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PROOF OF SOME CONJECTURES ON THE MEAN-VALUE  
OF TITCHMARSH SERIES WITH APPLICATIONS TO  
TITCHMARSH'S PHENOMENON

BY

K. RAMACHANDRA

§ 1. INTRODUCTION. This is a continuation of [R]<sub>1</sub> but no previous knowledge of this paper is necessary. In fact we improve these results a good deal. However for some applications of the main results of the present paper, a knowledge of the results of [BRS] is assumed. The main results of the present paper are the following two theorems on what I call weak Titchmarsh series. We begin with a definition.

**WEAK TITCHMARSH SERIES.** Let  $0 \leq \epsilon < 1, D \geq 1, C \geq 1$  and  $H \geq 10$ . Put  $R = H^\epsilon$ . Let  $a_1 = \lambda_1 = 1$  and  $\{\lambda_n\}$  ( $n = 1, 2, 3, \dots$ ) be any sequence of real numbers with  $\frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C$  ( $n = 1, 2, 3, \dots$ ) and  $\{a_n\}$  ( $n = 1, 2, 3, \dots$ ) any sequence of complex numbers satisfying

$$\sum_{\lambda_n \leq X} |a_n| \leq D(\log X)^R$$

for all  $X \geq 3C$ . Then for complex  $s = \sigma + it$  ( $\sigma > 0$ ) we define the analytic function  $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$  as a weak Titchmarsh series associated with the parameters occurring in the definition.

**THEOREM 1 (FOURTH MAIN THEOREM).** For a weak Titchmarsh series  $F(s)$  with  $H \geq 36C^2 H^\epsilon$ , we have

$$\liminf_{\sigma \rightarrow +0} \int_0^H |F(\sigma + it)| dt \geq H - 36C^2 H^\epsilon - 12CD.$$

**THEOREM 2 (FIFTH MAIN THEOREM).** *For a weak Titchmarsh series  $F(s)$  with  $\log H \geq 4320 C^2(1 - \varepsilon)^{-5}$ , we have,*

$$\liminf_{\sigma \rightarrow +0} \int_0^H |F(\sigma + it)|^2 dt \geq \sum_{n \leq M} \left( H - \frac{H}{\log H} - 100C^2n \right) |a_n|^2 - 2D^2$$

where  $M = (36C^2)^{-1} H^{1-\varepsilon} (\log H)^{-4}$ .

**REMARKS.** Theorems 1 and 2 have been referred to as the fourth and the fifth main theorems in  $[R]_2$ . Also we remark that it is not difficult to improve the conditions in the theorems slightly.

§ 2. **PROOF OF THEOREM 1.** We can argue with  $\sigma > 0$  and then pass to the limit as  $\sigma \rightarrow +0$ . But formally the notation is simplified if we treat as though  $F(s)$  is convergent absolutely if  $\sigma = 0$  and there is no loss of generality. Let  $r$  be a positive integer and  $0 < U \leq r^{-1}H$ . Then since  $|F(s)| \geq 1 + \operatorname{Re}(F(s))$ , we have (with  $\lambda = u_1 + \dots + u_r$ ),

$$\begin{aligned} \int_0^H |F(it)| dt &\geq U^{-r} \int_0^U du_r \dots \int_0^U du_1 \int_\lambda^{H-rU+\lambda} |F(it)| dt \\ &\geq U^{-r} \int_0^U du_r \dots \int_0^U du_1 \int_\lambda^{H-rU+\lambda} \{1 + \operatorname{Re}(F(it))\} dt \\ &\geq H - rU - 2^{r+1}U^{-r}J \end{aligned}$$

where  $J = \sum_{n=2}^{\infty} |a_n| (\log \lambda_n)^{-r-1}$ . Now  $J = S_0 + \sum_{j=1}^{\infty} S_j$  where  $S_0 = \sum_{\lambda_n \leq 3C} |a_n| (\log \lambda_n)^{-r-1}$  and  $S_j = \sum_{3^j C \leq \lambda_n \leq 3^{j+1}C} |a_n| (\log \lambda_n)^{-r-1}$ . In  $S_0$  we use  $\lambda_n \geq \lambda_2 \geq 1 + C^{-1}$  and so  $(\log \lambda_n)^{-r-1} \leq (2C)^{r+1}$  and we obtain  $S_0 \leq D(2C)^{r+1}(3C)^R$ . Also, we have,

$$\begin{aligned} S_j &\leq D(\log(3^{j+1}C))^R (\log(3^jC))^{-r-1} \\ &\leq D2^R (\log(3^jC))^{R-r-1}, (\text{since } 3^{j+1}C \leq (3^jC)^2), \\ &\leq D2^R j^{-2} \text{ by fixing } r = [3R]. \end{aligned}$$

Thus for  $r = [3R]$  we have

$$\begin{aligned} J &\leq D(2C)^{r+1}(3C)^R + 2D2^R, (\text{since } \sum_{j=1}^{\infty} j^{-2} < 2), \\ &\leq 3D(2C)^{r+1}(3C)^R. \end{aligned}$$

Collecting we have,

$$\begin{aligned} 2^{r+1}U^{-r}J &\leq 12CD(3C)^R\left(\frac{4C}{U}\right)^r \\ &\leq 12CD\left(\frac{12C^2}{U}\right)^R \text{ if } U \geq 4C \\ &\leq 12CD \text{ by fixing } U = 12C^2. \end{aligned}$$

The only condition which we have to satisfy is  $rU \leq H$  which is secured by  $H \geq 36C^2H^\epsilon$ . This completes the proof of Theorem 1.

**§ 3. PROOF OF THEOREM 2.** We write  $\lambda = u_1 + \dots + u_r$ , where  $0 \leq u_i \leq U$  and  $0 < U \leq r^{-1}H$ . We put  $M_1 = [M]$ ,  $A(s) = \sum_{m \leq M_1} a_m \lambda_m^{-s}$  and  $B(s) = \sum_{n \geq M_1+1} a_n \lambda_n^{-s}$  so that  $F(s) = A(s) + B(s)$ . For the moment we suppose  $M$  to be a free parameter with the restriction  $3 \leq M \leq H$ . We use

$$|F(it)|^2 \geq |A(it)|^2 + 2 \operatorname{Re}(A(it)\overline{B(it)}).$$

Now by a well-known theorem of H.L. Montgomery and R.C. Vaughan we have

$$\int_{\lambda}^{H-rU+\lambda} |A(it)|^2 dt \geq \sum_{n \leq M} (H - rU - 100C^2n) |a_n|^2$$

Next the absolute value of

$$2U^{-r} \int_0^U du_r \dots \int_0^U du_1 \int_{\lambda}^{H-rU+\lambda} (A(it)\overline{B(it)}) dt \quad (3.1)$$

does not exceed

$$\begin{aligned} &2^{r+2}U^{-r} \sum_{m \leq M_1, n \geq M_1+1} |a_m \overline{a_n}| \left(\log \frac{\lambda_n}{\lambda_m}\right)^{-r-1} \\ &\leq 2^{r+2}U^{-r} \left( \sum_{m \leq M_1} |a_m| \right) \left( \sum_{n \geq M_1+1} |a_n| \left(\log \frac{\lambda_n}{\lambda_{M_1}}\right)^{r+1} \right) \end{aligned}$$

Here the  $m$ -sum is  $\leq D(\log \lambda_{M_1})^R \leq D(\log(3MC))^R$ , since  $\lambda_{M_1} \leq M_1C \leq MC$ . It is enough to choose  $M \geq 1$  for the bound for the  $m$ -sum. The  $n$ -sum can be broken up into  $\lambda_n \leq 3\lambda_{M_1}$  and  $3^j\lambda_{M_1} < \lambda_n \leq 3^{j+1}\lambda_{M_1}$  ( $j =$

1, 2, 3, ...). Let us denote these sums by  $S_0$  and  $S_j$ . Now since  $\left(\log \frac{\lambda_n}{\lambda_{M_1}}\right) \geq \left(\log \frac{\lambda_{M_1+1}}{\lambda_{M_1}}\right) \geq \log \left(1 + \frac{1}{C\lambda_{M_1}}\right) \geq (2C\lambda_{M_1})^{-1} \geq (2C^2M)^{-1}$ , we obtain

$$S_0 \leq D(\log(3\lambda_{M_1}))^R (2C^2M)^{r+1} \leq D(\log(3MC))^R (2C^2M)^{r+1}.$$

Also

$$\begin{aligned} S_j &\leq D(\log(3^{j+1}\lambda_{M_1}))^R (j \log 3)^{-r-1} \\ &\leq D(j \log 3 + \log(3MC))^R j^{-r-1} \\ &\leq 2^R D(j \log 3)^R (\log(3MC))^R j^{-r-1} \\ &\leq 4^R D(\log(3MC))^R j^{-2}, \text{ if } r \geq R+1, \end{aligned}$$

and so (since  $\sum_{j=1}^{\infty} j^{-2} < 2$ ),

$$\left(\sum_m \dots\right) \left(\sum_n \dots\right) \leq D^2 (\log(3MC))^R (\log(3MC))^R Y$$

(where  $Y = (2C^2M)^{r+1} + 2(4^R)$ )

$$\leq D^2 (\log(3MC))^{2R} ((2C^2M)^{r+1} + 2(4^R)).$$

Hence the absolute value of the expression (3.1) does not exceed

$$\begin{aligned} &D^2 (\log(3MC))^{2R} \left( (8C^2M) \left( \frac{4C^2M}{U} \right)^r + 2 \left( \frac{4^R}{U^r} \right) \right) \quad (3.2) \\ &\leq D^2 \left\{ 8C^2M \left( \frac{4C^2M(\log(3MC))^2}{U} \right)^{R+\log(8C^2M)} + 2 \left( \frac{4}{U} \right)^R \right\} \end{aligned}$$

if  $U \geq 4C^2M$  and  $r \geq R + \log(8C^2M)$ . We put  $U = 12C^2M(\log(3MC))^2$  and obtain for (3.2) the bound  $D^2\{1+1\} \leq 2D^2$ . The conditions to be satisfied are  $M \geq 1$  and

$$12C^2M(\log(3MC))^2(R + \log(8C^2M) + 1) \leq H.$$

In fact we can satisfy  $Ur \leq \frac{H}{\log H}$  by requiring

$$12C^2M(\log(3MC))^2(R + \log(8C^2M) + 1) \leq \frac{H}{\log H}.$$

This is satisfied if

$$36C^2M(\log(8C^2M))^3R \leq H(\log H)^{-1}$$

Let  $8C^2M \leq H$ . Then  $36C^2MR \leq H(\log H)^{-4}$  gives what we want. We choose  $M = (36C^2)^{-1}H^{1-\varepsilon}(\log H)^{-4}$ . Clearly this satisfies  $8C^2M \leq H$ . In order to satisfy  $M \geq 1$  we have to secure that

$$(36C^2)^{-1} \frac{((1-\varepsilon)(\log H))^5}{120} (\log H)^{-4} \geq 1$$

i.e.  $\log H \geq 4320C^2(1-\varepsilon)^{-5}$ .

This completes the proof of Theorem 2.

§ 4. APPLICATIONS OF THEOREMS 1 AND 2. An immediate application of Theorem 1 is

**THEOREM 3.** Let  $\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}$  (where  $0 < a \leq 1$ ) be the Hurwitz zeta-function in  $\sigma > 1$  and consider its analytic continuation in  $\sigma \geq 1$ . Then

$$\min_{T \geq 1} \int_T^{T+H} |\zeta(1+it, a)| dt \geq \frac{1}{a}H + o(H).$$

Let  $(\zeta(s))^u = \sum_{n=1}^{\infty} d_u(n)n^{-s}$  where  $u$  is any complex constant. Consider the analytic continuation of  $(\zeta(s))^u$  in  $\sigma \geq 1, t \geq 1$ . An immediate corollary to Theorem 2 is

**THEOREM 4.** We have,

$$\min_{T \geq 1} \left( \frac{1}{H} \int_T^{T+H} |\zeta(1+it)^u|^2 dt \right) \geq \sum_{n=1}^{\infty} \frac{|d_u(n)|^2}{n^2} + o(1),$$

and in particular for  $u = 1$ , we have,

$$\min_{T \geq 1} \left( \frac{1}{H} \int_T^{T+H} |\zeta(1+it)|^2 dt \right) \geq \frac{\pi^2}{6} + o(1).$$

It is possible to prove by using Theorem 2 a very nice  $\Omega$  theorem for  $|\zeta(1+it)^z|$ , where  $z = e^{i\theta}$  ( $0 \leq \theta < 2\pi, \theta$  fixed) as  $t \rightarrow \infty$ . It is

**THEOREM 5.** *We have,*

$$\min_{T \geq 1} \max_{T \leq t \leq T+H} |(\zeta(1+it))^z| \geq e^{\gamma} \lambda(\theta) (\log \log H - \log \log \log H) + O(1), \quad (4.1)$$

where

$$\lambda(\theta) = \prod_p \left\{ \left(1 - \frac{1}{p}\right) \left( \frac{\sqrt{p^2 - \sin^2 \theta} + \cos \theta}{p - p^{-1}} \right)^{\cos \theta} \exp \left( \sin \theta \sin^{-1} \left( \frac{\sin \theta}{p} \right) \right) \right\}$$

**PROOF.** By Theorem 2 we have with  $\varepsilon = \frac{1}{3}$ ,  $u = kz$ ,

$$\frac{1}{H} \int_T^{T+H} |(\zeta(1+it))^u|^2 dt \geq \frac{1}{2} \sum_{n \leq H^{\frac{1}{4}}} \frac{|d_u(n)|^2}{n^2} \quad (4.2)$$

uniformly in  $T \geq 1$ , and  $k$  any positive integer satisfying  $1 \leq k \leq \log H$ , provided  $H$  exceeds an absolute constant. Denote by  $S$  the RHS in (4.2). Then  $S^{\frac{1}{2k}}$  has been studied in [BRS] as a function of  $H$  as  $k$  runs over  $1 \leq k \leq \log H$ . It has been proved (by considering the maximum term of the sum in  $S$ ) that

$$\max_{1 \leq k \leq \log H} \left( S^{\frac{1}{2k}} \right) \geq e^{\gamma} \lambda(\theta) (\log \log H - \log \log \log H) + O(1).$$

This completes the proof of Theorem 5.

These ideas are quite general (applicable to zeta and  $L$ -functions of algebraic number fields). For example for ordinary  $L$ -series  $L(s) = L(s, \chi)$  where  $\chi$  is a non-principal character mod  $q$ , we can prove

**THEOREM 6.** *We have,*

$$\min_T \max_{T \leq t \leq T+H} |(L(1+it))^z| \geq e^{\gamma} \lambda(\theta) \frac{\varphi(q)}{q} \{(\log \log H - \log \log \log H) + O(1)\}$$

uniformly in  $q \geq 3$ .

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## REFERENCES

- [BRS ] R. BALASUBRAMANIAN, K. RAMACHANDRA AND A. SANKARANARAYANAN, *On the frequency of Titchmarsh's phenomenon for  $\zeta(s)$ -VIII* (to appear).
- [R<sub>1</sub> ] K. RAMACHANDRA, *On the frequency of Titchmarsh's phenomenon for  $\zeta(s)$ -VII*, Annales Acad. Sci. Fenn. Ser AI, 14 (1989), 27-40.
- [R<sub>2</sub> ] K. RAMACHANDRA, *A short monograph on the Riemann zeta function* (a book to appear).

P.S. The theorems of § 4 can be stated for  $\sigma$  satisfying  $|\sigma - 1| \leq \psi(t)$  for suitable  $\psi(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

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