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THE NUMBER OF FINITE NON-ISOMORPHIC ABELIAN GROUPS IN MEAN SQUARE

Aleksandar Ivic

ABSTRACT. Let  $\Delta(x) = \sum_{n \leq x} a(n) - \sum_{j=1}^6 c_j x^{1/j}$  denote the error term in the abelian group problem. Using zeta-function methods it is proved that

$$\int_1^X \Delta^2(x) dx \ll X^{39/29} \log^2 X,$$

where the exponent  $39/29 = 1.344827\dots$  is close to the best possible exponent  $4/3$  in this problem.

Let as usual  $a(n)$  denote the number of non-isomorphic abelian groups with  $n$  elements. This is a well-known multiplicative function, whose generating Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a(n)n^{-s} = \zeta(s)\zeta(2s)\zeta(3s)\zeta(4s)\dots \quad (\text{Re } s > 1)$$

shows that methods from zeta-function theory may be used in various problems involving  $a(n)$ . A classical problem, investigated first by P. Erdős and G. Szekeres [2], is to determine the order of the error term in the asymptotic formula for  $A(x) = \sum_{n \leq x} a(n)$ .

Since  $F(s)$  has only simple poles for  $s = 1/j$  ( $j = 1, 2, \dots$ ), it is natural to expect that  $A(x)$  is well approximated by a finite sum of residues of  $F(s)x^s s^{-1}$  at  $s=1/j$ . With this in mind we define

$$\Delta(x) = \sum_{n \leq x} a(n) - \int_1^6 \operatorname{Res}_{s=1/j} F(s)x^s s^{-1} = \sum_{n \leq x} a(n) - \int_1^6 c_j x^{1/j},$$

where  $c_j = \prod_{k=1, k \neq j} \zeta(k/j)$ . Upper bound estimates for  $\Delta(x)$  have a long history (see [6], [7], and Ch. 14 of [3]), and the best published result

$$\Delta(x) \ll x^{97/381} \log^{35} x$$

is due to G. Kolesnik [5]. This is far from the bound  $\Delta(x) \ll x^{1/6+c}$ , which one may conjecture in this problem. In the other direction, W. Schwarz [6] proved  $\Delta(x) = \Omega(x^{1/6-\delta})$  for any  $\delta > 0$  under the truth of the Riemann hypothesis. Here as usual,  $f(x) = \Omega(g(x))$  means that  $f(x) = o(g(x))$  does not hold as  $x \rightarrow \infty$ . A simple unconditional proof of this result is given in Ch. 14 of [3], but a sharper result, namely  $\Delta(x) = \Omega(x^{1/6} \log^{1/2} x)$ , may be derived from the general method of R. Balasubramanian and K. Ramachandra [1]. It was kindly pointed to the author by K. Ramachandra that their methods lead to the slightly sharper

$$(1) \quad \int_1^X \Delta^2(x) dx = \Omega(X^{4/3} \log X),$$

which was independently proved in [4], where it was conjectured that the integral in (1) is asymptotic to  $CX^{4/3} \log X$  ( $C > 0, X \rightarrow \infty$ ). It was also stated in [1] without proof that

$$(2) \quad \int_1^X \Delta^2(x) dx \ll X^{4/3} \log^2 X$$

holds. However, in correspondence with Professor K. Ramachandra it transpired that this claim is, unfortunately, unsubstantiated. It

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seems that, at present, it is not possible to obtain (2) even if one assumes the Riemann hypothesis. If one assumes the Lindelöf hypothesis ( $\zeta(\frac{1}{2}+it) \ll t^c$ ; a well-known consequence of the Riemann hypothesis), then it is possible to obtain a result only a little weaker than (2), namely  $X^{4/3} \log^3 X$  in place of  $X^{4/3} \log^2 X$ . The author is very grateful to Professor K. Ramachandra for this and other remarks concerning this topic.

The aim of this note is to prove a good, unconditional bound for the mean square of  $\Delta(x)$ . This is contained in the following

Theorem.

$$(3) \quad \int_1^X \Delta^2(x) dx \ll X^{39/29} \log^2 X.$$

The exponent  $\frac{39}{29} = \frac{40-1}{30-1} = 1.34482758\dots$  is close to the best possible exponent  $\frac{4}{3}$ , and it could be still slightly decreased, but not beyond  $\frac{67}{50} = 1.34$  by existing results concerning power moments of  $\zeta(s)$ .

Proof of the Theorem. We suppose  $X/2 \leq x \leq X$ ,  $1/6 < \sigma < 1/5$ ,

$X^c \ll G \ll X^{1-c}$ ,  $F(s) = \zeta(s)\zeta(2s)\zeta(3s)\dots$ . An application of

Perron's formula (see the Appendix of [3]) gives

$$(4) \quad \Delta(x) - c_0 x^{1/6} = O(GX^c) + O\left(G \int_{\sigma}^{1+c} |F(\alpha+iXG^{-1})| x^{\alpha-1} d\alpha\right) \\ + (2\pi)^{-1} \int_{-X/G}^{X/G} F(\alpha+it) \frac{x^{\sigma+it}}{\sigma+it} dt.$$

The first step in the proof is to show that

$$(5) \quad X^c G + G \int_{\sigma}^{1+c} |F(\alpha+iXG^{-1})| x^{\alpha-1} d\alpha \ll X^{1/6}$$

for a suitable  $G$ , and the choice will be  $G = X^c$ . To bound  $F$  it will be sufficient to use

$$(6) \quad \zeta(\sigma+it) \ll (1+t)^{(1-\sigma)/3} \log t \quad (\sigma \geq 1/2, t \geq t_0)$$

and the functional equation  $\zeta(s) = \chi(s)\zeta(1-s)$ , where

$$t^{\frac{1}{2}-\sigma} \ll \chi(\sigma+it) \ll t^{\frac{1}{2}-\sigma} \quad (\text{see [3]}). \text{ Since the exponent of } t \text{ in (6) is}$$

a linear function of  $\sigma$  it is easily checked that  $|F(\alpha+iXG^{-1})|$  is majorized by bounds valid for  $\alpha=\sigma$  and  $\alpha=1+c$ , respectively. In the latter case the bound in question is trivially  $\ll X^c$ , while for  $\alpha = \sigma$  we have

$$\begin{aligned} F(\sigma+iXG^{-1}) &\ll \prod_{j=1}^5 |\zeta(j\sigma+iXG^{-1})| \\ &\ll (XG^{-1})^{1-3\sigma} |\zeta(1-\sigma+iXG^{-1})\zeta(1-2\sigma+2iXG^{-1})| \prod_{j=3}^5 |\zeta(j\sigma+iXG^{-1})| \\ &\ll (XG^{-1})^{1-3\sigma+c} (XG^{-1})^{(\sigma+2\sigma+1-3\sigma+1-4\sigma+1-5\sigma)/3} \ll X^{2-6\sigma+c}. \end{aligned}$$

Therefore the left-hand side of (5) is  $\ll X^{1-5\sigma+c} \ll X^{1/6}$  for  $c$  sufficiently small, since  $\sigma > 1/6$ . Thus from (4) with  $G=X^c$  we obtain

$$\begin{aligned} \int_{X/2}^X \Delta^2(x) dx &\ll X^{4/3} + \int_{-X^{1+c}}^{X^{1-c}} \int_{-X^{1-c}}^{X^{1-c}} \frac{F(\sigma+it)F(\sigma-iu)}{(\sigma+it)(\sigma-iu)} \int_{X/2}^X x^{2\sigma+it-iu} dx dt du \\ &\ll X^{4/3} + X^{2\sigma+1} \int_{-X^{1-c}}^{X^{1-c}} \int_{-X^{1-c}}^{X^{1-c}} \left| \frac{F(\sigma+it)F(\sigma+iu)}{(\sigma+it)(\sigma+iu)} \right| \frac{dt du}{1+|t-u|}. \end{aligned}$$

In view of  $|ab| \leq \frac{1}{2}(|a|^2 + |b|^2)$  the double integral does not exceed

$$\int_{-X^{1-c}}^{X^{1-c}} |F(\sigma+it)|^2 (\sigma^2+t^2)^{-1} \left( \int_{-X^{1-c}}^{X^{1-c}} \frac{du}{1+|t-u|} \right) dt$$

$$\ll \log X \int_{-X^{1-c}}^{X^{1-c}} |F(\sigma+it)|^2 (\sigma^2+t^2)^{-1} dt.$$

Using again the functional equation for  $\zeta(s)$  we obtain then

$$(7) \int_{X/2}^X \Delta^2(x) dx \ll X^{4/3} + X^{2\sigma+1} \log^2 X \left[ 1 + \sum_{1 \ll M \leq X^{1-c}} M^{-6\sigma} I_M \right],$$

where for brevity we have set

$$(8) I_M = \int_{M/2}^M |\zeta(1-\sigma+it)\zeta(1-2\sigma+2it)\zeta(3\sigma+3it)\sigma(4\sigma+4it)\zeta(5\sigma+5it)|^2 dt.$$

To bound  $I_M$  we shall use Hölder's inequality and power moments for  $\zeta(s)$ . Namely, for  $1/2 < \sigma < 1$  fixed we define  $m(\sigma) (\geq 4)$  as the supremum of all numbers  $m$  such that

$$\int_1^T |\zeta(\sigma+it)|^m dt \ll T^{1+c}$$

holds for any  $c > 0$ . We shall use the following bounds for  $m(\sigma)$  (Th.8.4 of [3]), which are hitherto the sharpest ones:

$m(\sigma) \geq 4/(3-4\sigma)$	for $1/2 < \sigma \leq 5/8$ ,
$m(\sigma) \geq 10/(5-6\sigma)$	for $5/8 \leq \sigma \leq 35/54$ ,
$m(\sigma) \geq 19/(6-6\sigma)$	for $35/54 \leq \sigma \leq 41/60$ ,
$m(\sigma) \geq 2112/(659-948\sigma)$	for $41/60 \leq \sigma \leq 3/4$ ,
$m(\sigma) \geq 12408/(4537-4830\sigma)$	for $3/4 \leq \sigma \leq 5/6$ ,
$m(\sigma) \geq 4324/(1031-1044\sigma)$	for $5/6 \leq \sigma \leq 7/8$ .

Now we choose  $\sigma = 5/29 = 0.172413793\dots$  and we obtain

$$\begin{aligned} \sigma_1 = 3\sigma = 0.51724137\dots & \quad m(\sigma_1) \geq 4.29629629\dots = p_1 \\ \sigma_2 = 1-2\sigma = 0.65517241\dots & \quad m(\sigma_2) \geq 9.18333333\dots = p_2 \\ \sigma_3 = 4\sigma = 0.68965517\dots & \quad m(\sigma_3) \geq 10.29205176\dots = p_3 \end{aligned}$$

$$\sigma_4 = 1 - \sigma = 0.82758620\dots \quad m(\sigma_4) \geq 25.31710406\dots = p_4.$$

$$\sigma_5 = 5\sigma = 0.86206896\dots \quad m(\sigma_5) \geq 33.00763359\dots = p_5.$$

It follows then by Hölder's inequality for integrals that

$$I_M \leq \prod_{j=1}^4 \left[ \int_{M/2}^M |\zeta(\sigma_j + it)|^{p_j} dt \right]^{2/p_j} \left[ \int_{M/2}^M |\zeta(\sigma_5 + 5it)|^{2A} dt \right]^{1/A}.$$

where

$$\frac{1}{A} + \frac{2}{p_1} + \frac{2}{p_2} + \frac{2}{p_3} + \frac{2}{p_4} = 1,$$

hence  $A = 23.05516\dots$ ,  $2A = 46.11033\dots$ ,  $2A - p_5 = 13.10270\dots$ .

Therefore by the choice of  $p_1, \dots, p_5$  we have

$$\begin{aligned} I_M &\ll M^{1+\epsilon} \max_{M/2 \leq t \leq M} |\zeta(\sigma_5 + 5it)|^{(2A-p_5)/A} \\ &\ll M^{1+(2A-p_5)(1-\sigma_5)/(5A)+\epsilon} = M^{1.01567\dots+\epsilon}, \end{aligned}$$

since  $\zeta(\sigma + it) \ll t^{(1-\sigma)/5} \log t$  for  $5/6 \leq \sigma \leq 1$ ,  $t \geq t_0$ . But

$-6\sigma = -1.03448\dots$ , hence (7) and (8) yield

$$\int_{X/2}^X \Delta^2(x) dx \ll X^{4/3+X^{2\sigma+1}} \log^2 X \ll X^{2\sigma+1} \log^2 X$$

for  $\sigma = 5/23$ . Replacing  $X$  by  $X/2$ ,  $X/2^2, \dots$  etc. and adding one easily obtains (3).

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