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ON THE FREQUENCY OF TITCHMARSH'S PHENOMENON FOR  $\zeta(s)$ -IV

By R. BALASUBRAMANIAN

§ 1. INTRODUCTION. The object of this paper is to prove the following

Theorem 1 : The following lower bound holds :

$$\max_{T \leq t \leq T+H} \left| \zeta \left( \frac{1}{2} + it \right) \right| \geq \text{Exp} \left[ \frac{3}{4} \left( \frac{\log H}{\log \log H} \right)^{1/2} \right]$$

where  $100 \log \log T \leq H \leq T$ , and  $T \geq T_0$ .

Remark 1. In [BR1] we proved a weaker version of this Theorem where  $\frac{3}{4}$  was replaced by a small constant. The constant  $\frac{3}{4}$  is a substantial improvement. It can be improved slightly and this seems to be the limit of our method (see Theorem 4 below).

Remark 2. The result  $\max_{0 < t < T} \left| \zeta \left( \frac{1}{2} + it \right) \right| > \text{Exp} \left[ \frac{1}{20} \left( \frac{\log T}{\log \log T} \right)^{1/2} \right]$  was obtained by Montgomery [M] on the assumption of Riemann Hypothesis. Our result is independent of any hypothesis.

Remark 3. A similar result can be proved for zeta function of number fields etc. one can refer, for example to Remark 3 of [BR1].

§ 2. TITCHMARSH SERIES. In this section, we state some results on the mean square of Titchmarsh series and these results are essentially due to Ramachandra.

Let  $A(>1)$  be a constant. Let  $\{a_n\}$  be a sequence of complex numbers, possibly depending on a parameter  $H(>10)$  such that  $a_1=1$ ; and  $|a_n| < (nH)^A$ . Let  $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$  be analytically continuable in the rectangle  $R(T,H)=\{\sigma \geq 0; T \leq t \leq T+H\}$  and maximum of  $|F(s)|$  within the rectangle be bounded by  $\exp \exp \left[ \frac{H}{100} \right]$ . Also assume that  $T \geq H \geq H_0$ , a large constant. Then we have

Theorem 2. There exists a constant  $c=c(A)>0$  such that

$$\frac{1}{H} \int_T^{T+H} |F(it)|^2 dt \geq c_A \sum_{n < \frac{H}{100}} |a_n|^2 \left[ 1 - \frac{\log n}{\log H} + \frac{1}{\log \log H} \right].$$

§ 3. MAIN THEOREM. In this section, we state the main theorem:

For  $l>0$ , let  $I(l) = \int_{2l}^{\infty} \frac{e^{-x}}{x} dx$

$$D(l) = \frac{1}{2}(I(l) + e^{-2l})\sqrt{2l} e^l \text{ and let}$$

$B = \max_{l>1} D(l)$ , the maximum being over all real  $l>0$ . Then we have,

Theorem 3.  $\max_{k>1} \frac{1}{2k} \left[ \frac{\log X}{\log \log X} \right]^{-1/2} \log \left[ \sum_{n < X} \frac{d_k^2(n)}{n} \right]$  tends to  $B$  as  $X \rightarrow \infty$

where the maximum is taken over all natural numbers  $k$ .

Having defined  $B$ , we can slightly strengthen Theorem 1 as

Theorem 4.  $\max_{T < t < T+H} \left| \zeta \left( \frac{1}{2} + it \right) \right| > \text{Exp} \left\{ B_1 \left[ \frac{\log H}{\log \log H} \right]^{1/2} \right\}$  , where

$100 \log \log T < H < T$  provided  $B_1 < B$  and  $T$  is sufficiently large.

§ 4. PROOFS OF THEOREMS 1, 2 AND 4.

The proof of Theorem 2, either in slightly weaker form (but sufficient for our purpose) or slightly stronger form can be found in [B] (Theorem 4), [BR1] [BR2] [R1] and [R2]. The proof of Theorem 3 will be given in the next few sections. To deduce

Theorem 4 from Theorems 2 and 3, we choose  $F(s) = \left[ \zeta \left( \frac{1+s}{2} \right) \right]^k$ . Then by Theorems 2 and 3, we have, for a suitable choice of  $k$ , and for any  $\epsilon > 0$ ,

$$\begin{aligned} \max_{T < t < T+H} \left| \zeta \left( \frac{1+it}{2} \right) \right| &\geq \left( \frac{1}{H} \int_T^{T+H} \left| \zeta \left( \frac{1+it}{2} \right) \right|^{2k} dt \right)^{1/2k} \\ &> \left[ c_A \sum_{n < H/100} \frac{d_k^2(n)}{n} \left( 1 - \frac{\log n}{\log H} + \frac{1}{\log \log H} \right) \right]^{1/2k} \\ &> \text{Exp} \left\{ (B-\epsilon) \left( \frac{\log H}{\log \log H} \right)^{1/2} \right\} \text{ and hence Theorem 4.} \end{aligned}$$

Even though the expression for  $B$  is unwieldy, it is possible to get an approximate value numerically and this gives that  $B > \frac{3}{4}$  and this proves Theorem 1. (In fact  $B = 0.75\dots$ )

§ 5. SOME PRELIMINARY LEMMAS. In this section, we prove some preliminary lemmas.

$$\text{Let } \epsilon = \epsilon(a) = \frac{1}{2} + \frac{5a}{\log k}$$

$$A_p = A_p(a) = \sum_{r=0}^{\infty} \frac{d_k^2(p^r)}{2r^\epsilon}$$

$$G(a) = \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\epsilon}}$$

$$M = 10k^2 e^{-10a};$$

We assume that  $a$  is less than  $\frac{\log k}{100}$  so that

$$\frac{1}{2} \leq \epsilon \leq \frac{3}{4} \text{ and } M \geq k^{1.9}.$$

Lemma 1. We have

$$(a) \quad \sum_{p < M} \frac{k}{p^\epsilon} = O \left( \frac{k^2}{\log k} e^{-15a} \right)$$

$$(b) \sum_{p \geq M} \frac{k^4}{p^{4\theta}} = O\left(\frac{k^2}{\log k} e^{-15a}\right)$$

$$(c) \sum_{p \geq M} \frac{k^2}{p^{2\theta}} = I(10a)k^2 + O\left(\frac{k^2}{\log k} e^{-15a}\right)$$

Proof. Since  $\sum_{p < M} \frac{k}{p^\theta} = O\left(\frac{kM^{1-\theta}}{\log M}\right)$ , (a) follows.

$$\sum_{p \geq M} \frac{k^4}{p^{4\theta}} = O\left(\frac{k^4 M^{1-4\theta}}{\log M}\right) \text{ and hence (b) follows.}$$

To prove (c), we use prime number theorem in the form

$$\vartheta(u) = \sum_{p \leq u} \log p = u + O\left(\frac{u}{(\log u)^{10}}\right).$$

Hence

$$\begin{aligned} \sum_{p \geq M} \frac{k^2}{p^{2\theta}} &= k^2 \int_M^\infty \frac{1}{u^{2\theta} \log u} d(\vartheta(u)) \\ &= k^2 \int_M^\infty \frac{1}{u^{2\theta} \log u} du + k^2 \int_M^\infty \frac{d(\vartheta(u)-u)}{u^{2\theta} \log u} \\ &= S_1 + S_2, \text{ say.} \end{aligned}$$

In  $S_1$ , we make a change of variable

$$v = \frac{10a}{\log k} \log u \text{ to get}$$

$$S_1 = k^2 \int_A^\infty \frac{e^{-v} dv}{v} \text{ where } A = 20a - \frac{100a^2}{\log k} + \frac{10a \log 10}{\log k}.$$

$$\text{Thus } S_1 = k^2 \int_{20a}^\infty \frac{e^{-v} dv}{v} + O\left(k^2 \int_A^{20a} \frac{e^{-v} dv}{v}\right).$$

Since  $A \geq 19a$ , the 0-term is  $O\left(k^2 \int_A^{20a} e^{-19a} dv\right) = O\left(\frac{a^2 k^2}{\log k} e^{-19a}\right)$ .

Thus  $S_1 = I(10a)k^2 + O\left(e^{-15a} \frac{k^2}{\log k}\right)$ . By integrating by parts, we check that  $S_2 = O\left(\frac{e^{-15a} k^2}{\log k}\right)$  and this completes the proof.

Lemma 2. We have  $\log G(a) = I(10a)k^2 + O\left(\frac{k^2}{\log k} e^{-15a}\right)$ .

Proof. Since  $G(a) = \prod_p A_p(a)$ , it follows that

$$\log G(a) = \sum_p \log A_p(a).$$

Since  $A_p \leq \left(\sum_{r=0}^{\infty} \frac{d_k(p^r)}{p^{r\theta}}\right)^2 = (1-p^{-\theta})^{-2k}$ ,  $\log A_p = O\left(\frac{k}{p^\epsilon}\right)$  and by

Lemma 1(a),  $\sum_{p \leq M} \log A_p = O\left(\frac{k^2}{\log k} e^{-15a}\right)$ . Since  $\frac{k}{2\theta} \leq \frac{1}{10}$  for  $p \geq M$ ,

$$A_p = 1 + \frac{k^2}{2\theta} + O\left(\frac{k^4}{p^{4\theta}}\right) \quad \text{for } p \geq M.$$

$$\text{Hence } \log A_p = \frac{k^2}{2\theta} + O\left(\frac{k^4}{p^{4\theta}}\right)$$

$$\sum_{p \geq M} \log A_p = \sum_{p \geq M} \frac{k^2}{2\theta} + O\left(\sum_{p \geq M} \frac{k^4}{p^{4\theta}}\right) \quad \text{and the result follows}$$

from Lemmas 1(b) and 1(c).

Lemma 3. If  $a$  is bounded above and below and

$$\frac{\log y}{k^2 \log k} \geq \frac{e^{-20a}}{10a} + (\log k)^{-1/\theta}, \quad \text{then}$$

$$\log \left[ \sum_{n \geq Y} \frac{d_k^2(n)}{n^{2\theta(a)}} \right] \leq I(10a)k^2 - \frac{k^2}{\sqrt{\log k}}$$

Proof. Let  $b = 2(\log k)^{-1/8}$ . Then

$$\begin{aligned} \sum_{n \geq Y} \frac{d_k^2(n)}{n^{2\theta(a)}} &\leq Y^{-\frac{10b}{\log k}} \sum_{n \geq Y} \frac{d_k^2(n)}{n^{2\theta(a-b)}} \\ &\leq Y^{-\frac{10b}{\log k}} G(a-b) \end{aligned}$$

$$\begin{aligned} \log \left[ \sum_{n \geq Y} \frac{d_k^2(n)}{n^{2\theta(a)}} \right] &\leq \log G(a-b) - \frac{10b}{\log k} \log Y \\ &= I(10a-10b) k^{2+0} \left[ \frac{k^2}{\log k} \right] - \frac{10b}{\log k} \log Y \\ &= (I(10a)-10bI'(10a)+0(b^2))k^2 - \frac{10b}{\log k} \log Y \\ &= \left[ I(10a)+(10b) \frac{e^{-20a}}{10a} + 0(b^2) - \frac{10b}{k^2 \log k} \log Y \right] k^2 \end{aligned}$$

and hence the result.

§6. The upper bound in Theorem 3.

We now prove that  $\frac{1}{2k} \log \left[ \sum_{n \leq X} \frac{d_k^2(n)}{n} \right] \left( \frac{\log \log X}{\log X} \right)^{1/2} \leq B + c$  for

any  $k; X$  sufficiently large. Suppose  $k \geq \sqrt{\log X} (\log \log X)^2$ . Then

$$d_k(n) \leq k^{\Omega(n)} \leq k^{2 \log n} \text{ so that}$$

$$\sum_{n \leq X} \frac{d_k^2(n)}{n} \leq \sum_{n \leq X} \frac{k^{4 \log n}}{n} \leq k^{4 \log X} (\log X + O(1)). \text{ Hence}$$

$$\frac{1}{2k} \log \left[ \sum_{n \leq X} \frac{d_k^2(n)}{n} \right] \leq \frac{5(\log X)(\log k) + \log \log X}{2k}$$

which is  $O\left(\frac{\sqrt{\log X}}{\log \log X}\right)$ .

Hence the result is true in this case. If  $k \leq \sqrt{\log X} (\log \log X)^2$ ,

define  $a$  by

$$\frac{\log X}{k^2 \log k} = \frac{e^{-20a}}{10a}. \text{ Then } a \leq \frac{\log k}{100}.$$

Further  $k \leq \sqrt{20a} e^{10a} \sqrt{\frac{\log X}{\log(10a \log X)}}$ .

$$\sum_{n \leq X} \frac{d_k^2(n)}{n} \leq X^{\frac{10a}{\log k}} \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\theta(a)}} = X^{\frac{10a}{\log k}} G(a),$$

$$\begin{aligned} \frac{1}{2k} \log \left[ \sum_{n \leq X} \frac{d_k^2(n)}{n} \right] &\leq \frac{10a}{2k \log k} \log X + \frac{\log G(a)}{2k} \\ &= \frac{e^{-20a}}{2} k + \frac{I(10a)k^2}{2k} + O\left(\frac{k e^{-15a}}{\log k}\right) \\ &\leq \frac{1}{2} k (e^{-20a} + I(10a)) + O\left(\frac{k e^{-15a}}{\log k}\right). \end{aligned}$$

Substituting the values of  $k$ , the  $O$ -term is easily seen to be

$$O\left(\sqrt{\frac{\log X}{\log \log X}}\right) \text{ and the main term is at most } (B+c) \sqrt{\frac{\log X}{\log \log X}}.$$

### §7. The lower bound in Theorem 3.

In this section, we prove that for a suitable choice of  $k$ ,



$$\frac{1}{2k} \log \left[ \sum_{n \leq X} \frac{d_k^2(n)}{n} \right] \geq (B-c) \sqrt{\frac{\log X}{\log \log X}}.$$

Choose  $a$  to be a real number such that  $D(l)$  takes its maximum at  $l = 10a$ ; then  $a$  is bounded above and below. Choose the largest integer  $k$  such that

$$\frac{\log X}{k^2 \log k} \geq \frac{e^{-20a}}{10a} + (\log k)^{-1/8}$$

$$\text{Now } G(a) = \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\theta(a)}} = \text{Exp} \left[ I(10a)k^2 + O\left(\frac{k^2}{\log k}\right) \right]. \text{ By Lemma 3,}$$

$$\sum_{n \leq X} \frac{d_k^2(n)}{n^{2\theta(a)}} \text{ is small. Hence}$$

$$\sum_{n \leq X} \frac{d_k^2(n)}{n^{2\theta(a)}} = \text{Exp} \left[ I(10a)k^2 + O\left(\frac{k^2}{\log k}\right) \right].$$

$$\begin{aligned} \text{Hence } \sum_{n \leq X} \frac{d_k^2(n)}{n} &\geq X^{(2\theta-1)} \text{Exp} \left[ I(10a)k^2 + O\left(\frac{k^2}{\log k}\right) \right]. \\ &= \text{Exp} \left[ \frac{10a}{\log k} \log X + I(10a)k^2 + O\left(\frac{k^2}{\log k}\right) \right]. \end{aligned}$$

Substituting the value of  $k$ , the result follows.

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expression for B and getting the lower bound  $\frac{3}{4}$  for B.

A REMARK. The author had proved this result immediately after the results of [BR1] were discovered. But due to various reasons the result of the present paper could not be published earlier.

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