



HAL
open science

On Waring's Problem: $g(4) \leq 20$.

R Balasubramanian

► **To cite this version:**

R Balasubramanian. On Waring's Problem: $g(4) \leq 20$. Hardy-Ramanujan Journal, 1985, Volume 8 - 1985, pp.1-40. 10.46298/hrj.1985.114 . hal-01104695

HAL Id: hal-01104695

<https://hal.science/hal-01104695>

Submitted on 19 Jan 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On Waring's Problem :

$$g(4) < 20$$

R. BALASUBRAMANIAN

§ 1. In [1], I proved that every integer is a sum of at most 21 biquadrates. The object of this paper is to prove a refinement namely

Theorem : Every positive integer is a sum of not more than twenty biquadrates.

Remarks :

(1) Since 79 is not expressible as a sum of eighteen biquadrates, nineteen, if true, is best possible ; but we are unable to improve the theorem further at present.

(2) All numbers upto 10^{310} are sums of 19 biquadrates. This is proved by Henry Thomas Jr. using extensive numerical calculation ([7], Th. 3.3). Our method yields that all numbers bigger than 10^{700} are sums of 19 biquadrates.

(3) For the history of the problem, we refer the reader to [1]

Our work is based upon the papers of Chen Jing run [2] and Davenport [4]. The extensive computer work necessary was done by Thomas ([6] and [7]) and we have freely borrowed these results.

Acknowledgement :

The author wishes to thank Professor K. Ramachandra for his encouragement and the interest he evinced at every stage of the work. The author also wishes to thank Professor H. Halberstam for his encouragement and for the interest he showed in the work. Part of the work was done when the author was visiting the Institute For Advanced Study, Princeton, (supported in part by NSF grant MCS 77-18723 AO4) and the University of Illinois, Urbana. The author is thankful to these institutions for their hospitality.

§ 2. Notation :

The following notation will be used throughout this paper.

$$e(x) = e^{2\pi ix}$$

$$S_{a,q} = \sum_{x=1}^q e\left(\frac{ax^4}{q}\right).$$

Let $N (> 10^{400})$ be a given integer to be represented as a sum of twenty biquadrates.

$$P = [N^{1/4}];$$

$$T(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^4)$$

Let m be an integer. (In § 9, we shall choose $m = 10$; in other sections, m can either be 9 or 10).

The singular series $S(n)$

$$= \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left(\frac{S_{a,q}}{q} \right)^m e\left(-\frac{an}{q}\right)$$

The truncated singular series

$$S_1(n) = \sum_{q < P^{1/2}} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left(\frac{S_{a,q}}{q} \right)^m e\left(-\frac{an}{q}\right)$$

$$\Psi(d) = \int_0^P e(dx^4) dx$$

The major arc m is defined by

$$\left\{ d : \left| d - \frac{h}{q} \right| < \frac{1}{8qP^3} \text{ for some } q < P^{1/2} \text{ and } (h,q) = 1 \right\}$$

The minor arc m is defined by

$$\left\{ d : \left| d - \frac{h}{q} \right| < \frac{1}{8qP^3} \text{ for some } q, P^{1/2} < q < 8P^3 \right. \\ \left. \text{and } (h,q) = 1 \right\}$$

$$W_0(N_0) = \int_m (T(d))^m e(-dN_0) dd$$

$$W(N_0) = \int_0^1 (T(d))^m e(-dN_0) dd$$

$$R(N_0) = \int_{-\infty}^{\infty} (\Psi(x))^m e(-xN_0) dx$$

Each section contains a proposition, which is the main result of the section. Lemmas are subsidiary results needed to prove the proposition.

§ 3. An upper bound for $S_{a, q}$

We recall that $S_{a, q} = \sum_{x=1}^q e\left(\frac{ax^4}{q}\right)$. We define

$$S(q) = \max_a (S_{a, q}).$$

$$(a, q) = 1$$

In this section, we prove

Prop. 1: *There holds $|S(q)| < (4.3)q^{3/4}$*

In order to prove the proposition, we need

Lemma 1 (a) $S(q)$ is a multiplicative function of q

(b) If $p \neq 2$, and $a \not\equiv 0 \pmod{p}$,

$$|S_{a, p}| < (\delta - 1)p^{1/2}$$

where $\delta = (4, p-1)$

(c) If $p \neq 2$, $|S_{a, p}^{\nu}| = p^{\nu-1}$
if $2 < \nu < 4$

(d) If $p > 2$, $|S_{a, p}^{\nu}| = p^3 |S_{a, p}^{\nu-4}|$
if $\nu > 4$

Proof :

A proof can be found in Davenport [3] (Lemma 6 (page 31) Lemma 12 (page 42), Lemma 13 (page 43) and Lemma 14 (page 44))

From Lemma 1, it follows that

$$\begin{aligned} \frac{S(q)}{q^{3/4}} &= \prod_{p \parallel q} \pi \frac{S(p^{\alpha})}{p^{3/4 \alpha}} \\ &< \pi \max_{p} (1, \max_{\alpha > 0} \frac{S(p^{\alpha})}{p^{3/4 \alpha}}) \end{aligned}$$

which being a finite product can be evaluated and this gives the proposition. For further details, we refer the reader to Theorem 2.1 in page 38 of Thomas [6].

§ 4. A lower bound for $R(N_0)$

Let us recall that

$$\Psi(\alpha) = \int_0^P e(\alpha x^4) dx$$

$$R(N_0) = \int_{-\infty}^{\infty} (\Psi(\alpha))^m e(-\alpha N_0) d\alpha$$

$$W(N_0) = \int_0^1 (T(\alpha))^m e(-\alpha N_0) d\alpha$$

$$\text{Define } B = B(\alpha) = \begin{cases} P & \text{if } |\alpha| < P^{-4} \\ 2|\alpha|^{-1/4} & \text{if } |\alpha| > P^{-4} \end{cases}$$

Lemma 2 :

Let $f(x)$ be a real function which is twice differentiable in $A < x < B$; suppose that, in the interval $A < x < B$, we have $0 < f'(x) < \frac{1}{2}$ and $f''(x) > 0$. Then

$$\sum_{A < n \leq B} e(f(n)) = \int_A^B e(f(x)) dx + 4\theta$$

Proof :

This is Lemma 13 (page 34) in Vinogradov [9]

Lemma 3

We have $|\Psi(d)| < B(d)$

Proof :

Clearly $|\Psi(d)| < P$.

To prove that $|\Psi(d)| < 2|d|^{-1/4}$, it suffices to prove the result for $d > 0$. Now a change of variable $dx^4 = y$ transforms the integral to

$$\Psi(d) = \frac{1}{4d^{1/4}} \int_0^{dP^4} \frac{e(y) dy}{y^{3/4}}$$

and the result is immediate.

Lemma 4 :

If $N_0 - p^{3/4} < N_1 < N_0$, then

$$W(N_1) = R(N_0) + \int_{\frac{1}{8P^3}}^{1 - \frac{1}{8P^3}} (T(\alpha))^m e(-\alpha N_1) d\alpha \\ + \theta 10^6 p^{m-5+3/4}$$

Proof :

By Lemma 2, if $|\alpha| < \frac{1}{8P^3}$, we have

$$T(\alpha) = \sum_{1 < x < P} e(\alpha x^4) = \int_0^P e(\alpha x^4) dx + 4\theta \\ = \Psi(\alpha) + 4\theta$$

Hence by Lemma 3,

$$|(T(\alpha))^m - (\Psi(\alpha))^m| < 4m (\max(T(\alpha), \Psi(\alpha)))^{m-1} \\ < 4m (B + 4)^{m-1}$$

$$W(N_1) = \int_0^1 (T(\alpha))^m e(-\alpha N_1) d\alpha \\ = \int_{-\frac{1}{8P^3}}^{1 - \frac{1}{8P^3}} (T(\alpha))^m e(-\alpha N_1) d\alpha$$

$$= \int_{|\alpha| \leq \frac{1}{8p^3}} \dots + \int_{\frac{1}{8p^3} < |\alpha| \leq \frac{1}{4p^3}} \dots$$

In the first integral, we replace $(T(\alpha))^m$ by $(\Psi(\alpha))^m$ with an error E_1 . Then we replace $e(-\alpha N_1)$ by $e(-\alpha N_0)$ with an error E_2 . Now we extend the range of integration to $[-\infty, \infty]$ with an error E_3 . Hence

$$\begin{aligned} W(N_1) &= \int_{-\infty}^{\infty} (\Psi(\alpha))^m e(-\alpha N_0) d\alpha \\ &= \int_{-\frac{1}{8p^3}}^{\frac{1}{8p^3}} (T(\alpha))^m e(-\alpha N_1) d\alpha + E_1 + E_2 + E_3 \\ &= R(N_0) + \int_{-\frac{1}{8p^3}}^{\frac{1}{8p^3}} (T(\alpha))^m e(-\alpha N_1) d\alpha \\ &\quad + E_1 + E_2 + E_3. \end{aligned}$$

$$\begin{aligned} \text{Now } |E_1| &\leq \\ &\left| \int_{|\alpha| \leq \frac{1}{8p^3}} (T(\alpha))^m - (\Psi(\alpha))^m e(-\alpha N_1) d\alpha \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left| \int_{|\alpha| \leq \frac{1}{8P^3}} (4m) (B+4)^{m-1} d\alpha \right| \\
&\leq \left| \int_{|\alpha| \leq P^{-4}} (4m) (P+4)^{m-1} d\alpha \right| \\
&+ \left| \frac{1}{8P^3} \int_{|\alpha| \geq P^{-4}} (4m) (2|\alpha|^{-\frac{1}{2}} + 4)^{m-1} d\alpha \right| \\
&\leq 10^5 P^{m-5}, \text{ since } m \text{ is either } 9 \text{ or } 10
\end{aligned}$$

Similarly,

$$\begin{aligned}
|E_2| &\leq \left| \int_{|\alpha| < \frac{1}{8P^3}} (\Psi(\alpha))^m \left| e(-\alpha N_1) - e(-\alpha N_0) \right| d\alpha \right| \\
&\leq \int_{|\alpha| < \frac{1}{8P^3}} B^m (2\pi) |\alpha| |N_1 - N_0| d\alpha \\
&\leq P^{3\frac{3}{4}} (2\pi) \int_{|\alpha| < P^{-4}} P^m |\alpha| d\alpha +
\end{aligned}$$

$$+ P^{3\frac{3}{4}} (2\pi) \int_{|\alpha| > P^{-4}} (2|\alpha|^{-\frac{1}{4}})^m |\alpha| d\alpha$$

$$< 10^5 P^{m-5+\frac{3}{4}}$$

$$|E_3| < \int_{|\alpha| > \frac{1}{8P^3}} |\Psi(\alpha)|^m d\alpha$$

$$< \int_{|\alpha| > \frac{1}{8P^3}} B^m d\alpha$$

$$\leq 10^5 P^{\frac{3}{4}m-3}$$

This proves the result.

Lemma 5 :

$$\text{With } M = \left[\frac{P^{3\frac{3}{4}}}{2} \right]$$

$$\sum_{1 \leq A \leq M} \sum_{1 \leq B \leq M} W(N_0 - A - B) = M^2 R(N_0)$$

$$+ o(10^7 M^2 P^{m-5+\frac{3}{4}})$$

Proof :

In lemma 4, we take $N_1 = N_0 - A - B$ and sum over $1 \leq A \leq M$ and $1 \leq B \leq M$. This gives

$$\sum_A \sum_B W(N_0 - A - B) = M^2 R(N_0)$$

$$+ \int \frac{1 - \frac{1}{8P^3}}{\frac{1}{8P^3}} (T(\alpha))^m e(-\alpha N_0) \left(\sum_A e(\alpha A) \right)^2 d\alpha$$

$$+ \theta 10^6 M^2 P^{m-5+\frac{1}{2}}$$

Since $\left| \sum_A e(\alpha A) \right| \leq \frac{1}{\|\alpha\|}$, the integral appearing on the right, is bounded, in absolute value, by

$$\int \frac{1 - \frac{1}{8P^3}}{\frac{1}{8P^3}} P^m \left(\frac{1}{\|\alpha\|} \right)^2 d\alpha \leq 10^4 P^{m+3}$$

Lemma 6

We have

$$\sum_{1 \leq A \leq M} \sum_{1 \leq B \leq M} W(N_0 - A - B)$$

$$> T_m \left(\frac{m-1}{4} \right) M^2 N_0^{\frac{m}{4} - 1}$$

$$\text{where } T_m = \frac{(\Gamma(5/4))^m}{\Gamma(1 + \frac{m}{4})}$$

Proof :

By lemma 3 (page 22) of Vinogradov [9], the number of integer solutions $K_r(N)$ of $x_1^4 + x_2^4 + \dots + x_r^4 \leq N$ is given by

$$K_r(N) = T_r N^{\frac{r}{4}} - \theta_r N^{\frac{r-1}{4}} \quad \text{where}$$

$$T_r = \frac{(\Gamma(5/4))^r}{\Gamma(1 + \frac{r}{4})}$$

Now,

$$\begin{aligned} & \sum_{1 \leq B \leq M} W(N_0 - A - B) \\ &= \sum_B (K_m(N_0 - A - B) - K_m(N_0 - A - B - 1)) \\ &= K_m(N_0 - A - 1) - K_m(N_0 - A - M - 1) \\ &= T_m [(N_0 - A - 1)^{\frac{m}{4}} - (N_0 - A - M - 1)^{\frac{m}{4}}] + 2m\theta N_0^{\frac{m-1}{4}} \\ &> T_m \frac{m-1}{4} M N_0^{\frac{m}{4} - 1} \end{aligned}$$

and hence the lemma.

Prop. 2

We have, for $m = 9$ or 10 ,

$$R(N_0) \geq 0.25 N_0^{\frac{m}{4} - 1}$$

Proof :

From lemmas 5 and 6, it follows that

$$R(N_0) \geq \left(\frac{T_m(m-1)}{4} - 10^{-5} \right) N_0^{\frac{m}{4} - 1}$$

and hence the result.

§ 5 : A lower bound for $S_1(N_0)$:

Let us recall that $S(n) = S(n, m)$ is given by

$$\begin{aligned} S(n) &= \sum_{q=1}^{\infty} \sum_{a=1}^q \left(\frac{S_{a,q}}{q} \right)^m e\left(-\frac{an}{q}\right) \\ &= \sum_q A(n; q) \text{ say} \end{aligned}$$

$$\text{Write } X_p(n) = \chi_p(n; m) = \sum_{i=1}^{\infty} A(n; p^i).$$

Lemma 7 :

$A(n; q)$ is a multiplicative function of q and hence

$$S(n) = \prod_p (1 + \chi_p(n))$$

Proof :

We refer the reader to Lemma 2.11 (page 20) of Vaughan [8]

Lemma 8 :

(a) For any prime p , and natural number λ , let $N_m(p^\lambda, n)$ denote the number of solutions of

$$x_1^4 + x_2^4 + \dots + x_m^4 \equiv n \pmod{p^\lambda},$$

$$1 \leq x_i \leq p^\lambda \text{ and not all } x_i \equiv 0 \pmod{p}$$

Further set $\gamma = 1$ if p is odd and $\gamma = 4$ if $p = 2$. Let p^{4r+s} exactly divide n , $0 \leq s \leq 3$.

$$k_0 = \max(4r + s + 1, 4r + \gamma). \text{ Then}$$

$$A(n, p^\lambda) = 0 \text{ for } \lambda > k_0 \text{ and}$$

$$1 + \chi_p(n, m) = p^{-(m-1)\gamma} N_m(p^\gamma, 0) \left\{ \sum_{\tau=0}^{r-1} p^{\tau(4-m)} \right\} \\ + p^{(4-m)r - (m-1)\gamma} N_m(p^\gamma, np^{-4r})$$

where the empty sum is understood to be zero.

(b) Let $d = \text{g.c.d. of } (4, p-1)$. Then for $p \neq 2$,

$$\left| N_m(p, n) - p^{m-1} \right| \leq \left(1 - \frac{1}{p}\right) (d-1)^m p^{\frac{m}{2}}$$

Proof :

For the proof of Lemma 8 (a), we refer the reader to Hilfsatz 293 of LANDAU [5] or Prop. 4. 3 of Thomas [6]. For the proof of lemma 8 (b), we refer to lemma 41 (page 91) of Thomas [6].

From Lemma 8, we have

Lemma 9 :

$$\text{Let } K > 100; \quad S_2 = \{ p > K; \quad p \equiv 3 \pmod{4} \}$$

$$S_4 = \{ p > K; \quad p \equiv 1 \pmod{4} \}. \quad \text{Then for } d = 2 \text{ or } 4$$

$$\begin{aligned} & \prod_{p \in S_d} (1 + \chi_p(n)) \\ & > \exp \left(-2 (d-1)^m K^{1 - \frac{m}{2}} \left(1 + \frac{K}{2m-8} \right) \right) \end{aligned}$$

Lemma 10 :

$$\text{We have } \prod_{p > 114} (1 + \chi_p(n)) > 0.97$$

$$\text{Since } \prod_{p > 114} (1 + \chi_p(n))$$

$$\begin{aligned} &= \prod_{\substack{p > 127 \\ p \equiv 3 \pmod{4}}} (1 + \chi_p(n)) \prod_{\substack{p > 137 \\ p \equiv 1 \pmod{4}}} (1 + \chi_p(n)) \end{aligned}$$

the result follows from Lemma 9.

Lemma 11 :

$$\text{We have } \prod_{2 < p < 113} (1 + \chi_p(n)) > 0.297.$$

Proof :

This is easily verified using Lemma 8 (a). For details, we refer the reader to Thomas [6]

Lemma 12 :

If $n \equiv \tilde{n} \pmod{16}$, $2 < \tilde{n} < m-2$, then

$$1 + X_2(n) = \binom{m}{\tilde{n}} 2^{-m+4} \text{ and hence}$$

$$1 + X_2(n) > \frac{45}{64}$$

Here $\binom{m}{\tilde{n}}$ is the binomial coefficient

Proof :

This is straight forward. For details, we refer the reader to Theorem 4.2 of Thomas [6].

Lemma 13 :

We have $S(n) > 0.2025$

provided $n \equiv 2, 3, 4, 5, 6$ or $7 \pmod{16}$

Proof :

This follows from lemmas 7, 10, 11 and 12.

Lemma 14 :

We have $S(n) - S_1(n) \leq 0.002$

Proof :

Since $|S(n) - S_1(n)| =$

$$\left| \sum_{q > p^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a, q) = 1}}^q \left(\frac{S_{a, q}}{q} \right)^m e\left(-\frac{an}{q}\right) \right|$$

$$< \sum_{q > p^{\frac{1}{2}}} \sum_{a=1}^q \left(4.3 q^{-\frac{1}{4}} \right)^m,$$

we are through.

Prop 3 :

There holds the inequality

$$S_1(n) > 0.2$$

if $n \equiv l \pmod{16}$, and $l \in \{2, 3, 4, 5, 6, 7\}$.

§ 6 : A lower bound for $W_0(N_0)$:

Let us recall that $W_0(N_0) =$

$$\int_m (T(\alpha))^m e(-\alpha N_0) d\alpha.$$

Lemma 15 :

If $|\beta| < \frac{1}{8qP^3}$, then

$$\begin{aligned} \left(T\left(\frac{a}{q} + \beta\right)\right)^m &= (\Psi(\beta) \frac{S_{a,q}}{q})^m \\ &+ \theta \cdot 4qm (4.3 B(\beta)q^{-\frac{1}{2}} + 4q)^{m-1} \end{aligned}$$

Proof :

We have,

$$\begin{aligned} T\left(\frac{a}{q} + \beta\right) &= \sum_{x=1}^P e\left(\left(\frac{a}{q} + \beta\right)x^4\right) \\ &= \sum_{y=0}^{q-1} \sum_{-\frac{y}{q} < t < \frac{P-y}{q}} e\left(\left(\frac{a}{q} + \beta\right)(qt + y)^4\right) \\ &= \sum_y \sum_t e\left(\frac{ay^4}{q} + \beta (qt + y)^4\right) \\ &= \sum_y e\left(\frac{ay^4}{q}\right) D_y(z), \text{ say} \end{aligned}$$

Now, using Lemma 2,

$$\begin{aligned}
 D_y(z) &= \int_{-\frac{y}{q}}^{\frac{P-y}{q}} e(\beta(qt+y)^4) dt + 4\theta \\
 &= \frac{1}{q} \int_0^P e(\beta x^4) dx + 4\theta \\
 &= \frac{1}{q} \Psi(\beta) + 4\theta.
 \end{aligned}$$

$$\text{Hence } T\left(\frac{a}{q} + \beta\right) = \frac{S_{a,q}}{q} \Psi(\beta) + 4\theta q$$

The result follows from the following inequalities.

$$\left| \frac{S_{a,q}}{q} \right| < (4.3) q^{-\frac{1}{4}}$$

$$|\Psi(\beta)| < B(\beta) \text{ and}$$

$$a^m - b^m < m(a-b) \max(|a|^{m-1}, |b|^{m-1})$$

Lemma 16 :

We have

$$W_0(N_0) = S_1(N_0) R(N_0) + \theta 10^{13} P^{m-\frac{9}{2}}$$

Proof :

Since the proof is similar to that of Lemma 4, we give only the sketch of the proof.

$$\begin{aligned}
 W_0(N_0) &= \int_m (T(\alpha))^m e(-\alpha N_0) d\alpha \\
 &= \sum_{q < P^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a, q)=1}}^q \int \frac{\frac{a}{q} + \frac{1}{8qP^3}}{\frac{a}{q} - \frac{1}{8qP^3}} (T(\alpha))^m e(-N_0 \alpha) d\alpha \\
 &= \sum_{q < P^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a, q)=1}}^q \int \frac{1}{8qP^3} \left(T\left(\frac{a}{q} + \beta\right) \right)^m e\left(-N_0\left(\frac{a}{q} + \beta\right)\right) d\beta - \frac{1}{8qP^3}
 \end{aligned}$$

First we replace $\left(T\left(\frac{a}{q} + \beta\right)\right)^m$ by $\left(\psi(\beta) \frac{S_{a, q}}{q}\right)^m$

and the error is, by Lemma 15, at most $10^{12} P^{m-9/2}$. Now we extend the range of integration of the integral to $[-\infty, \infty]$ which gives an error at most $10^{12} P^{m-9/2}$. Hence

$$\begin{aligned}
 W_0(N_0) &= \sum_{q < P^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a, q)=1}}^q \int_{-\infty}^{\infty} (\psi(\beta))^m \left(\frac{S_{a, q}}{q}\right)^m \\
 &\quad e\left(-N_0\left(\frac{a}{q} + \beta\right)\right) d\beta + \theta 10^{13} P^{m-9/2} \\
 &= S_1(N_0) R(N_0) + \theta 10^{13} P^{m-9/2}
 \end{aligned}$$

Prop 4 : *We have*

$$W_0(N_0) \geq 0.05 N_0^{\frac{m}{4} - 1}$$

provided $N_0 \equiv 2, 3, 4, 5, 6$ or $7 \pmod{16}$

Proof :

This follows from Prop. 2, Prop. 3 and Lemma 16.

§ 7 : *The estimate on the minor arc :*

We define $f(n) = n^{-a} \prod_{p|n} (1 - p^{-a})^{-1}$ where $a (> 0)$

is a constant. Ultimately we shall choose $a = 0.1$

$$g(n) = \sqrt{f(n)}; \quad h(n) = \sum_{\substack{l|n \\ l \leq P; \frac{n}{l} \leq P}} \frac{1}{g(l)}$$

$$C_a = \prod_p \left(1 + \frac{1}{p(P^a - 1)} \right)$$

$$D_a = \prod_p \left(1 + \frac{(1 - P^{-a})^{-\frac{1}{2}} - 1}{P} \right)$$

$$k(m) = \sum_{d|m} \frac{1}{h(d)}$$

Lemma 17

Let $\lambda(n)$ be a non negative multiplicative function with $\prod_p \left(1 + \frac{\lambda(p)}{p} \right)$ convergent. Then

$$\sum_{n \leq X} (n^{-\beta} \frac{\pi}{p/n} (1 + \lambda(p))) < 2 \frac{\pi}{p} \left(1 + \frac{\lambda(p)}{p} \right) X^{1-\beta}$$

if $|\beta| < \frac{1}{2}$

Proof:

$$\begin{aligned} \sum_{n \leq X} (n^{-\beta} \frac{\pi}{p/n} (1 + \lambda(p))) &= \sum_{n \leq X} n^{-\beta} \sum_{d/n} \mu^2(d) \lambda(d) \\ &= \sum_d \mu^2(d) \lambda(d) \sum_{\substack{n \leq X \\ n \equiv 0 \pmod{d}}} n^{-\beta} \\ &< 2 X^{1-\beta} \sum_d \frac{\mu^2(d) \lambda(d)}{d} \\ &< 2 X^{1-\beta} \frac{\pi}{p} \left(1 + \frac{\lambda(p)}{p} \right) \end{aligned}$$

Lemma 18

We have

$$(a) \quad \sum_{n \leq X} f(n) < 2 C_a X^{1-a}$$

$$(b) \quad \sum_{n \leq X} g(n) < 2 D_a X^{1-\frac{a}{2}}$$

$$(c) \quad \sum_{n \leq X} g(n) n^{a/2} < 2 D_a X$$

$$(d) \quad \sum_{n \leq X} \frac{1}{g(n)} < X^{1+\frac{a}{2}}$$

$$(d) \quad \sum_{n \leq X} \frac{1}{f(n)} < X^{1+a}$$

Proof:

(a), (b), and (c) follow from Lemma 17 by the proper choice of $\lambda(n)$ and β ; since $\frac{1}{g(n)} \leq n^{a/2}$ and $\frac{1}{f(n)} \leq n^a$, (d) and (e) follow.

Lemma 19: *We have*

$$\sum_{n \leq P^2} f(n) (h(n))^2 \leq 20P^{2-a} C_a D_a^2 (\log P)$$

Proof :

$$\begin{aligned} \sum_{n \leq P^2} f(n) h^2(n) &= \sum_{n \leq P^2} f(n) \left(\sum_{\substack{l \leq P, \\ n/l \leq P}} \frac{1}{g(l)} \right)^2 \\ &< 2 \sum_{l_1 \leq P} \frac{1}{g(l_1)} \sum_{\substack{l_2 \leq l_1 \\ n \leq P/2 \\ n \equiv 0 \pmod{[l_1, l_2]}}} \frac{1}{g(l_2)} \\ &\quad \sum_{m \leq \frac{P/2}{[l_1, l_2]}} f(m) f([l_1, l_2]) \end{aligned}$$

$$\leq 4 C_a \sum_{l_1} \frac{1}{g(l_1)} \sum_{l_2} \frac{f([l_1, l_2])}{g(l_2)} \left(\frac{P/l_2}{[l_1, l_2]} \right)^{1-a}$$

$$\leq 4 C_a P^{1-a} \sum_d \sum_{(l_1, l_2) = d} \frac{1}{g(l_1)} \frac{g^2 \frac{(l_1/l_2)}{d}}{g(l_2)} \left(\frac{d}{l_1} \right)^{1-a}$$

$$\leq 4 C_a P^{1-a} \sum_d \sum_{(l_1, l_2) = d} \frac{1}{g(l_1) g(l_2)} \left(\frac{g(l_1) g(l_2)}{g(d)} \right)^2 \frac{d^{1-a}}{l_1^{1-a}}$$

$$\leq 4 C_a P^{1-a} \sum_d \frac{d^{1-a}}{g^2(d)} \sum_{\substack{l_1 \equiv 0 \pmod{d} \\ l_2 \equiv 0 \pmod{d} \\ l_2 < l_1}} \frac{g(l_1)}{l_1^{1-a}} \sum_{l_2 \equiv 0 \pmod{d}} g(l_2)$$

$$\leq 8 D_a P^{1-a} \sum_d \frac{1}{d^{a/2} g(d)} \sum_{\substack{l_1 \leq P \\ l_1 \equiv 0 \pmod{d}}} g(l_1) l_1^{a/2}$$

$$\leq 16 C_a D_a^2 P^{2-a} \sum_d \frac{1}{d}$$

$$< 20 C_a D_a^2 P^{2-a} (\log P).$$

Lemma 20 :

$$\text{If } f(x) = a_k x^k + a_{k-1} x^{k-1} + \dots + a_0 \text{ and}$$

$$g_l(x) = f(x+l) - f(x) = l a_k x^{k-1} + \dots,$$

$$\text{then } \left| \sum_{x=1}^P e(f(x)) \right|^2 < \\ 2 \sum_{l=1}^P \left| \sum_{x=1}^{P-l} e(g_l(x)) \right| + P$$

Proof :

$$\left| \sum_{x=1}^P e(f(x)) \right|^2 = \sum_x \sum_y e(f(y) - f(x)) \\ < \left| \sum_{\substack{x \\ x=y}} \sum_y e(f(y) - f(x)) \right| \\ + 2 \left| \sum_{\substack{x \\ y > x}} \sum_y e(f(y) - f(x)) \right|$$

We write $y = x + l$ in the second sum and simplify. This gives the result.

Lemma 21 :

$$\text{If } \left| a - \frac{h}{q} \right| < \frac{1}{8qP^3} \text{ and } P < q < 8P^3, \text{ then}$$

$$|T(d)| \leq 4 C_a^{1/8} D_a^{1/2} P^{\frac{7+a}{8}} (\log P)^{\frac{1}{4}}$$

Proof :

By Lemma 20,

$$|T(d)|^2 \leq 2 \sum_{l=1}^P \left| \sum_{x=1}^{P-l} e(4d/x^3 + \dots) \right| + P$$

Hence

$$T_1(d) = |T(d)|^2 - P$$

$$\leq 2 \sum_{l=1}^P \left| \sum_{x=1}^{P-l} e(4d/x^3 + \dots) \right|$$

$$|T_1(d)|^2 \leq 4 \left(\sum_{l=1}^P g(l) \right) \left(\sum_{l=1}^P \frac{1}{g(l)} \left| \sum_x e(4d/x^3 + \dots) \right|^2 \right)$$

$$\leq 8 D_a P^{1 - \frac{a}{2}} \left(\sum_{l=1}^P \frac{1}{g(l)} \left| \sum_x e(4d/x^3 + \dots) \right|^2 \right)$$

By Lemma 20 and 19 (d),

$$T_2(\mathfrak{d}) = \frac{|T_1(\mathfrak{d})|^2}{8D_a P^{1-\frac{a}{2}}} < \sum_{l=1}^P \frac{1}{g(l)}$$

$$\left\{ 2 \sum_{l_1=1}^{P-l} \left| \sum_{x=1}^{P-l-l_1} e(12\mathfrak{d} //_1 x^2 + \dots) \right| + P \right\}$$

$$< 2 \sum_{l=1}^P \frac{1}{g(l)} \sum_{l_1=1}^{P-l} \left| \sum_{x=1}^{P-l-l_1} e(12\mathfrak{d} //_1 x^2 + \dots) \right| + P^{2+a}$$

$$T_3(\mathfrak{d}) = T_2(\mathfrak{d}) - P^{2+a} < 2 \sum_{l, l_1} \frac{1}{g(l)}$$

$$\left| \sum_{x=1}^{P-l-l_1} e(12\mathfrak{d} //_1 x^2 + \dots) \right|$$

$$< 2 \sum h(n)$$

$$n < \frac{P^2}{4}$$

$$P-l-l_1$$

$$\max_{l, l_1 = n} \left| \sum_{x=1} e(12\mathfrak{d} //_1 x^2 + \dots) \right|$$

$$|T_3(\mathbf{d})|^2 < 4 \left(\sum_n f(n) h^2(n) \right) \left(\sum_n \frac{1}{f(n)} \right. \\ \left. \max_{l_1 = n} \left| \sum_x e(12 \mathbf{d} // l_1 x^2 + \dots) \right|^2 \right)$$

$$< \frac{(80 P^{2-a} C_a D_a^2 \log P)}{2^{2-a}} \left(\sum_n \frac{1}{f(n)} \right. \\ \left. \max_{l_1 = n} \left| \sum_x e(12 \mathbf{d} // l_1 x^2 + \dots) \right|^2 \right)$$

$$T_4(\mathbf{d}) = \frac{2^{2-a} T_3(\mathbf{d})^2}{80 P^{2-a} C_a D_a^2 \log P} < \sum_n \frac{1}{f(n)}$$

$$\max_{l_1 = n} \left| \sum_x e(12 \mathbf{d} // l_1 x^2 + \dots) \right|^2$$

$$< \sum_n \frac{1}{f(n)} \max_{l_1 = n} \left\{ 2 \sum_{l_2 = 1}^{P - l_1} \right.$$

$$\left. \left| \sum_{x=1}^{P - l_1 - l_2} e(24 \mathbf{d} // l_1 l_2 x + \dots) \right| + P \right\}$$

$$< 2 \sum_n \frac{1}{f(n)}$$

$$\max_{l_1 = n} \sum_{l_2 = 1}^{P - l_1} \min \left(P, \frac{1}{2 \lfloor 24 \mathbf{d} // l_1 l_2 \rfloor} \right) + P^{3+2a}$$

$$< 2 \sum_m k(m) \cdot \min \left(P, \frac{1}{2 \lfloor 24 \alpha m \rfloor} \right) + P^{3+2a}$$

One easily checks that $k(m) < m^a$ and $m \approx 1/2 < \frac{P^3}{27}$;

We also have $a = 0.1$. Hence

$$T_4(\alpha) < 2 \sum_{m < \frac{P^3}{27}} m^a \min \left(P, \frac{1}{2 \lfloor 24 \alpha m \rfloor} \right) + P^{3+2a}$$

$$< 2 \left(\frac{P^3}{27} \right)^a \sum_{m < \frac{P^3}{27}} \min \left(P, \frac{1}{2 \lfloor 24 \alpha m \rfloor} \right) + P^{3+2a}$$

For a given α , we know that there exist h and q , $(h, q) = 1$ such that $\left| \alpha - \frac{h}{q} \right| < \frac{1}{8qP^3}$, $q < 8P^3$.

Let $q' = \frac{q}{(24, q)}$. The m -sum is divided into at most $\left(\frac{P^3}{27q'} + 1 \right)$ subintervals, each of length q' . In each subinterval, we check that the sum is at most $2P + q' \log q' + q'$.

Hence

$$\begin{aligned} |T_4(\alpha)| &< 2 \frac{P^{0.3}}{(27)^{0.3}} \left(\frac{P^3}{27q'} + 1 \right) \\ &\quad (2P + q' \log q' + q') + P^{3+2a} \\ &< 25 P^{3.3} \log P \end{aligned}$$

$$\text{Hence } \left| T_3(\alpha) \right|^2 < 10^3 P^{5.2} C_a D_a^2 (\log P)^2$$

$$\text{Hence } \left| T_2(\alpha) \right| < (31.63) P^{2.6} C_a^{\frac{1}{2}} D_a (\log P)$$

$$\text{Hence } \left| T_1(\alpha) \right| < (15.91) P^{\frac{7.1}{4}} C_a^{\frac{1}{4}} D_a (\log P)^{\frac{1}{4}}$$

which proves the result.

Lemma 22 :

$$\text{If } \left| \alpha - \frac{h}{q} \right| < \frac{1}{8 q P^3} \text{ and } P^{\frac{1}{8}} < q \leq P,$$

$$\text{then } |T(\alpha)| < 100 P^{7/8}.$$

Proof :

Arguing as in Lemma 9.5 of Thomas [6] or Lemma 9 of Davenport [3], we have the result. (Lemma 9.5 of Thomas [6] is proved under the restriction $\left| \alpha - \frac{h}{q} \right| < \frac{1}{64 q P^3}$ but the same proof holds in our case).

Proof :

$$\text{On the minor arc } |T(\alpha)| < 40 P^{\frac{71}{8}} (\log P)^{\frac{1}{4}}$$

Proof :

The proposition follows from Lemmas 21 and 22

§ 8 : A lower bound for the number of integers upto X representable as a sum of five fourth powers

Lemma 23 :

Let $P \geq 10^{10}$. Let M be the number of solutions of the equation $x^4 + u_h = y^4 + u_j$, subject to (i) $P < x, y < 2P$ (ii) $x \equiv y \pmod{2}$ (iii) $u_h, u_j < P^{3+\mu}$ and (iv) $u_h, u_j \in \mathcal{U}$. Then

$$M < PU + P \frac{1+3\mu}{4} U^{5/4} B.$$

Here $0 < \mu < \frac{1}{2}$; \mathcal{U} is a set of integers in $[0, P^{3+\mu}]$. U is the number of elements of \mathcal{U} .

$$B^4 = \frac{1}{4} \max d_4(m)$$

$$M < \frac{P^{3+\mu}}{192}$$

Proof :

Even though this has not been explicitly stated anywhere, this is essentially contained in Lemma 1 of Davenport [4] and Lemma 7.1 of Thomas [6].

Lemma 24 : The number of solutions of the equation

$$a^4 + b^4 = x^4 + y^4, \text{ with } a, b, x, y < A;$$

$a \equiv x \pmod{2}$; $b \equiv y \pmod{2}$ is at most $A^2 (\log A)^4$ provided $A > 10^{10}$.

Proof :

We write $x + a = 2k$; $x - a = 2l$; $y + b = 2m$
 $y - b = 2n$.

Then we have to find the number of solutions of the equation
 $k/(k^2 + l^2) = m(m^2 + n^2)$, with $k, m \leq A$; $l, n \leq A/2$.
 The number of solutions are

$$\begin{aligned}
 &= \sum_{k, l} \sum_{m, n} 1 \\
 &\quad mn(m^2 + n^2) = k/(k^2 + l^2) \\
 &= \sum_{k, l} d(k/(k^2 + l^2)) \\
 &< \sum \frac{d^2(k) + d^2(l)}{2} d(k^2 + l^2) \\
 &< \sum_{k \leq A} d^2(k) d(k^2 + l^2) \\
 &< 2 \sum_{k \leq A} d^2(k) \sum_l \sum_{\substack{j/k^2 + l^2 \\ j \leq 2A}} 1 \\
 &< 2 \sum_{k \leq A} d^2(k) \sum_{j \leq 2A} \sum_{\substack{k^2 + l^2 \equiv 0 \pmod{j} \\ l \leq A}} 1 \\
 &< 2 \sum_{k \leq A} d^2(k) \sum_{j \leq 2A} \rho(j) \left(\frac{A}{j} + 1 \right)
 \end{aligned}$$

where $P(j)$ is the number of solutions of $k^2 + l^2 \equiv 0 \pmod{j}$, $1 \leq l \leq j$, for a fixed k . We observe that $P(j) \leq \sum_{d|j} \chi(d)$,

where χ is the Dirichlet character $\pmod{4}$ and hence we easily check that

$$\sum_{j \leq 2A} \frac{P(j)}{j} \leq \log A$$

$$\sum_{j \leq 2A} P(j) \leq 4A$$

$$\text{and } \sum_{k \leq A} d^2(k) \leq \frac{A}{3} \log^3(A+2)$$

This proves the result.

Lemma 25 :

Let l be 0, 1 or 2. The number of integers upto X , representable as a sum of two biquadrates and belonging to $l \pmod{16}$ is at least $X^{0.45}$ provided $X \geq 10^{188}$.

Proof :

Choose f_0 and $f_1 \in (0, 1)$ such that $f_0 + f_1 = l$.

Let $r(m)$ be the number of solutions of $m = x^4 + y^4$,

$$x, y \leq \left(\frac{X}{2}\right)^{\frac{1}{4}}; \quad x \equiv f_0 \pmod{2}; \quad y \equiv f_1 \pmod{2}$$

$$\text{Then } \sum r(m) \geq \left(\left(\frac{X}{2}\right)^{\frac{1}{4}} - 1\right)^2$$

By lemma 24, $\sum r^2(m) < \left(\frac{X}{2}\right)^{\frac{1}{2}} (\log X^{\frac{1}{2}})^4$
 $< X^{0.5424}$.

Hence the required estimate

$$\sum_{r(m) \neq 0} 1 \geq \frac{(\sum r(m))^2}{\sum r^2(m)}$$

and we are through.

Lemma 26 :

Let $X > 0$; $P = \left[\frac{X}{17}\right]^{\frac{1}{2}} + 1$; Let μ, B and U_0

satisfy the following conditions

(i) $0 < \mu < \frac{1}{2}$

(ii) $B^4 = \frac{1}{2} \cdot \max_{m < \frac{1}{192} P^{3+\mu}} d_4(m)$

(iii) The number of integers upto $P^{3+\mu}$ which are sums of l biquadrates and belonging to $f_0 \pmod{16}$ is atleast U_0 for every $f_0 \in (0, 1, 2, \dots, l)$

(iv) $B^4 < U_0 P^3 - 3\mu$.

Then the number of integers upto X which are sum of $(l+1)$ biquadrates and belonging to $f \pmod{16}$ is atleast $\frac{PU_0}{8}$ for every $f \in (0, 1, 2, \dots, (l+1))$.

Proof :

Let $f_0 \in (0, 1, 2, \dots, l)$ and $f_1 \in (0, 1)$ such that $f_0 + f_1 = f$. Let $\mathcal{U} = \{n < P^{3+\mu} : n \text{ is sum of } l \text{ biquadrates and } n \equiv f_0 \pmod{16}\}$; U be the number of elements of \mathcal{U} ; By (iii), $U > U_0$.

Let $r(m)$ be the number of solutions of $m = x^4 + u_h$, $P < x < 2P$; $x \equiv f_1 \pmod{2}$ and $u_h \in \mathcal{U}$. The number N of integers upto X , which are representable as a sum of $(l+1)$ biquadrates and $\equiv f \pmod{16}$ is at least

$$\sum_m 1 \\ r(m) \neq 0 \\ > \left(\sum r(m) \right)^2 / \left(\sum r^2(m) \right)$$

Now, using lemma 23

$$N > \left(\frac{PU}{2} \right)^2 / \left(PU + P \frac{1+3\mu}{4} U^{5/4} B \right) \\ > \left(\frac{PU_0}{2} \right)^2 / \left(PU_0 + P \frac{1+3\mu}{4} U_0^{5/4} B \right)$$

$$\text{By (iv), } P \frac{1+3\mu}{4} U_0^{5/4} B < PU_0$$

$$\text{Hence } N > \left(\frac{PU_0}{2} \right)^2 / 2PU_0 > \frac{PU_0}{8}$$

Lemma 27 :

Let f be 0, 1, 2, or 3. Then the number of integers upto X , representable as a sum of three biquadrates belonging to $f \pmod{16}$ is atleast $X^{0.603}$ provided $X > 10^{236}$.

Proof :

We choose $l = 2$; $\mu = 0.20096$ and $U_0 = P^{(3+\mu)}$ (0.45)

in lemma 26; since $B^4 < 10^{12} P^{(3+\mu)}$ (0.235), we verify (iv) in lemma 26. This gives that the required estimate

$$> PU_0 > \frac{1}{8} \left(\frac{X}{17} \right)^{0.610108} > X^{0.603} .$$

Lemma 28 :

Let f be 0, 1, 2, 3 or 4. Then the number of integers upto X , representable as a sum of 4 biquadrates and belonging to $f \pmod{16}$ is atleast $X^{0.7095}$ provided $X > 10^{307}$

Proof :

We choose $l = 3$; $\mu = 0.0869$; $U_0 = P^{(3+\mu)}$ (0.603)

in lemma 26. We have to use $B^4 < 10^{14} P^{(3+\mu)}$ (0.225)

This yields the result.

Prop 6 :

Let f be 0, 1, 2, 3, 4 or 5. Then the number of integers upto X representable as a sum of 5 biquadrates and belonging to $f \pmod{16}$ is atleast $X^{0.7795}$ provided $X > 10^{410}$

Proof :

We take $l = 4$; $\mu = 0.01133$; and $U_0 = P^{(3+\mu)} (0.7095)$.

We use $B^4 \leq (1.5) \times 10^{23}$. This gives the result.

§ 9. Conclusion of the proof :

Lemma 30 :

If $N \geq 10^{412}$, then N can be written as a sum of at most twenty biquadrates.

Proof :

Choose $f \in (0, 1, 2, 3, 4, 5)$ and

$l \in (2, 3, 4, 5, 6, 7)$ such that

$$N - 2f \equiv l \pmod{16}$$

$$P = [N^{\frac{1}{4}}]$$

Let $\mathcal{U} = \left\{ n \leq \frac{N}{4} ; n \text{ is representable as a sum of} \right.$

five biquadrates : $n \equiv f \pmod{16} \left. \right\}$

Let $U(\mathcal{A}) = \sum_{u \in \mathcal{U}} e(u\mathcal{A})$

$$r(N) = \int_0^1 (T(\mathcal{A}))^{10} \cdot (u(\mathcal{A}))^2 e(-N\mathcal{A}) d\mathcal{A}$$

It suffices to prove that $r(N) > 0$

$$\text{Now } r(N) = \int_m + \int_{\mathbb{E}}$$

Now, using Proposition 4,

$$\begin{aligned} \int_m &= \sum_{u_1, u_2} \int_m (T(d))^{10} e^{-(N - u_1 - u_2)d} dd \\ &\geq \sum_{u_1, u_2} W_0(N - u_1 - u_2) \\ &> 0.01 U^2 P^6 \end{aligned}$$

on the other hand, using Proposition 5 and 6,

$$\begin{aligned} &\left| \int_{\mathbb{E}} (T(d))^{10} (U(d))^2 e^{(-Nd)} dd \right| \\ &\leq \max_{d \in \mathbb{E}} |T(d)|^{10} \int_0^1 |u(d)|^2 dd \\ &\leq (40)^{10} P^{\frac{71}{8}} \cdot (\log P)^{2.5} U \\ &\leq (40)^{10} P^{\frac{71}{8}} \cdot (\log P)^{2.5} U^2 / \left(\frac{N}{4}\right)^{0.7795} \\ &\leq 4^{11} 10^{10} P^{5.757} U^2 (\log P)^{2.5} \end{aligned}$$

It suffices to verify that, for $P \leq 10^{102}$,

$$4^{11} 10^{10} P^{5.757} (\log P)^{2.5} < 0.01 P^6$$

and this is true.

Lemma : 31

If $N \leq 10^{412}$, then N can be written as a sum of at most 20 biquadrates.

Proof :

If $N \leq 10^{412}$, then $N - [N^{\frac{1}{4}}]^4 < 10^{310}$ and hence representable as a sum of at most 19 biquadrates by Th. 3.3 of Thomas [7]

Thus the proof of the theorem is complete.

Note :

Some doubts have been raised about the correctness of lemma 31 and the author has not verified the calculations given in [7]. But even a weaker result, compared to lemma 31, will suffice for our purpose and we shall return to this subject at a later date.

References

1. **R. Balasubramanian**, *On Waring's problem: , $g(4) < 21$* , Hardy Ramanujan Journal, Vol. 2 (1979) (1–31).
2. **Chen jing run**, *Waring's problem for $g(5) = 37$* Chinese Mathematical Acta 6 (1965) (105–127)
3. **B. Davenport**, *Analytic Methods for Diophantine equations and Diophantine inequalities*, Ann. Arbor, Ann. Arbor Publishers, MICHIGAN (1962).
4. **H. Davenport**, *On Waring's problem for fourth powers*, Annals of Mathematics, 40 (1939), (731–747)
5. **Lardau, E.** *Vorlesungen uber zahlen theorie, Part 6, Das Waringshe problem*, New York, Chelsea, 1955.
6. **Thomas. H. E.**, *A numerical approach to Waring's problem for fourth powers*, Dissertation, Univ. of Michigan, Ann. Arbor, Michigan 1973.
7. **Thomas. H.E.**, *Waring's problem for twenty two biquadrates*, Transactions of the American Math. Society, 193 (1974), (427–430).
8. **R. C. Vaughan**, *The Hardy-Littlewood Method*, Cambridge University Press, CAMBRIDGE (1957)
9. **I. M. Vinogradov**, *The method of trigonometrical sums in the theory of numbers*, translated from the Russian, revised and annotated by **Davenport. A.** and **Roth K. F.**; New York, Inter science (1954).

School of Mathematics,
Tata Institute of Fundamental Research,
Homi Bhabha Road.
Bombay-400 005. (India)