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PROOF OF SOME CONJECTURES ON THE  
MEAN-VALUE OF TITCHMARSH SERIES-I

BY

R. BALASUBRAMANIAN AND K. RAMACHANDRA

§ 1. INTRODUCTION. When we are integrating a function related to a series which we call TITCHMARSH SERIES, (a function of a real variable  $t$ )  $|F(it)|$ , or  $|F(it)|^2$  from  $t = 0$  to  $t = H$  ( $H \geq 10$ ) we encounter the following situation. Let  $a_1 = \lambda_1 = 1$  and  $\{a_n\}$  ( $n = 1, 2, 3, \dots$ ) be a sequence of complex numbers and  $\{\lambda_n\}$  ( $n = 1, 2, 3, \dots$ ) an increasing sequence of real numbers with  $\frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C$  for  $n \geq 1$ , where  $C$  is a positive constant. We suppose  $a_n$  to depend on  $n$  and  $H$  such that  $|a_n| \leq (nH)^A$  for  $n \geq 2$  and more generally we suppose that  $|a_n| n^{-A}$  is bounded above by a suitable big function (of  $A$  and)  $H$ , where  $A$  is a positive integer constant. (Also in the paper that follows K. Ramachandra  $[R]_2$  considers the case where instead of these conditions  $\sum_{n \leq X} |a_n|$  is bounded above by suitable functions of  $X$  and  $H$  for all  $X \geq 2$ ). We refer to all such series ( $F(s) = \sum_{n=1}^{\infty} (a_n \lambda_n^{-s})$ ,  $s = \sigma + it$ ,  $\sigma \geq A+2$ ) as TITCHMARSH SERIES. Trivially  $F(s)$  is analytic in  $\sigma \geq A+2$  and we suppose that  $F(s)$  can be continued analytically in  $(\sigma \geq 0, 0 \leq t \leq H)$  some times with some "growth conditions on certain lines". We put  $B = A+2$ . We prove in all five main theorems (the last two are in  $[R]_2$ , the paper that follows) on Titchmarsh series. Theorems 2 and 3 are sharper versions of two conjectures (stated by K. Ramachandra  $[R]_1$  in Durham Conference held in 1979). The last two

main theorems essentially due to K. Ramachandra [ $R$ ]<sub>2</sub> are published in the next paper. The first three are jointly due to R. Balasubramanian and K. Ramachandra and are published in the present paper. We begin by stating a main lemma.

§ 2. **MAIN LEMMA.** Let  $r$  be a positive integer  $H \geq (r+5)U$ ,  $U \geq 2^{70}(16B)^2$  and  $N$  and  $M$  positive integers subject to  $N > M \geq 1$ . Let  $b_m$  ( $m \leq M$ ) and  $c_n$  ( $n \geq N$ ) be complex numbers and  $A(s) = \sum_{m \leq M} b_m \lambda_m^{-s}$ . Let  $B(s) = \sum_{n \geq N} c_n \lambda_n^{-s}$  be absolutely convergent in  $\sigma \geq A+2$  and continuable analytically in  $\sigma \geq 0$ . Write  $g(s) = A(-s)B(s)$ ,

$$G(s) = U^{-r} \int_0^U du_r \cdots \int_0^U du_1 (g(s+i\lambda))$$

(here and elsewhere  $\lambda = u_1 + u_2 + \cdots + u_r$ ). Assume that there exist real numbers  $T_1$  and  $T_2$  with  $0 \leq T_1 \leq U$ ,  $H-U \leq T_2 \leq H$ , such that

$$|g(\sigma+iT_1)| + |g(\sigma+iT_2)| \leq \text{ExpExp}\left(\frac{U}{16B}\right)$$

uniformly in  $0 \leq \sigma \leq B$ . (As stated already  $B = A+2$ ). Let

$$S_1 = \sum_{m \leq M, n \geq N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^B 2^r (U \log \frac{\lambda_n}{\lambda_m})^{-r},$$

and

$$S_2 = \sum_{m \leq M, n \geq N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^B.$$

Then

$$\begin{aligned} \left| \int_{2U}^{H-(r+3)U} G(it) dt \right| &\leq \left| U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} g(it) dt \right| \\ &\leq 2B^2 U^{-10} + 54BU^{-1} \int_0^H |g(it)| dt \\ &\quad + (H + 64B^2) S_1 + 16B^2 \text{Exp}\left(-\frac{U}{8B}\right) S_2. \end{aligned}$$

To prove this main lemma we need five lemmas. After proving these we complete the proof of the main lemma.

**LEMMA 2.1.** Let  $z = x + iy$  be a complex variable with  $|x| \leq \frac{1}{4}$ . Then,

we have,

$$(a) \quad | \text{Exp}((\text{Sin } z)^2) | \leq e^{\frac{1}{2}} < 2 \text{ for all } y$$

and

$$(b) \quad \text{If } | y | \geq 2,$$

$$| \text{Exp}((\text{Sin } z)^2) | \leq e^{\frac{1}{2}} (\text{ExpExp } | y |)^{-1} < 2(\text{ExpExp } | y |)^{-1}.$$

**PROOF.** We have

$$\begin{aligned} \text{Re}(\text{Sin } z)^2 &= -\frac{1}{4} \text{Re} \left\{ \left( e^{i(x+iy)} - e^{-i(x+iy)} \right)^2 \right\} \\ &= -\frac{1}{4} \text{Re} \{ e^{2ix-2y} + e^{-2ix+2y} - 2 \} \\ &= \frac{1}{2} - \frac{1}{4} \{ (e^{-2y} + e^{2y}) \cos(2x) \}. \end{aligned}$$

But in  $| x | \leq \frac{1}{4}$ , we have  $\cos(2x) = \cos(| 2x |) \geq \cos \frac{1}{2} \geq \cos \frac{\pi}{6} \geq \frac{\sqrt{3}}{2}$ . The rest of the proof is trivial since (i)  $\cosh y$  is an increasing function of  $| y |$  and (ii) for  $| y | \geq 2$

$$\text{Exp}\left(-\frac{\sqrt{3}}{8} e^{2|y|}\right) \leq (\text{ExpExp } | y |)^{-1}$$

since  $e^2 > (2.7)^2$  and  $\frac{8}{\sqrt{3}} < \frac{8 \times 1.8}{3} = 4.8$  and so  $e^2 > \frac{8}{\sqrt{3}}$ . The lemma is completely proved.

**LEMMA 2.2.** For any two real numbers  $k$  and  $\sigma$  with  $0 < | \sigma | \leq 2B$ , we have,

$$\int_{-\infty}^{\infty} | \text{Exp} \left( \text{Sin}^2 \left( \frac{ik - \sigma - iu_1}{8B} \right) \right) \frac{du_1}{ik - \sigma - iu_1} | \leq 12 + 4 \log \left| \frac{2B}{\sigma} \right|.$$

**PROOF.** Split the integral into three parts  $J_1, J_2$  and  $J_3$  corresponding to  $| u_1 - k | \geq 2B, | \sigma | \leq | u_1 - k | \leq 2B$  and  $| u_1 - k | \leq \sigma$ . The contribution to  $J_1$  from  $| u_1 - k | \geq 16B$  is (by (b) of Lemma 2.1)

$$\begin{aligned} &\leq \frac{2e^{\frac{1}{2}}}{16B} \int_{16B}^{\infty} \text{Exp}\left(-\frac{u_1}{8B}\right) du_1 \\ &= e^{\frac{1}{2}} \int_2^{\infty} \text{Exp}(-u_1) du_1 = \text{Exp}\left(-\frac{3}{2}\right). \end{aligned}$$

The contribution to  $J_1$  from  $2B \leq |u_1 - k| \leq 16B$  is (by (a) of Lemma 2.1)

$$\leq e^{\frac{1}{2}} \int_{2B \leq |u_1 - k| \leq 16B} |u_1 - k|^{-1} du_1 = 2e^{\frac{1}{2}} \log 8 = 6e^{\frac{1}{2}} \log 2.$$

Now

$$6e^{\frac{1}{2}} \log 2 + \text{Exp} \left( -\frac{3}{2} \right) < 6 \left( 1 + \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{6 \cdot 2^2} \right) \left( \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^2} \right) \\ + \left( \frac{1}{2 \cdot 7} \right)^{3/2} < 8.$$

Thus  $|J_1| \leq 8$ . Using (a) of Lemma 2.1 we have  $|J_2| \leq 4 \log \left| \frac{2B}{\sigma} \right|$ . In  $J_3$  the integrand is at most  $e^{\frac{1}{2}} \sigma^{-1}$  in absolute value and so  $|J_3| \leq 2e^{\frac{1}{2}} \leq 4$ . Hence the lemma is completely proved.

**LEMMA 2.3.** *If  $n > m$ , we have, for all real  $k$ ,*

$$\left| \int_0^U du_r \cdots \int_0^U du_1 \left( \frac{\lambda_m}{\lambda_n} \right)^{i(k+\lambda)} \right| \leq 2^r \left( \log \frac{\lambda_n}{\lambda_m} \right)^{-}$$

**PROOF.** Trivial.

**LEMMA 2.4.** *For all real  $t$  and all  $D \geq B$ , we have,*

$$|G(D + it)| \leq S_1 \text{ and } |g(D + it)| \leq S_2.$$

**PROOF.** We have, trivially,

$$|g(D + it)| \leq \sum_{m \leq M, n \geq N} |b_m c_n| \left( \frac{\lambda_m}{\lambda_n} \right)^D$$

and the second result follows on observing that  $\frac{\lambda_m}{\lambda_n} < 1$  and so  $\left( \frac{\lambda_m}{\lambda_n} \right)^D \leq \left( \frac{\lambda_m}{\lambda_n} \right)^B$ .

Next

$$G(D + it) = U^{-r} \int_0^U du_r \cdots \int_0^U du_1 (g(D + it + i\lambda)) \\ = U^{-r} \sum_{m \leq M, n \geq N} b_m c_n \left( \frac{\lambda_m}{\lambda_n} \right)^D \int_0^U du_r \cdots \int_0^U du_1 \left( \frac{\lambda_m}{\lambda_n} \right)^{i(t+\lambda)}$$

Using Lemma 2.3 and observing  $\left(\frac{\lambda_m}{\lambda_n}\right)^D \leq \left(\frac{\lambda_m}{\lambda_n}\right)^B$  the first result follows.

**LEMMA 2.5** *Let  $0 < \sigma \leq B$  and  $2U \leq t \leq H - (r + 3)U$ . Then, for  $H \geq (r + 5)U$  and  $U \geq (20)!(16B)^2$ , we have,*

$$|G(\sigma + it)| \leq BU^{-10} + U^{-1} \left(2 + 4 \log \frac{2B}{\sigma}\right) \int_0^H |g(it)| dt \\ + 16 S_1 \log(2B) + 8BS_2 \text{Exp}\left(-\frac{U}{8B}\right).$$

**REMARK.**  $(20)! < 2^{70}$ .

**PROOF.** We note, by Cauchy's theorem, that

$$2\pi i g(\sigma + it + i\lambda) = \int_{iT_1}^{B+1+iT_1} + \int_{B+1+iT_1}^{B+1+iT_2} - \int_{iT_2}^{B+1+iT_2} - \int_{iT_1}^{iT_2} \\ \left\{ g(w) \text{Exp}\left(\text{Sin}^2\left(\frac{w-\sigma-it-i\lambda}{8B}\right)\right) \right\} \frac{dw}{w-\sigma-it-i\lambda} \\ J_1 + J_2 - J_3 - J_4 \text{ say.}$$

We write

$$2\pi i G(\sigma + it) = 2\pi i U^{-r} \int_0^U du_r \cdots \int_0^U du_1 (g(\sigma + it + i\lambda)) \\ = U^{-r} \int_0^U du_r \cdots \int_0^U du_1 (J_1 + J_2 - J_3 - J_4) \\ = J_5 + J_6 - J_7 - J_8 \text{ say.}$$

Let us look at  $J_5$ . In  $J_1$  (also in  $J_3$ )  $|g(w)| \leq \text{ExpExp}\left(\frac{U}{16B}\right)$  (by the definition of  $T_1$  and  $T_2$ ). Also by using Lemma 2.1 (b) (since  $|Re w - \sigma| \leq B + 1 \leq 2B$ , and  $|Im(w - it - i\lambda)| \geq U \geq (20)!(16B)^2$ ), we have,

$$\left| \text{Exp}\left(\left(\frac{w - \sigma - it - i\lambda}{8B}\right)\right) \right| \leq 2 \text{Exp}\left(-\frac{U}{8B}\right).$$

Hence

$$|J_1| \leq \frac{2(B+1)}{U} \text{Exp}\left(\text{Exp} \frac{U}{16B} - \text{Exp} \frac{U}{8B}\right) \\ \leq \frac{2(B+1)}{U} \text{Exp}\left(-\left(\text{Exp} \frac{U}{16B}\right) \left(\text{Exp} \frac{U}{16B} - 1\right)\right) \\ \leq \frac{B}{2} U^{-10},$$

since  $U \geq (20)!(16B)^2$  and so  $\text{Exp} \frac{U}{16B} - 1 \geq 1$  and  $\text{Exp} \left( -\text{Exp} \frac{U}{16B} \right) \leq \text{Exp} \left( -\text{Exp} U^{\frac{1}{2}} \right) \leq \text{Exp} \left( -U^{\frac{1}{2}} \right) \leq (20)!U^{-10}$ . Thus  $|J_6| \leq \frac{1}{2}BU^{-10}$ .

Similarly,  $|J_7| \leq \frac{1}{2}BU^{-10}$ . Next

$$J_8 = U^{-r} \int_{iT_1}^{iT_2} g(w)dw \int_0^U du_r \cdots \int_0^U du_2 \int_0^U \text{Exp} \left( \sin^2 \left( \frac{w - \sigma - it - i\lambda}{8B} \right) \right) \frac{du_1}{w - \sigma - it - i\lambda}.$$

We note that  $w - \sigma - it - i\lambda = ik - \sigma - iu_1$  where  $k = Im w - t - u_2 \cdots - u_r$ . Hence the  $u_1$ -integral is in absolute value (by Lemma 2.2)

$$\leq 12 + 4 \log \frac{2B}{\sigma}.$$

This shows that

$$\begin{aligned} |J_8| &\leq U^{-r} \int_{iT_1}^{iT_2} |g(w)dw| \left\{ U^{r-1} \left( 12 + 4 \log \frac{2B}{\sigma} \right) \right\} \\ &\leq U^{-1} \left( 12 + 4 \log \frac{2B}{\sigma} \right) \int_0^H |g(it)| dt \end{aligned}$$

Finally we consider  $J_6$ .

$$\begin{aligned} J_6 &= U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{B+1+iT_1}^{B+1+iT_2} g(w) \text{Exp} \left( \sin^2 \left( \frac{w - \sigma - it - i\lambda}{8B} \right) \right) \frac{dw}{w - \sigma - it} \\ &= U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{B+1-\sigma+iT_1-it-i\lambda}^{B+1-\sigma+iT_2-it-i\lambda} g(w + \sigma + it + i\lambda) \\ &\quad \text{Exp} \left( \sin^2 \left( \frac{w}{8B} \right) \right) \frac{dw}{w}. \end{aligned}$$

Using Lemma 2.1 (b) we extend the range of integration of  $w$  to  $(B + 1 - \sigma - i\infty, B + 1 - \sigma + i\infty)$  and this gives an error which is at most

$$\begin{aligned} U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{|Im w| \geq U, Re w = B+1-\sigma} |g(w + \sigma + it + i\lambda) \\ \text{Exp} \left( \sin^2 \left( \frac{w}{8B} \right) \right) \frac{dw}{w}|. \end{aligned}$$

By Lemma 2.4 this is

$$\leq S_2 U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{|Im w| \geq U, Re w = B+1-\sigma} | \text{Exp} \left( \sin^2 \frac{w}{8B} \right) \frac{dw}{w} |.$$

Here the innermost integral is (by Lemma 2.1(b))

$$\leq \frac{4}{U} \int_U^\infty \text{Exp}\left(-\frac{u}{8B}\right) du \leq \int_U^\infty \text{Exp}\left(-\frac{u}{8B}\right) du = 8B \text{Exp}\left(-\frac{U}{8B}\right).$$

Thus the error does not exceed  $8BS_2 \text{Exp}\left(-\frac{U}{8B}\right)$  and so

$$\begin{aligned} |J_6| &\leq |U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{B+1-\sigma-i\infty}^{B+1-\sigma+i\infty} g(w+\sigma+it+i\lambda) \text{Exp}\left(\text{Sin}^2\left(\frac{w}{8B}\right)\right) \frac{dw}{w}| \\ &\quad + 8BS_2 \text{Exp}\left(-\frac{U}{8B}\right) \\ &= |U^{-r} \int_{B+1-\sigma-i\infty}^{B+1-\sigma+i\infty} \text{Exp}\left(\text{Sin}^2\left(\frac{w}{8B}\right)\right) \frac{dw}{w} \int_0^U du_r \cdots \int_0^U du_1 g(w+\sigma+it+i\lambda)| + 8BS_2 \text{Exp}\left(-\frac{U}{8B}\right) \\ &= \left| \int_{B+1-\sigma-i\infty}^{B+1-\sigma+i\infty} G(w+\sigma+it) \text{Exp}\left(\text{Sin}^2\left(\frac{w}{8B}\right)\right) \frac{dw}{w} \right| + 8BS_2 \text{Exp}\left(-\frac{U}{8B}\right). \end{aligned}$$

Using the first part of Lemma 2.4 we obtain

$$\begin{aligned} |J_6| &\leq S_1 \int_{B+1-\sigma-i\infty}^{B+1-\sigma+i\infty} \left| \text{Exp}\left(\text{Sin}^2\left(\frac{w}{8B}\right)\right) \frac{dw}{w} \right| + 8BS_2 \text{Exp}\left(-\frac{U}{8B}\right) \\ &\leq S_1 \left(12 + 4 \log \frac{2B}{B+1-\sigma}\right) + 8BS_2 \text{Exp}\left(-\frac{U}{8B}\right) \end{aligned}$$

by using Lemma 2.2. Thus

$$|J_6| \leq 16S_1 \log(2B) + 8BS_2 \text{Exp}\left(-\frac{U}{8B}\right).$$

This completes the proof of the lemma.

We are now in a position to complete the proof of the main lemma. We first remark that

$$\begin{aligned} 4 \int_0^B \log \frac{2B}{\sigma} d\sigma &= 4B \log 2 + 4\sqrt{2} \int_0^B \left(\frac{B}{\sigma}\right)^{\frac{1}{2}} d\sigma \\ &< 4\left(\frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^2}\right)B + (8 \times 1.415)B < 15B. \end{aligned}$$

By Cauchy's theorem, we have,

$$\begin{aligned} \int_{2U}^{H-(r+3)U} G(it)idt &= \int_{i(2U)}^{i(H-(r+3)U)} G(s)ds \\ &= \int_{i(2U)}^{B+i(2U)} G(s)ds + \int_{B+i(2U)}^{B+i(H-(r+3)U)} G(s)ds - \\ &\quad - \int_{i(H-(r+3)U)}^{B+i(H-(r+3)U)} G(s)ds \\ &= J_1 + J_2 - J_3 \text{ say.} \end{aligned}$$



Using the estimate given in Lemma 2.5, we see that

$$\begin{aligned} |J_1| &\leq \int_0^B (BU^{-10} + \frac{(12+4 \log \frac{2B}{\sigma})}{U}) \int_0^H |g(it)| dt \\ &\quad + 16(\log(2B))S_1 + 8BS_2 \text{Exp}(-\frac{U}{8B})d\sigma \\ &\leq B^2U^{-10} + \frac{12B+15B}{U} \int_0^H |g(it)| dt + 16BS_1 \log(2B) \\ &\quad + 8B^2S_2 \text{Exp}(-\frac{U}{8B}). \end{aligned}$$

The same estimate holds for  $|J_3|$  also. For  $|J_2|$  we use the estimate given in Lemma 2.4 to get

$$|J_2| \leq HS_1.$$

This completes the proof of the main lemma.

**§ 3. FIRST MAIN THEOREM.** Let  $A, B, C$  be as before  $0 < \epsilon \leq \frac{1}{2}, r \geq [(200A + 200)\epsilon^{-1}]$ ,  $|a_n| \leq n^A H^{\frac{r\epsilon}{8}}$ . Then  $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$  is analytic in  $\sigma \geq A + 2$ . Let  $K \geq 30, U = H^{1-\frac{\epsilon}{2}} + 50B \log \log K_1$ . Assume that

$$H \geq (120B^2C^{2A+4}(4rC^2)^r)^{\frac{400}{\epsilon}} + (100rB)^{20} \log \log K_1,$$

and that there exist  $T_1, T_2$  with  $0 \leq T_1 \leq U, H - U \leq T_2 \leq H$  such that

$$|F(\sigma + iT_1)| + |F(\sigma + iT_2)| \leq K$$

uniformly in  $0 \leq \sigma \leq B$  where  $F(s)$  is assumed to be analytically continuable in  $\sigma \geq 0$ . Then

$$\int_0^H |F(it)|^2 dt \geq (H - 10rC^2H^{1-\frac{\epsilon}{2}} - 100rB \log \log K_1) \sum_{n \leq H^{1-\epsilon}} |a_n|^2,$$

where

$$K_1 = \left( \sum_{n \leq H^{1-\epsilon}} |a_n| \lambda_n^B \right) K + \left( \sum_{n \leq H^{1-\epsilon}} |a_n| \lambda_n^B \right)^2.$$

**REMARK 1.** We need the conditions  $H \geq (r + 5)U, U \geq 2^{70}(16B)^2$  in the application of the main lemma. All such conditions are satisfied by our lower bound choice for  $H$ . We have not attempted to obtain economical

lower bounds.

**REMARK 2.** Taking  $F(s) = (\zeta(\frac{1}{2} + it + iT))^k$  in the first main theorem we obtain the following as an immediate corollary. Let  $C(\varepsilon, k) \log \log T \leq H \leq T$ . Then for all integers  $k \geq 1$

$$\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k} dt \geq (1-\varepsilon) \sum_{n \leq H^{1-\varepsilon}} (d_k(n))^2 n^{-1} \geq (C'_k - 2\varepsilon)(\log H)^{k^2},$$

where

$$C'_k = (\Gamma(k^2 + 1))^{-1} \prod_p \left\{ (1 - p^{-1})^{k^2} \sum_{m=0}^{\infty} \left( \frac{\Gamma(k+m)}{\Gamma(k)m!} \right)^2 p^{-m} \right\}.$$

(This is because it is well-known that

$$\sum_{n \leq X} (d_k(n))^2 n^{-1} = \left\{ C'_k + O\left(\frac{1}{\log X}\right) \right\} (\log X)^{k^2}.$$

Our third main theorem gives a sharpening of this. The third main theorem is sharper than the conjecture (stated by K. Ramachandra [R]<sub>1</sub> in Durham conference 1979). The conjecture (as also the weaker form of the conjecture proved by him in the conference) would only give

$$\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k} dt \gg_k (\log H)^{k^2} \text{ in } C(k) \log \log T \leq H \leq T.$$

But the third main Theorem gives

$$\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k} dt \geq C'_k (\log H)^{k^2} + O\left(\frac{\log \log T}{H} (\log H)^{k^2}\right) + O((\log H)^{k^2-1})$$

where the  $O$ -constants depend only on  $k$ .

**REMARK 3.** The first main theorem gives a lower bound for  $\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k} dt$  uniformly in  $1 \leq k \leq \log H, T \geq H \geq 30$  and  $C \log \log T \leq H \leq T$ . From this it follows (as was shown in [B]<sub>1</sub>) that for  $C \log \log T \leq H \leq T$  we have uniformly

$$\max_{T \leq t \leq T+H} |\zeta(\frac{1}{2} + it)| > \text{Exp} \left( \frac{3}{4} \sqrt{\frac{\log H}{\log \log H}} \right)$$

if  $C$  is chosen to be a large positive constant. On Riemann hypothesis we can deduce from the first main theorem the following more general result. Let  $\theta$  be fixed and  $0 \leq \theta < 2\pi$ . Put  $z = e^{i\theta}$ . Then (on Riemann hypothesis), we have,

$$\max_{T \leq t \leq T+H} |(\zeta(\frac{1}{2} + it))^z| > \text{Exp} \left( \frac{3}{4} \sqrt{\frac{\log H}{\log \log H}} \right)$$

where the LHS is interpreted as  $\lim_{\sigma \rightarrow \frac{1}{2}+0}$  of the same expression with  $\frac{1}{2} + it$  replaced by  $\sigma + it$ . This result with  $\theta = \frac{\pi}{2}$  and  $\frac{3\pi}{2}$  gives a quantitative improvement of some results of J.H. Mueller [M].

PROOF. Write  $M = [H^{1-\epsilon}]$ ,  $N = M + 1$ ,  $A(s) = \sum_{m \leq M} \bar{a}_m \lambda_m^{-s}$ ,  $\bar{A}(s) = \sum_{m \leq M} a_m \lambda_m^{-s}$ ,  $B(s) = \sum_{n > N} a_n \lambda_n^{-s}$ . Then we have, in  $\sigma \geq A + 2$ ,

$$F(s) = \bar{A}(s) + B(s).$$

Also,

$$\begin{aligned} |F(it)|^2 &= |\bar{A}(it)|^2 + 2 \operatorname{Re}(A(-it)B(it)) + |B(it)|^2 \\ &\geq |\bar{A}(it)|^2 + 2 \operatorname{Re}(g(it)) \end{aligned}$$

where  $g(s) = A(-s)B(s)$ . Hence

$$\begin{aligned} \int_0^H |F(it)|^2 dt &\geq U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} |F(it)|^2 dt \\ &\geq U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} (|\bar{A}(it)|^2 + 2 \operatorname{Re} g(it)) dt \\ &= J_1 + 2J_2 \text{ say.} \end{aligned}$$

Now  $\log \left( \frac{\lambda_{n+1}}{\lambda_n} \right) = -\log \left( 1 - \left( 1 - \frac{\lambda_n}{\lambda_{n+1}} \right) \right) \geq \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} \geq (2nC^2)^{-1}$ . Hence by Montgomery-Vaughan theorem,

$$\begin{aligned} J_1 &\geq \int_{2U}^{H-(r+3)U} |\bar{A}(it)|^2 dt \\ &\geq \sum_{n \leq M} (H - (r+5)U - 100C^2n) |a_n|^2. \end{aligned}$$

We have

$$\begin{aligned}
 |g(s)| &= |A(-s)B(s)| = |A(-s)(F(s) - A(s))| \\
 &\leq \left( \sum_{n \leq H^{1-\epsilon}} |a_n| \lambda_n^B \right) K + \left( \sum_{n \leq H^{1-\epsilon}} |a_n| \lambda_n^B \right)^2 \\
 &= K_1.
 \end{aligned}$$

By the main lemma, we have,

$$\begin{aligned}
 |J_2| &\leq |U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} g(it) dt| \\
 &\leq \frac{2B^2}{U^{10}} + \frac{54B}{U} \int_0^H |g(it)| dt + (H + 64B^2)S_1 + 16B^2 S_2 \text{Exp}\left(-\frac{U}{8B}\right). \quad (3.1)
 \end{aligned}$$

We simplify the last expression in (3.1). We can assume that  $\int_0^H |F(it)|^2 dt \leq H \sum_{n \leq H^{1-\epsilon}} |a_n|^2$  (otherwise the result is trivially true). Hence

$$\begin{aligned}
 \int_0^H |g(it)| dt &= \int_0^H |A(-it)B(it)| dt \\
 &\leq \int_0^H |A(-it)|^2 dt + \int_0^H |B(it)|^2 dt \\
 &\leq \int_0^H |A(-it)|^2 dt + \int_0^H |F(it) - \bar{A}(it)|^2 dt \\
 &\leq 3 \int_0^H |A(-it)|^2 dt + 2 \int_0^H |F(it)|^2 dt \\
 &\leq 3 \sum_{n \leq M} (H + 100C^2 n) |a_n|^2 + 2H \sum_{n \leq M} |a_n|^2 \\
 &\leq (300C^2 + 5)H \sum_{n \leq M} |a_n|^2.
 \end{aligned}$$

$$\begin{aligned}
 S_2 &\leq \sum_{m \leq M, n \geq N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^{A+2} \\
 &\leq \sum_{m \leq M, n \geq N} |a_m a_n| \left(\frac{\lambda_m}{\lambda_n}\right)^{A+2} \\
 &\leq \sum_{m \leq M, n \geq N} m^A H^{\frac{\pi\epsilon}{8}} n^A H^{\frac{\pi\epsilon}{8}} (C^2 mn^{-1})^{A+2} \\
 &\leq H^{\frac{\pi\epsilon}{4}} C^{2A+4} \sum_{m \leq M} m^{2A+2} \sum_{n \geq N} n^{-2} \\
 &\leq H^{\frac{\pi\epsilon}{4} + 2A+3} C^{2A+4} \text{ since } \frac{\pi^2}{6} - 1 < 1.
 \end{aligned}$$

Now

$$S_1 \leq \left( U \log \frac{\lambda_N}{\lambda_M} \right)^{-r} 2^r S_2$$

and

$$\log \frac{\lambda_N}{\lambda_M} \geq \frac{1}{2} \frac{\lambda_N - \lambda_M}{\lambda_M} \geq (2C^2 M)^{-1},$$

$$U \log \left( \frac{\lambda_N}{\lambda_M} \right) \geq (2C^2)^{-1} H^{\frac{1}{2}}.$$

Thus

$$\begin{aligned} |J_2| &\leq \frac{2B^2}{U^{10}} + 54B(300C^2 + 5)HU^{-1} \sum_{n \leq M} |a_n|^2 \\ &\quad + (H + 64B^2)H^{-\frac{r}{4} + 2A + 3} 2^r (2C^2)^r C^{2A+4} \\ &\quad + 16B^2 \text{Exp} \left( -\frac{U}{8B} \right) H^{\frac{r}{4} + 2A + 3} C^{2A+4}. \text{(Note } a_1 = \lambda_1 = 1). \end{aligned}$$

So

$$\begin{aligned} &(\tau + 5)U + 100C^2 H^{1-\epsilon} + 2|J_2| \left( \sum_{n \leq M} |a_n|^2 \right)^{-1} \\ &\leq (\tau + 5)H^{1-\frac{1}{2}} + 100C^2 H^{1-\epsilon} + 100B r \log \log K_1 \\ &\quad + \frac{4B^2}{H^5} + 108B(300C^2 + 5)H^{\frac{1}{2}} + 128(2^r)(2C^2)^r B^2 H^{2A+4-50A} C^{2A+4} \\ &\quad + 32B^2 C^{2A+4} r! (8B)^r H^{2A+3+\frac{r}{4}-\frac{1}{2}} \\ &\leq 100 B r \log \log K_1 + r C^2 H^{1-\frac{1}{4}} \left\{ \frac{r+5}{r C^2 H^{\frac{1}{4}}} + \frac{100C^2}{H^{\frac{3}{4}}} \right\} \\ &\quad + \frac{4B^2}{H^5} + \frac{108B(300C^2+5)}{H^{1-\frac{1}{4}}} + 128(2^r)(2C^2)^r B^2 H^{-1} C^{2A+4} + 32B^2 C^{2A+4} r! (8B)^r H^{-1} \} \\ &\leq 100B r \log \log K_1 + 10C^2 r H^{1-\frac{1}{4}}. \end{aligned}$$

This completes the proof of the theorem.

§ 4. **SECOND MAIN THEOREM.** *We assume the same conditions as in the first main theorem except that we change the definition of  $U$  to  $U = H^{\frac{1}{2}} + 50B \log \log K_2$ . Then there holds*

$$\int_0^H |F(it)| dt \geq H - 10 r H^{\frac{1}{2}} - 100r B \log \log K_2,$$

where  $K_2 = K + 1$ .

**REMARK.** Conditions like  $H \geq (\tau + 5)U, U \geq 2^{70}(16B)^2$  are taken care of by the inequality for  $H$ .

**PROOF.** We have,

$$\begin{aligned} \int_0^H |F(it)| dt &\geq U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(\tau+3)U+\lambda} |F(it)| dt \\ &\geq U^{-r} \operatorname{Re} \left( \int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(\tau+3)U+\lambda} F(it) dt \right) \\ &= U^{-r} \operatorname{Re} \left\{ \int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(\tau+3)U+\lambda} (1 + A(-it)B(it)) dt \right\} \end{aligned}$$

(where  $A(s) \equiv 1$  (i.e.  $a_1 = 1 = M$ ) and  $B(s) = F(s) - 1 = J_1 + \operatorname{Re} J_2$  say. Clearly  $J_1 \geq H - (\tau + 5)U$ . For  $J_2$  we use the main lemma.

$$|J_2| \leq \frac{2B^2}{U^{10}} + \frac{54B}{U} \int_0^H |g(it)| dt + (H + 64B^2)S_1 + 16B^2 \operatorname{Exp} \left( -\frac{U}{8B} \right) S_2. \quad (4.1)$$

As in the proof of the first main theorem we can assume  $\int_0^H |F(it)| dt \leq H$  and so  $\int_0^H |g(it)| dt \leq 2H$ . We have  $|g(s)| \leq K + 1 = K_2$ . Now

$$S_2 \leq H^{\frac{\tau}{4}+2A+3} C^{2A+4},$$

and  $U \log \left( \frac{\lambda N}{\lambda M} \right) = U \log \lambda_2 \geq (2C)^{-1}U$ ,

$$S_1 \leq 2^r S_2 (U \log \lambda_2)^{-r} \leq 2^r S_2 ((2C)^{-1}U)^{-r}.$$

This shows that

$$\begin{aligned} &(\tau + 5)U + |J_2| \\ &\leq (\tau + 5)U + \frac{2B^2}{U^{10}} + \frac{54B}{U} 2H + \frac{(H+64B^2)}{(2C^{-1}U)^r} 2^r C^{2A+4} H^{\frac{\tau}{4}+2A+3} \\ &+ 16B^2 \operatorname{Exp} \left( -\frac{U}{8B} \right) H^{\frac{\tau}{4}+2A+3} C^{2A+4} \\ &\leq 100rB \log \log K_2 + rH^{\frac{7}{8}} \left\{ \frac{\tau+5}{r} + \frac{2B^2}{rH^{\frac{8}{9}}} + \frac{108B}{H^{\frac{1}{4}}} \right\} \\ &+ (H + 64B^2) 2^r C^{2A+4} H^{\frac{\tau}{4}+2A+3+\frac{r}{16}-(\tau+1)\frac{7}{8}} \end{aligned}$$

$$\begin{aligned}
 &+16B^2C^{2A+4}(8B)^r r! H^{\frac{r}{4}+2A+3-\frac{r}{8}} \\
 &\leq 10rH^{\frac{1}{8}} + 100rB \log \log K_2,
 \end{aligned}$$

when  $H$  satisfies the inequality of the theorem.

**§ 5. THIRD MAIN THEOREM.** Let  $\{a_n\}$  and  $\{\lambda_n\}$  be as in the introduction and  $|a_n| \leq (nH)^A$  where  $A \geq 1$  is an integer constant. Then  $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$  is analytic in  $\sigma \geq A+2$ . Suppose  $F(s)$  is analytically continuable in  $\sigma \geq 0$ . Assume that (for some  $K \geq 30$ ) there exist  $T_1$  and  $T_2$  with  $0 \leq T_1 \leq H^{\frac{1}{8}}$ ,  $H - H^{\frac{1}{8}} \leq T_2 \leq H$  such that  $|F(\sigma + iT_1)| + |F(\sigma + iT_2)| \leq K$  uniformly in  $0 \leq \sigma \leq A+2$ . Let

$$H \geq (4C)^{9000A^2} + 520000 A^2 \log \log K_3.$$

Then

$$\int_0^H |F(it)|^2 dt \geq \sum_{n \leq \alpha H} (H - (3C)^{1000A} H^{\frac{1}{8}} - 130000 A^2 \log \log K_3 - 100C^2 n) |a_n|^2,$$

where  $\alpha = (200C^2)^{-1} 2^{-8A-20}$  and

$$K_3 = \left( \sum_{n \leq H} |a_n| \lambda_n^B \right) K + \left( \sum_{n \leq H} |a_n| \lambda_n^B \right)^2.$$

To prove this theorem we need the following two lemmas.

**LEMMA 5.1.** In the interval  $[\alpha H, (1600C^2)^{-1} H]$  there exists an  $X$  such that

$$\sum_{X \leq n \leq X+H^{\frac{1}{4}}} |a_n|^2 \leq H^{-\frac{1}{4}} \sum_{n \leq X} |a_n|^2,$$

provided  $H \geq 2^{1000A^2} C^{50A}$ .

**PROOF.** Assume that such an  $X$  does not exist. Then for all  $X$  in  $[\alpha H, (1600C^2)^{-1} H]$ ,

$$\sum_{X \leq n \leq X+H^{\frac{1}{4}}} |a_n|^2 > H^{-\frac{1}{4}} \sum_{n \leq X} |a_n|^2. \quad (5.1)$$

Let  $L = \alpha H$ ,  $I_j = [2^{j-1}L, 2^jL]$  for  $j = 1, 2, \dots, 8A + 17$ . Also let  $I_0 = [1, L]$ . Put  $S_j = \sum_{n \in I_j} |a_n|^2$  ( $j = 0, 1, 2, \dots, 8A + 17$ ). For  $j \geq 1$  divide the interval  $I_j$  into maximum number of disjoint sub-intervals each of length  $H^{\frac{1}{2}}$  (discarding the bit at one end). Since the lemma is assumed to be false the sum over each sub-interval is  $\geq H^{-\frac{1}{2}}S_{j-1}$ . The number of sub-intervals is  $\geq [2^{j-1}LH^{-\frac{1}{2}}] - 1 \geq 2^{j-2}LH^{-\frac{1}{2}}$  (provided  $2^{j-1}LH^{-\frac{1}{2}} - 2 \geq 2^{j-2}LH^{-\frac{1}{2}}$ , i.e.  $2^{j-2}LH^{-\frac{1}{2}} \geq 2$  i.e.  $\alpha H^{\frac{3}{2}} \geq 4$  i.e.  $H \geq (4\alpha^{-1})^{\frac{2}{3}}$ ). It follows that  $S_j \geq 2^{j-2}LH^{-\frac{1}{2}}S_{j-1}$ . By induction  $S_j \geq (\frac{1}{2}LH^{-\frac{1}{2}})^j S_0$ . Since  $S_0 \geq 1$  we have in particular

$$S_{8A+17} \geq \left(\frac{1}{2}\alpha H^{\frac{1}{2}}\right)^{8A+17} \geq \left(\frac{1}{2}\alpha\right)^{8A+17} H^{4A+\frac{1}{2}\cdot 17}.$$

On the other hand

$$S_{8A+17} = \sum_{\alpha_1 H \leq n \leq \alpha_2 H} |a_n|^2 \leq \sum_{n \leq \alpha_2 H} (nH)^{2A},$$

where  $\alpha_1 = 16^{-1}(200C^2)^{-1}$  and  $\alpha_2 = 8^{-1}(200C^2)^{-1}$ . Thus  $S_{8A+17} \leq H^{4A+1}$ . Combining the upper and lower bounds we are led to

$$H^{\frac{1}{2}\cdot 15} \leq (2\alpha^{-1})^{8A+17} \tag{5.2}$$

provided  $H \geq (4\alpha^{-1})^{\frac{4}{3}}$  (the latter condition is satisfied by the inequality for  $H$  prescribed by the Lemma). But (5.2) contradicts the inequality prescribed for  $H$  by the lemma. This contradiction proves the Lemma.

From now on we assume that  $X$  is as given by Lemma 5.1.

**LEMMA 5.2.** Let  $\bar{A}(s) = \sum_{n \leq X} a_n \lambda_n^{-s}$ ,  $E(s) = \sum_{X \leq n \leq X+H^{\frac{1}{2}}} a_n \lambda_n^{-s}$  and  $B(s) = F(s) - \bar{A}(s) - E(s)$ . Clearly in  $\sigma \geq A+2$  we have  $B(s) = \sum_{n \geq X+H^{\frac{1}{2}}} a_n \lambda_n^{-s}$ . Let

$$H \geq 2^{1000A^2} C^{50A}, U = H^{\frac{1}{2}} + 100 B \log \log K_3, K_3 \geq 30 \text{ and } H \geq (2r+5)U.$$

Then we have the following five inequalities.

(a)  $\int_0^H |\bar{A}(it)|^2 dt \leq 100C^2 H \sum_{n \leq X} |a_n|^2,$



$$(b) \int_{2U+rU}^{H-(r+3)U} |\overline{A}(it)|^2 dt \geq \sum_{n \leq X} (H - (2r+5)U - 100C^2n) |a_n|^2,$$

$$(c) \int_0^H |E(it)|^2 dt \leq 100C^2 H^{\frac{1}{2}} \sum_{n \leq X} |a_n|^2,$$

$$(d) \int_0^H |B(it)|^2 dt \leq 1000C^2 H \sum_{n \leq X} |a_n|^2,$$

and finally

$$(e) \int_0^H |A(-it)B(it)| dt \leq 400C^2 H \sum_{n \leq X} |a_n|^2,$$

where (d) and (e) are true provided

$$\int_0^H |F(it)|^2 dt \leq H \sum_{n \leq X} |a_n|^2.$$

**PROOF.** The inequalities (a) and (b) follow from the Montgomery-Vaughan theorem. From the same theorem

$$\begin{aligned} \int_0^H |E(it)|^2 dt &\leq \sum_{X \leq n \leq X+H^{\frac{1}{2}}} (H + 100C^2n) |a_n|^2 \\ &\leq 100C^2 H \sum_{X \leq n \leq X+H^{\frac{1}{2}}} |a_n|^2 \end{aligned}$$

and hence (c) follows from Lemma 5.1. Since

$$|B(it)|^2 \leq 9(|F(it)|^2 + |\overline{A}(it)|^2 + |E(it)|^2)$$

the inequality (d) follows from (a) and (c). Lastly (e) follows from (a) and (d). Thus the lemma is completely proved.

We are now in a position to prove the theorem. We write (with  $\lambda = u_1 + u_2 + \dots + u_r$  as usual)

$$\int_0^H |F(it)|^2 dt \geq U^{-r} \int_0^U du_r \dots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} |F(it)|^2 dt$$

(where  $(r+5)U \leq H$  and  $0 \leq u_i \leq U$ . In fact we assume  $(2r+5)U \leq H$ ).

Now

$$|F(it)|^2 \geq |\overline{A}(it)|^2 + 2\operatorname{Re}(A(-it)B(it)) + 2\operatorname{Re}(A(-it)E(it)) + 2\operatorname{Re}(\overline{B}(-it)E(it)),$$

where  $\overline{B}(s)$  is the analytic continuation of  $\sum_{n \geq X+H^{\frac{1}{2}}} a_n \lambda_n^{-s}$ . Accordingly

$$\int_0^H |F(it)|^2 dt \geq J_1 + J_2 + J_3 + J_4 \quad (5.3)$$

where

$$J_1 = \int_0^H |\overline{A}(it)|^2 dt, J_2 = 2 \operatorname{Re} \int_0^H (A(-it)B(it)) dt,$$

$$J_3 = 2 \operatorname{Re} \int_0^H (A(-it)E(it)) dt \text{ and } J_4 = 2 \operatorname{Re} \int_0^H (\overline{B}(-it)E(it)) dt.$$

By Lemma 5.2(b), we have,

$$J_1 \geq \sum_{n \leq X} (H - (2r + 5)U - 100C^2n) |a_n|^2.$$

Also by Lemma 5.2 ((a) and (c)), we have,

$$|J_3| \leq 2 \int_0^H |A(-it)E(it)| dt \leq 200C^2 H^{\frac{7}{8}} \sum_{n \leq X} |a_n|^2.$$

Similarly by Lemma 5.2 ((c) and (d)),

$$|J_4| \leq 800C^2 H^{\frac{7}{8}} \sum_{n \leq X} |a_n|^2.$$

For  $J_2$  we use the main lemma. We choose  $U = H^{\frac{7}{8}} + 100 B \log \log K_3$ . We have  $g(s) = A(-s)B(s)$ . We have

$$|g(s)| \leq \left( \sum_{n \leq H} |a_n| \lambda_n^B \right) K + \left( \sum_{n \leq H} |a_n| \lambda_n^B \right)^2 = K_3.$$

By Lemma 5.2 ((e)) we have

$$\int_0^H |g(it)| dt \leq 400C^2 H \sum_{n \leq X} |a_n|^2$$

Again

$$\begin{aligned} S_2 &\leq \sum_{m \leq X, n \geq X+H^{\frac{1}{4}}} |a_m| |a_n| \left(\frac{\lambda_m}{\lambda_n}\right)^{A+2} \\ &\leq \sum_{m \leq X, n \geq X+H^{\frac{1}{4}}} (mH)^A (nH)^A (C^2 mn^{-1})^{A+2} \\ &\leq C^{2A+4} H^{4A+3}. \end{aligned}$$

Put  $x = \frac{\lambda_N}{\lambda_M} - 1$  where  $N = [X + H^{\frac{1}{4}}]$ ,  $M = [X]$ . Then  $0 < x < \frac{2C(N-M)}{C^{-1}M} < \frac{3C^2 H^{\frac{1}{4}}}{\alpha H} < \frac{1}{2}$  under the conditions on  $H$  imposed in the theorem. Hence

$$U \log \left( \frac{\lambda_N}{\lambda_M} \right) \geq \frac{U}{2} \left( \frac{\lambda_N - \lambda_M}{\lambda_M} \right) \geq \frac{U}{2} \left( \frac{N - M - 3}{C^2 M} \right) \geq \frac{1}{2} H^{\frac{1}{8}} \left( \frac{H^{\frac{1}{4}} - 3}{C^2 H} \right) \geq \frac{H^{\frac{1}{8}}}{3C^2},$$

(under the conditions on  $H$  imposed in the theorem). Thus

$$S_1 \leq 2^r S_2 H^{-\frac{7}{8}} (3C^2)^r.$$

We choose  $r = 100A + 100$  and check that  $U \geq 2^{70}(16B)^2$ , and that  $H \geq (2r + 5)U$ . Thus by applying the main Lemma we obtain

$$\begin{aligned} \left| \frac{1}{2} J_2 \right| &\leq \left\{ \frac{2B^2}{U^{10}} + \frac{54B}{U} (400C^2 H) + \frac{(H + 64B^2) 2^r C^{2A+4} H^{4A+3}}{((3C^2)^{-1} H^{\frac{1}{8}})^r} \right. \\ &\quad \left. + 16B^2 \text{Exp} \left( -\frac{U}{8B} \right) C^{2A+4} H^{4A+3} \right\} \sum_{n \leq X} |a_n|^2. \end{aligned}$$

Hence

$$\int_0^H |F(it)|^2 \geq \sum_{n \leq \alpha H} (H - D - 100C^2 n) |a_n|^2,$$

where

$$\begin{aligned} D &= (2r + 5)U + 1000C^2 H^{\frac{7}{8}} + \frac{4B^2}{U^{10}} + \frac{43200C^2 B H}{U} \\ &+ (H + 64B^2) 2^{r+1} C^{2A+4} (3C^2)^r H^{4A+3-\frac{7}{8}} \\ &+ 32B^2 \text{Exp} \left( -\frac{U}{8B} \right) C^{2A+4} H^{4A+3} \\ &< 130000A^2 \log \log K_3 + 405AH^{\frac{7}{8}} + 1000C^2 H^{\frac{7}{8}} + 36A^2 H^{\frac{7}{8}} \end{aligned}$$

$$\begin{aligned}
 &+43200C^2(3A)H^{\frac{7}{8}} \\
 &+600A^2H(2^{100A+101})C^{2A+4}3^{100A+100}C^{200A+200}H^{4A+3-12A-12} \\
 &+300A^2C^{2A+4}(720)(56)(24A)^8H^{\frac{7}{8}} \\
 &\leq 130000A^2\log\log K_3 + H^{\frac{7}{8}}\{405A + 1000C^2 + 36A^2 + 129600AC^2 \\
 &+ 600A^2C^{408A}3^{401A} + 3^{58}A^{10}C^{6A}\} \\
 &\leq 130000A^2\log\log K_3 + (3C)^{1000A}.
 \end{aligned}$$

This proves the theorem completely.

The next two theorems due to K. Ramachandra belong to a different class in the sense that restrictions of bounds like those involving  $K$  do not appear. His paper follows ours.

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