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HYBRID MEAN SQUARE OF L-FUNCTIONS

By M. J. Narlikar

§1. Introduction. Let  $k \geq 1$  be an integer and  $\chi$  run through the set of characters mod  $k$ . Let  $T \geq 0$  and  $T_0 = T+2$ . Then R. Balasubramanian and K. Ramachandra proved the following theorem. For the history of the theorem see [1].

Theorem 1. We have

$$\frac{1}{2\pi\phi(k)} \sum_{\chi \pmod k} \int_0^T |L(\frac{1}{2}+it, \chi)|^2 dt = \frac{\phi(k)}{k^2} A(k, T) + O\left(\frac{\phi(k)}{k} T^{1/2} \log T_0\right)$$

where  $A(k, T) = \frac{kT}{2\pi} \log \frac{kT}{2\pi} + \left(2\gamma-1 + \sum_{p|k} \frac{\log p}{p-1}\right) \frac{kT}{2\pi}$ ,  $\gamma$  being the

Euler's constant.

In this paper I prove the following

Theorem 2. For  $T \geq 2$  and  $k \geq 3$ ,

$$\begin{aligned} \frac{1}{2\pi\phi(k)} \sum_{\chi \pmod k} \int_0^T |L(\frac{1}{2}+it, \chi)|^2 dt &= \frac{\phi(k)}{k^2} A(k, T) + \\ &+ B(k)T^{1/2} + O\left(\frac{\phi(k)}{k} T^{5/12} (\log T)^2\right) \end{aligned}$$

where  $B(k)$  is a negative constant and depends only on  $k$ . Moreover

$$|B(k)| \gg \frac{\phi(k)}{k} \quad \text{and} \quad |B(k)| \ll \frac{\phi(k)}{k}.$$

§2. Proof of the Theorem 2.

Lemma 1. We have

$$\frac{1}{\phi(k)} \sum_{\chi \bmod k} |L(\frac{1}{2}+it, \chi)|^2 = \frac{1}{k} \sum_{\substack{(a,k)=1 \\ 1 \leq a \leq k}} |\zeta(\frac{1}{2}+it, \frac{a}{k})|^2$$

where  $\zeta(s, \alpha)$  for  $0 < \alpha < 1$  is the Hurwitz Zeta-function defined by

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n+\alpha)^{-s} \quad \text{for} \quad \text{Re } s > 1.$$

Proof. Follows from the fact that if  $F(a)$  is a complex number for every residue class  $a \bmod k$  with  $(a, k) = 1$ , then

$$\frac{1}{\phi(k)} \sum_{\chi \bmod k} \left| \sum_{\substack{(a,k)=1 \\ 1 \leq a \leq k}} \chi(a)F(a) \right|^2 = \sum_{\substack{(a,k)=1 \\ 1 \leq a \leq k}} |F(a)|^2.$$

Lemma 2. Theorem 1 gives Theorem 2 for  $1 \leq T \leq 2$  and hence it is sufficient to prove that for  $T \geq 2$

$$\frac{1}{2\pi\phi(k)} \sum_{\chi \bmod k} \int_1^T |L(\frac{1}{2}+it, \chi)|^2 dt = \frac{\phi(k)}{k^2} \{A(k, T) - A(k, 1)\} + B(k)T^{1/2} + O\left(\frac{\phi(k)}{k} T^{5/12} (1 + O(T^{-2}))\right).$$

where  $B(k)$  is as in Theorem 2.

Lemma 3. For  $T \geq 2$ ,  $0 < \alpha < 1$ ,

$$\frac{1}{2\pi} \int_1^T |\zeta(\frac{1}{2}+it, \alpha)|^2 dt = \frac{T}{2\pi} \log \frac{T}{2\pi} + \frac{T}{2\pi} [C(\alpha) + \gamma - 1] - \frac{1}{2\pi\alpha} A_1(\alpha) T^{1/2} + O(T^{5/12} (\log T)^2) + O\left(\frac{\log T}{\sqrt{2\alpha}}\right).$$

where  $\|2\alpha\|$  denotes the distance of  $2\alpha$  from the nearest integer and

$$C(\alpha) = \lim_{N \rightarrow \infty} \left[ \sum_{n=1}^N \frac{1}{(n+\alpha)} - \log N \right] \text{ and}$$

$$A_1(\alpha) = \frac{1}{\sqrt{2\pi}} \left[ -2H(\alpha) + \int_{1-\alpha}^1 \frac{\cos\{\pi(u^2-u-1/4)\}}{\cos \pi u} + 2 \sum_{r=1}^{\infty} \int_r^{\infty} \cos\{\pi(u^2-1/4)\} du \right],$$

$$H(\alpha) = \int_{\frac{1-\alpha}{2}}^{\frac{1+\alpha}{2}} G^2(u) du + \int_{\alpha/2}^{1-\alpha/2} G^2(u) du$$

where  $G(u) = \frac{\cos\{2\pi(u^2-u-1/16)\}}{\cos(2\pi u)}$ .

Finally,  $A_1(\alpha) < -\frac{1}{\sqrt{2\pi}} H(\alpha)$

Proof. See [2].

Lemma 4. We have

$$B(k) = \frac{1}{\sqrt{2\pi} k} \sum_{\substack{(a, k)=1 \\ 1 \leq a \leq k}} \left[ -2H\left(\frac{a}{k}\right) + \int_{1-\frac{a}{k}}^1 \frac{\cos\{\pi(u^2-u-1/4)\}}{\cos \pi u} du \right. \\ \left. + 2 \sum_{r=1}^{\infty} \int_r^{\infty} \cos\{\pi(u^2-1/4)\} du \right]$$

and  $|B(k)| \ll \frac{\phi(k)}{k}$ .

Proof. Follows from the expression for  $A_1(\alpha)$  and the last part of

Lemma 3.

Lemma 5. In Lemma 3, the error term  $O\left(\frac{\log T}{\sqrt{|2\alpha|}}\right)$  can be replaced by

$$O\left[\min\left\{\frac{\log T}{\sqrt{|2\alpha|}}, T^{1/2} \log T\right\}\right].$$

Proof. The sums in Lemmas 11 to 14 of [2] which contribute to this O-term add up to

$$[\sqrt{T/2\pi}] \cdot 2 \operatorname{Re} \sum_{m=1}^{\lfloor \sqrt{T/2\pi} \rfloor} \sum_{0 \leq r \leq m} \frac{1}{4} \operatorname{Exp} \left[ 2\pi i (2m\alpha + 2r\alpha + \alpha^2) \right] \int_{r+\alpha}^{\infty} \operatorname{Exp} \left[ i\pi \left( e^{2-\frac{1}{4}} \right) \right] d\theta.$$

Now it is easy to see that this sum is  $O(T^{1/2} \log T)$ . We only have to integrate by parts and obtain an estimate  $O\left(\frac{1}{r+\alpha}\right)$  for  $r \geq 1$  and  $O(1)$  for  $r=0$ , for the expression inside the summation sign.

Lemma 6. We have

$$\frac{1}{k} \sum_{\substack{(a,k)=1 \\ 1 \leq a < k}} \min \left[ \frac{\log T}{\sqrt{|2a/k|}}, T^{1/2} \log T \right] \ll_c \frac{\phi(k)}{k} T^c.$$

Proof. Put  $S = \frac{1}{\phi(k)} \sum_{\substack{(a,k)=1 \\ 1 \leq a < k}} \min \left[ \frac{\log T}{\sqrt{|2a/k|}}, T^{1/2} \log T \right]$

and  $S_U = \frac{1}{\phi(k)} \sum_{\substack{1/2U \leq \sqrt{|2a/k|} < 1/U \\ (a,k)=1, 1 \leq a < k}} \min \left[ \frac{\log T}{\sqrt{|2a/k|}}, T^{1/2} \log T \right]$

where  $U$  runs through powers  $2^r$  of 2,  $2^r < k$ . We divide the totality of  $S_U$  into 3 parts according as the integer nearest to  $\frac{2a}{k}$  is 0, 1 or 2. In each case the number of integers  $a$  such that  $\frac{1}{2U} \leq \left| \frac{2a}{k} \right| \leq \frac{1}{U}$  is  $\ll \frac{\phi(k)}{U} + 2^{\omega(k)}$ , where  $\omega(k)$  is the number of different prime factors of  $k$ . Hence, if  $U \leq X$ , taking the first term in the

minimum,

$$S_U \ll \left[ 1 + U \frac{2^{\varphi(k)}}{\varphi(k)} \right] \log T.$$

For  $U > X$ , we take the second term in the minimum, so that

$$S_U \ll T^{1/2} \log T \left[ \frac{1 + 2^{\omega(k)}}{U + \varphi(k)} \right].$$

Now  $2^{\omega(k)} \ll k^c$ , hence

$$\frac{S}{\log T} \ll \log X + \frac{X}{X^{1-c}} + \frac{T^{1/2}}{k^{1-c}}.$$

If  $k < T^{10}$  we take  $X=k$  and we do not get the last two terms as the case  $U > X$  does not arise. If  $k \geq T^{10}$ , take  $X=T^{1/2}$  so that

$$\frac{S}{\log T} \ll T^c \text{ in either case.}$$

This proves the Lemma and Theorem 2 is now completely proved.

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ADDED IN THE END. Since the submission of this paper the following paper has appeared :- Tom Meurman, A generalisation of Atkinson's formula to L-functions, Acta Arith., XLVII (1986), p.351-370.