2(4^R)).$$

Hence the absolute value of the expression (3.1) does not exceed

$$\begin{aligned} &D^2 (\log(3MC))^{2R} \left((8C^2M) \left(\frac{4C^2M}{U} \right)^r + 2 \left(\frac{4^R}{U^r} \right) \right) \quad (3.2) \\ &\leq D^2 \left\{ 8C^2M \left(\frac{4C^2M(\log(3MC))^2}{U} \right)^{R+\log(8C^2M)} + 2 \left(\frac{4}{U} \right)^R \right\} \end{aligned}$$

if $U \geq 4C^2M$ and $r \geq R + \log(8C^2M)$. We put $U = 12C^2M(\log(3MC))^2$ and obtain for (3.2) the bound $D^2\{1+1\} \leq 2D^2$. The conditions to be satisfied are $M \geq 1$ and

$$12C^2M(\log(3MC))^2(R + \log(8C^2M) + 1) \leq H.$$

In fact we can satisfy $Ur \leq \frac{H}{\log H}$ by requiring

$$12C^2M(\log(3MC))^2(R + \log(8C^2M) + 1) \leq \frac{H}{\log H}.$$

This is satisfied if

$$36C^2M(\log(8C^2M))^3R \leq H(\log H)^{-1}$$

Let $8C^2M \leq H$. Then $36C^2MR \leq H(\log H)^{-4}$ gives what we want. We choose $M = (36C^2)^{-1}H^{1-\varepsilon}(\log H)^{-4}$. Clearly this satisfies $8C^2M \leq H$. In order to satisfy $M \geq 1$ we have to secure that

$$(36C^2)^{-1} \frac{((1-\varepsilon)(\log H))^5}{120} (\log H)^{-4} \geq 1$$

i.e. $\log H \geq 4320C^2(1-\varepsilon)^{-5}$.

This completes the proof of Theorem 2.

§ 4. APPLICATIONS OF THEOREMS 1 AND 2. An immediate application of Theorem 1 is

THEOREM 3. Let $\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}$ (where $0 < a \leq 1$) be the Hurwitz zeta-function in $\sigma > 1$ and consider its analytic continuation in $\sigma \geq 1$. Then

$$\min_{T \geq 1} \int_T^{T+H} |\zeta(1+it, a)| dt \geq \frac{1}{a}H + o(H).$$

Let $(\zeta(s))^u = \sum_{n=1}^{\infty} d_u(n)n^{-s}$ where u is any complex constant. Consider the analytic continuation of $(\zeta(s))^u$ in $\sigma \geq 1, t \geq 1$. An immediate corollary to Theorem 2 is

THEOREM 4. We have,

$$\min_{T \geq 1} \left(\frac{1}{H} \int_T^{T+H} |\zeta(1+it)^u|^2 dt \right) \geq \sum_{n=1}^{\infty} \frac{|d_u(n)|^2}{n^2} + o(1),$$

and in particular for $u = 1$, we have,

$$\min_{T \geq 1} \left(\frac{1}{H} \int_T^{T+H} |\zeta(1+it)|^2 dt \right) \geq \frac{\pi^2}{6} + o(1).$$

It is possible to prove by using Theorem 2 a very nice Ω theorem for $|\zeta(1+it)^z|$, where $z = e^{i\theta}$ ($0 \leq \theta < 2\pi, \theta$ fixed) as $t \rightarrow \infty$. It is

THEOREM 5. *We have,*

$$\min_{T \geq 1} \max_{T \leq t \leq T+H} |(\zeta(1+it))^z| \geq e^{\gamma} \lambda(\theta) (\log \log H - \log \log \log H) + O(1), \quad (4.1)$$

where

$$\lambda(\theta) = \prod_p \left\{ \left(1 - \frac{1}{p}\right) \left(\frac{\sqrt{p^2 - \sin^2 \theta} + \cos \theta}{p - p^{-1}} \right)^{\cos \theta} \exp \left(\sin \theta \sin^{-1} \left(\frac{\sin \theta}{p} \right) \right) \right\}$$

PROOF. By Theorem 2 we have with $\varepsilon = \frac{1}{3}$, $u = kz$,

$$\frac{1}{H} \int_T^{T+H} |(\zeta(1+it))^u|^2 dt \geq \frac{1}{2} \sum_{n \leq H^{\frac{1}{2}}} \frac{|d_u(n)|^2}{n^2} \quad (4.2)$$

uniformly in $T \geq 1$, and k any positive integer satisfying $1 \leq k \leq \log H$, provided H exceeds an absolute constant. Denote by S the RHS in (4.2). Then $S^{\frac{1}{2k}}$ has been studied in [BRS] as a function of H as k runs over $1 \leq k \leq \log H$. It has been proved (by considering the maximum term of the sum in S) that

$$\max_{1 \leq k \leq \log H} \left(S^{\frac{1}{2k}} \right) \geq e^{\gamma} \lambda(\theta) (\log \log H - \log \log \log H) + O(1).$$

This completes the proof of Theorem 5.

These ideas are quite general (applicable to zeta and L -functions of algebraic number fields). For example for ordinary L -series $L(s) = L(s, \chi)$ where χ is a non-principal character *mod* q , we can prove

THEOREM 6. *We have,*

$$\min_T \max_{T \leq t \leq T+H} |(L(1+it))^z| \geq e^{\gamma} \lambda(\theta) \frac{\varphi(q)}{q} \{(\log \log H - \log \log \log H) + O(1)\}$$

uniformly in $q \geq 3$.

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P.S. The theorems of § 4 can be stated for σ satisfying $|\sigma - 1| \leq \psi(t)$ for suitable $\psi(t) \rightarrow 0$ as $t \rightarrow \infty$.

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