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HYBRID MEAN SQUARE OF L-FUNCTIONS

By M. J. Narlikar

§1. Introduction. Let $k \geq 1$ be an integer and χ run through the set of characters mod k . Let $T \geq 0$ and $T_0 = T+2$. Then R. Balasubramanian and K. Ramachandra proved the following theorem. For the history of the theorem see [1].

Theorem 1. We have

$$\frac{1}{2\pi\phi(k)} \sum_{\chi \pmod{k}} \int_0^T |L(\frac{1}{2}+it, \chi)|^2 dt = \frac{\phi(k)}{k^2} A(k, T) + O\left(\frac{\phi(k)}{k} T^{1/2} \log T_0\right)$$

where $A(k, T) = \frac{kT}{2\pi} \log \frac{kT}{2\pi} + \left[2\gamma-1 + \sum_{p|k} \frac{\log p}{p-1}\right] \frac{kT}{2\pi}$, γ being the

Euler's constant.

In this paper I prove the following

Theorem 2. For $T \geq 2$ and $k \geq 3$,

$$\begin{aligned} \frac{1}{2\pi\phi(k)} \sum_{\chi \pmod{k}} \int_0^T |L(\frac{1}{2}+it, \chi)|^2 dt &= \frac{\phi(k)}{k^2} A(k, T) + \\ &+ B(k)T^{1/2} + O\left(\frac{\phi(k)}{k} T^{5/12} (\log T)^2\right) \end{aligned}$$

where $B(k)$ is a negative constant and depends only on k . Moreover

$$|B(k)| \gg \frac{\phi(k)}{k} \quad \text{and} \quad |B(k)| \ll \frac{\phi(k)}{k}.$$

§2. Proof of the Theorem 2.

Lemma 1. We have

$$\frac{1}{\phi(k)} \sum_{\chi \bmod k} |L(\frac{1}{2}+it, \chi)|^2 = \frac{1}{k} \sum_{\substack{(a,k)=1 \\ 1 \leq a \leq k}} |\zeta(\frac{1}{2}+it, \frac{a}{k})|^2$$

where $\zeta(s, \alpha)$ for $0 < \alpha < 1$ is the Hurwitz Zeta-function defined by

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n+\alpha)^{-s} \quad \text{for} \quad \text{Re } s > 1.$$

Proof. Follows from the fact that if $F(a)$ is a complex number for every residue class $a \bmod k$ with $(a, k) = 1$, then

$$\frac{1}{\phi(k)} \sum_{\chi \bmod k} \left| \sum_{\substack{(a,k)=1 \\ 1 \leq a \leq k}} \chi(a)F(a) \right|^2 = \sum_{\substack{(a,k)=1 \\ 1 \leq a \leq k}} |F(a)|^2.$$

Lemma 2. Theorem 1 gives Theorem 2 for $1 \leq T \leq 2$ and hence it is sufficient to prove that for $T \geq 2$

$$\frac{1}{2\pi\phi(k)} \sum_{\chi \bmod k} \int_1^T |L(\frac{1}{2}+it, \chi)|^2 dt = \frac{\phi(k)}{k^2} \{A(k, T) - A(k, 1)\} + B(k)T^{1/2} + O\left(\frac{\phi(k)}{k} T^{5/12} (1 \log T)^2\right).$$

where $B(k)$ is as in Theorem 2.

Lemma 3. For $T \geq 2$, $0 < \alpha < 1$,

$$\frac{1}{2\pi} \int_1^T |\zeta(\frac{1}{2}+it, \alpha)|^2 dt = \frac{T}{2\pi} \log \frac{T}{2\pi} + \frac{T}{2\pi} [C(\alpha) + \gamma - 1] - \frac{1}{2\pi\alpha} A_1(\alpha) T^{1/2} + O(T^{5/12} (\log T)^2) + O\left(\frac{\log T}{\sqrt{2\alpha}}\right).$$

where $\|2\alpha\|$ denotes the distance of 2α from the nearest integer and

$$C(\alpha) = \lim_{N \rightarrow \infty} \left[\sum_{n=1}^N \frac{1}{(n+\alpha)} - \log N \right] \text{ and}$$

$$A_1(\alpha) = \frac{1}{\sqrt{2\pi}} \left[-2H(\alpha) + \int_{1-\alpha}^1 \frac{\cos\{\pi(u^2-u-1/4)\}}{\cos \pi u} + 2 \sum_{r=1}^{\infty} \int_r^{\infty} \cos\{\pi(u^2-1/4)\} du \right],$$

$$H(\alpha) = \int_{\frac{1-\alpha}{2}}^{\frac{1+\alpha}{2}} G^2(u) du + \int_{\alpha/2}^{1-\alpha/2} G^2(u) du$$

where $G(u) = \frac{\cos\{2\pi(u^2-u-1/16)\}}{\cos(2\pi u)}$.

Finally, $A_1(\alpha) < -\frac{1}{\sqrt{2\pi}} H(\alpha)$

Proof. See [2].

Lemma 4. We have

$$B(k) = \frac{1}{\sqrt{2\pi} k} \sum_{\substack{1 \leq a \leq k \\ (a, k)=1}} \left[-2H\left(\frac{a}{k}\right) + \int_{1-\frac{a}{k}}^1 \frac{\cos\{\pi(u^2-u-1/4)\}}{\cos \pi u} du \right. \\ \left. + 2 \sum_{r=1}^{\infty} \int_r^{\infty} \cos\{\pi(u^2-1/4)\} du \right]$$

and $|B(k)| \ll \frac{\phi(k)}{k}$.

Proof. Follows from the expression for $A_1(\alpha)$ and the last part of

Lemma 3.

Lemma 5. In Lemma 3, the error term $O\left(\frac{\log T}{\sqrt{2\alpha}}\right)$ can be replaced by

$$O\left[\min\left\{\frac{\log T}{\sqrt{2\alpha}}, T^{1/2} \log T\right\}\right].$$

Proof. The sums in Lemmas 11 to 14 of [2] which contribute to this O-term add up to

$$[\sqrt{T/2\pi}] \cdot 2 \operatorname{Re} \sum_{m=1}^{\infty} \sum_{0 \leq r \leq m} \frac{1}{4} \operatorname{Exp}\left[2\pi i(2m\alpha + 2r\alpha + \alpha^2)\right] \int_{r+\alpha}^{\infty} \operatorname{Exp}\left[i\pi\left(e^{2-\frac{1}{4}}\right)\right] d\theta.$$

Now it is easy to see that this sum is $O(T^{1/2} \log T)$. We only have to integrate by parts and obtain an estimate $O\left(\frac{1}{r+\alpha}\right)$ for $r \geq 1$ and $O(1)$ for $r=0$, for the expression inside the summation sign.

Lemma 6. We have

$$\frac{1}{k} \sum_{\substack{(a,k)=1 \\ 1 \leq a < k}} \min\left[\frac{\log T}{\sqrt{2a/k}}, T^{1/2} \log T\right] \ll_c \frac{\phi(k)}{k} T^c.$$

Proof. Put $S = \frac{1}{\phi(k)} \sum_{\substack{(a,k)=1 \\ 1 \leq a < k}} \min\left[\frac{\log T}{\sqrt{2a/k}}, T^{1/2} \log T\right]$

and $S_U = \frac{1}{\phi(k)} \sum_{\substack{1/2U \leq \sqrt{2a/k} < 1/U \\ (a,k)=1, 1 \leq a < k}} \min\left[\frac{\log T}{\sqrt{2a/k}}, T^{1/2} \log T\right]$

where U runs through powers 2^r of 2, $2^r < k$. We divide the totality of S_U into 3 parts according as the integer nearest to $\frac{2a}{k}$ is 0, 1 or 2. In each case the number of integers a such that $\frac{1}{2U} \leq \sqrt{\frac{2a}{k}} \leq \frac{1}{U}$ is $\ll \frac{\phi(k)}{U} + 2^{\omega(k)}$, where $\omega(k)$ is the number of different prime factors of k . Hence, if $U \leq X$, taking the first term in the

minimum,

$$S_U \ll \left[1 + U \frac{2^{\varphi(k)}}{\varphi(k)} \right] \log T.$$

For $U > X$, we take the second term in the minimum, so that

$$S_U \ll T^{1/2} \log T \left[\frac{1 + 2^{\omega(k)}}{U + \varphi(k)} \right].$$

Now $2^{\omega(k)} \ll k^c$, hence

$$\frac{S}{\log T} \ll \log X + \frac{X}{X^{1-c}} + \frac{T^{1/2}}{k^{1-c}}.$$

If $k < T^{10}$ we take $X=k$ and we do not get the last two terms as the case $U > X$ does not arise. If $k \geq T^{10}$, take $X=T^{1/2}$ so that

$$\frac{S}{\log T} \ll T^c \text{ in either case.}$$

This proves the Lemma and Theorem 2 is now completely proved.

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ADDED IN THE END. Since the submission of this paper the following paper has appeared :- Tom Meurman, A generalisation of Atkinson's formula to L-functions, Acta Arith., XLVII (1986), p.351-370.