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# On Waring's Problem :

$$g(4) < 20$$

R. BALASUBRAMANIAN

§ 1. In [1], I proved that every integer is a sum of at most 21 biquadrates. The object of this paper is to prove a refinement namely

**Theorem :** Every positive integer is a sum of not more than twenty biquadrates.

**Remarks :**

(1) Since 79 is not expressible as a sum of eighteen biquadrates, nineteen, if true, is best possible ; but we are unable to improve the theorem further at present.

(2) All numbers upto  $10^{310}$  are sums of 19 biquadrates. This is proved by Henry Thomas Jr. using extensive numerical calculation ([7], Th. 3.3). Our method yields that all numbers bigger than  $10^{700}$  are sums of 19 biquadrates.

(3) For the history of the problem, we refer the reader to [1]

Our work is based upon the papers of Chen Jing run [2] and Davenport [4]. The extensive computer work necessary was done by Thomas ([6] and [7]) and we have freely borrowed these results.

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**§ 2. Notation :**

The following notation will be used throughout this paper.

$$e(x) = e^{2\pi ix}$$

$$S_{a,q} = \sum_{x=1}^q e\left(\frac{ax^4}{q}\right).$$

Let  $N (> 10^{400})$  be a given integer to be represented as a sum of twenty biquadrates.

$$P = [N^{1/4}];$$

$$T(d) = \sum_{1 \leq x \leq P} e(dx^4)$$

Let  $m$  be an integer. (In § 9, we shall choose  $m = 10$ ; in other sections,  $m$  can either be 9 or 10).

The singular series  $S(n)$

$$= \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left( \frac{S_{a,q}}{q} \right)^m e\left(-\frac{an}{q}\right)$$

The truncated singular series

$$S_1(n) = \sum_{q < P^{1/2}} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left( \frac{S_{a,q}}{q} \right)^m e\left(-\frac{an}{q}\right)$$

$$\Psi(d) = \int_0^P e(dx^4) dx$$

The major arc  $m$  is defined by

$$\left\{ d : \left| d - \frac{h}{q} \right| < \frac{1}{8qP^3} \text{ for some } q < P^{1/2} \text{ and } (h,q) = 1 \right\}$$

The minor arc  $m$  is defined by

$$\left\{ d : \left| d - \frac{h}{q} \right| < \frac{1}{8qP^3} \text{ for some } q, P^{1/2} < q < 8P^3 \right. \\ \left. \text{and } (h,q) = 1 \right\}$$

$$W_0(N_0) = \int_m (T(d))^m e(-dN_0) dd$$

$$W(N_0) = \int_0^1 (T(d))^m e(-dN_0) dd$$

$$R(N_0) = \int_{-\infty}^{\infty} (\Psi(\alpha))^m e(-\alpha N_0) d\alpha$$

Each section contains a proposition, which is the main result of the section. Lemmas are subsidiary results needed to prove the proposition.

§ 3. An upper bound for  $S_{a, q}$

We recall that  $S_{a, q} = \sum_{x=1}^q e\left(\frac{ax}{q}\right)^4$ . We define

$$S(q) = \max_a (S_{a, q}).$$

$$(a, q) = 1$$

In this section, we prove

**Prop. 1:** *There holds  $|S(q)| < (4.3)q^{3/4}$*

In order to prove the proposition, we need

**Lemma 1** (a)  $S(q)$  is a multiplicative function of  $q$

(b) If  $p \neq 2$ , and  $a \not\equiv 0 \pmod{p}$ ,

$$|S_{a, p}| < (\delta - 1) p^{1/2}$$

where  $\delta = (4, p-1)$

(c) If  $p \neq 2$ ,  $|S_{a, p}^{\nu}| = p^{\nu-1}$   
if  $2 < \nu < 4$

(d) If  $p > 2$ ,  $|S_{a, p}^{\nu}| = p^3 |S_{a, p}^{\nu-4}|$   
if  $\nu > 4$

**Proof :**

A proof can be found in Davenport [3] (Lemma 6 (page 31) Lemma 12 (page 42), Lemma 13 (page 43) and Lemma 14 (page 44) )

From Lemma 1, it follows that

$$\begin{aligned} \frac{S(q)}{q^{3/4}} &= \prod_{p \parallel q} \pi \frac{S(p^{\alpha})}{p^{3/4 \alpha}} \\ &< \pi \max_{p} (1, \max_{\alpha > 0} \frac{S(p^{\alpha})}{p^{3/4 \alpha}}) \end{aligned}$$

which being a finite product can be evaluated and this gives the proposition. For further details, we refer the reader to Theorem 2.1 in page 38 of Thomas [6].

#### § 4. A lower bound for $R(N_0)$

Let us recall that

$$\Psi(\alpha) = \int_0^P e(\alpha x^4) dx$$

$$R(N_0) = \int_{-\infty}^{\infty} (\Psi(\alpha))^m e(-\alpha N_0) d\alpha$$

$$W(N_0) = \int_0^1 (T(\alpha))^m e(-\alpha N_0) d\alpha$$

$$\text{Define } B = B(\alpha) = \begin{cases} P & \text{if } |\alpha| < P^{-4} \\ 2|\alpha|^{-1/4} & \text{if } |\alpha| > P^{-4} \end{cases}$$

**Lemma 2 :**

Let  $f(x)$  be a real function which is twice differentiable in  $A < x < B$ ; suppose that, in the interval  $A < x < B$ , we have  $0 < f'(x) < \frac{1}{2}$  and  $f''(x) > 0$ . Then

$$\sum_{A < n \leq B} e(f(n)) = \int_A^B e(f(x)) dx + 4\theta$$

**Proof :**

This is Lemma 13 (page 34) in Vinogradov [9]

**Lemma 3**

We have  $|\Psi(d)| < B(d)$

**Proof :**

Clearly  $|\Psi(d)| < P$ .

To prove that  $|\Psi(d)| < 2|d|^{-1/4}$ , it suffices to prove the result for  $d > 0$ . Now a change of variable  $dx^4 = y$  transforms the integral to

$$\Psi(d) = \frac{1}{4d^{1/4}} \int_0^{dP^4} \frac{e(y) dy}{y^{3/4}}$$

and the result is immediate.

**Lemma 4 :**

If  $N_0 - p^{3/4} < N_1 < N_0$ , then

$$W(N_1) = R(N_0) + \int_{\frac{1}{8P^3}}^{1 - \frac{1}{8P^3}} (T(\alpha))^m e(-\alpha N_1) d\alpha + \theta 10^6 p^{m-5+3/4}$$

**Proof :**

By Lemma 2, if  $|\alpha| < \frac{1}{8P^3}$ , we have

$$\begin{aligned} T(\alpha) &= \sum_{1 < x < P} e(\alpha x^4) = \int_0^P e(\alpha x^4) dx + 4\theta \\ &= \Psi(\alpha) + 4\theta \end{aligned}$$

Hence by Lemma 3,

$$\begin{aligned} |(T(\alpha))^m - (\Psi(\alpha))^m| &< 4m (\max(T(\alpha), \Psi(\alpha)))^{m-1} \\ &< 4m (B + 4)^{m-1} \end{aligned}$$

$$\begin{aligned} W(N_1) &= \int_0^1 (T(\alpha))^m e(-\alpha N_1) d\alpha \\ &= \int_{-\frac{1}{8P^3}}^{1 - \frac{1}{8P^3}} (T(\alpha))^m e(-\alpha N_1) d\alpha \end{aligned}$$



$$= \int_{|\alpha| \leq \frac{1}{8P^3}} \dots + \int_{\frac{1 - \frac{1}{8P^3}}{-\frac{1}{8P^3}}} \dots$$

In the first integral, we replace  $(T(\alpha))^m$  by  $(\Psi(\alpha))^m$  with an error  $E_1$ . Then we replace  $e(-\alpha N_1)$  by  $e(-\alpha N_0)$  with an error  $E_2$ . Now we extend the range of integration to  $[-\infty, \infty]$  with an error  $E_3$ . Hence

$$\begin{aligned} W(N_1) &= \int_{-\infty}^{\infty} (\Psi(\alpha))^m e(-\alpha N_0) d\alpha \\ &= \int_{-\frac{1}{8P^3}}^{\frac{1 - \frac{1}{8P^3}}{-\frac{1}{8P^3}}} (T(\alpha))^m e(-\alpha N_1) d\alpha + E_1 + E_2 + E_3 \\ &= R(N_0) + \int_{\frac{1}{8P^3}}^{\frac{1 - \frac{1}{8P^3}}{-\frac{1}{8P^3}}} (T(\alpha))^m e(-\alpha N_1) d\alpha \\ &\quad + E_1 + E_2 + E_3. \end{aligned}$$

$$\begin{aligned} \text{Now } |E_1| &\leq \\ &\left| \int_{|\alpha| \leq \frac{1}{8P^3}} (T(\alpha))^m - (\Psi(\alpha))^m e(-\alpha N_1) d\alpha \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left| \int_{|\alpha| \leq \frac{1}{8P^3}} (4m) (B+4)^{m-1} d\alpha \right| \\
&\leq \left| \int_{|\alpha| \leq P^{-4}} (4m) (P+4)^{m-1} d\alpha \right| \\
&+ \left| \frac{1}{8P^3} \int_{|\alpha| \geq P^{-4}} (4m) (2|\alpha|^{-\frac{1}{2}} + 4)^{m-1} d\alpha \right| \\
&\leq 10^5 P^{m-5}, \text{ since } m \text{ is either } 9 \text{ or } 10
\end{aligned}$$

Similarly,

$$\begin{aligned}
|E_2| &\leq \left| \int_{|\alpha| < \frac{1}{8P^3}} (\Psi(\alpha))^m \left| e(-\alpha N_1) - e(-\alpha N_0) \right| d\alpha \right| \\
&\leq \int_{|\alpha| < \frac{1}{8P^3}} B^m (2\pi) |\alpha| |N_1 - N_0| d\alpha \\
&\leq P^{3\frac{3}{4}} (2\pi) \int_{|\alpha| < P^{-4}} P^m |\alpha| d\alpha +
\end{aligned}$$

$$+ P^{3\frac{3}{4}} (2\pi) \int_{|\alpha| > P^{-4}} (2|\alpha|^{-\frac{1}{4}})^m |\alpha| d\alpha$$

$$< 10^5 P^{m-5+\frac{3}{4}}$$

$$|E_3| < \int_{|\alpha| > \frac{1}{8P^3}} |\Psi(\alpha)|^m d\alpha$$

$$< \int_{|\alpha| > \frac{1}{8P^3}} B^m d\alpha$$

$$\leq 10^5 P^{\frac{3}{4}m-3}$$

This proves the result.

**Lemma 5 :**

$$\text{With } M = \left[ \frac{P^{3\frac{3}{4}}}{2} \right]$$

$$\sum_{1 \leq A \leq M} \sum_{1 \leq B \leq M} W(N_0 - A - B) = M^2 R(N_0)$$

$$+ o(10^7 M^2 P^{m-5+\frac{3}{4}})$$

**Proof :**

In lemma 4, we take  $N_1 = N_0 - A - B$  and sum over  $1 \leq A \leq M$  and  $1 \leq B \leq M$ . This gives

$$\sum_A \sum_B W(N_0 - A - B) = M^2 R(N_0)$$

$$+ \int \frac{1 - \frac{1}{8P^3}}{\frac{1}{8P^3}} (T(\alpha))^m e(-\alpha N_0) \left( \sum_A e(\alpha A) \right)^2 d\alpha$$

$$+ \theta 10^6 M^2 P^{m-5+\frac{1}{2}}$$

Since  $\left| \sum_A e(\alpha A) \right| \leq \frac{1}{\|\alpha\|}$ , the integral appearing on the right, is bounded, in absolute value, by

$$\int \frac{1 - \frac{1}{8P^3}}{\frac{1}{8P^3}} P^m \left( \frac{1}{\|\alpha\|} \right)^2 d\alpha \leq 10^4 P^{m+3}$$

### Lemma 6

We have

$$\sum_{1 \leq A \leq M} \sum_{1 \leq B \leq M} W(N_0 - A - B)$$

$$> T_m \left( \frac{m-1}{4} \right) M^2 N_0^{\frac{m}{4} - 1}$$

$$\text{where } T_m = \frac{(\Gamma(5/4))^m}{\Gamma(1 + \frac{m}{4})}$$

**Proof :**

By lemma 3 (page 22) of Vinogradov [9], the number of integer solutions  $K_r(N)$  of  $x_1^4 + x_2^4 + \dots + x_r^4 \leq N$  is given by

$$K_r(N) = T_r N^{\frac{r}{4}} - \theta_r N^{\frac{r-1}{4}} \quad \text{where}$$

$$T_r = \frac{(\Gamma(5/4))^r}{\Gamma(1 + \frac{r}{4})}$$

Now,

$$\begin{aligned} & \sum_{1 \leq B \leq M} W(N_0 - A - B) \\ &= \sum_B (K_m(N_0 - A - B) - K_m(N_0 - A - B - 1)) \\ &= K_m(N_0 - A - 1) - K_m(N_0 - A - M - 1) \\ &= T_m [(N_0 - A - 1)^{\frac{m}{4}} - (N_0 - A - M - 1)^{\frac{m}{4}}] + 2m\theta N_0^{\frac{m-1}{4}} \\ &> T_m \frac{m-1}{4} M N_0^{\frac{m}{4} - 1} \end{aligned}$$

and hence the lemma.

**Prop. 2**

We have, for  $m = 9$  or  $10$ ,

$$R(N_0) \geq 0.25 N_0^{\frac{m}{4} - 1}$$

**Proof :**

From lemmas 5 and 6, it follows that

$$R(N_0) \geq \left( \frac{T_m(m-1)}{4} - 10^{-5} \right) N_0^{\frac{m}{4} - 1}$$

and hence the result.

§ 5 : A lower bound for  $S_1(N_0)$ :

Let us recall that  $S(n) = S(n, m)$  is given by

$$\begin{aligned} S(n) &= \sum_{q=1}^{\infty} \sum_{a=1}^q \left( \frac{S_{a,q}}{q} \right)^m e\left(-\frac{an}{q}\right) \\ &= \sum_q A(n; q) \text{ say} \end{aligned}$$

$$\text{Write } X_p(n) = \chi_p(n; m) = \sum_{i=1}^{\infty} A(n; p^i).$$

**Lemma 7 :**

$A(n; q)$  is a multiplicative function of  $q$  and hence

$$S(n) = \prod_p (1 + \chi_p(n))$$

**Proof :**

We refer the reader to Lemma 2.11 (page 20) of Vaughan [8]

**Lemma 8 :**

(a) For any prime  $p$ , and natural number  $\lambda$ , let  $N_m(p^\lambda, n)$  denote the number of solutions of

$$x_1^4 + x_2^4 + \dots + x_m^4 \equiv n \pmod{p^\lambda},$$

$$1 \leq x_i \leq p^\lambda \text{ and not all } x_i \equiv 0 \pmod{p}$$

Further set  $\gamma = 1$  if  $p$  is odd and  $\gamma = 4$  if  $p = 2$ . Let  $p^{4r+s}$  exactly divide  $n$ ,  $0 \leq s \leq 3$ .

$$k_0 = \max(4r + s + 1, 4r + \gamma). \text{ Then}$$

$$A(n, p^\lambda) = 0 \text{ for } \lambda > k_0 \text{ and}$$

$$1 + \chi_p(n, m) = p^{-(m-1)\gamma} N_m(p^\gamma, 0) \left\{ \sum_{\tau=0}^{r-1} p^{\tau(4-m)} \right\} \\ + p^{(4-m)r - (m-1)\gamma} N_m(p^\gamma, np^{-4r})$$

where the empty sum is understood to be zero.

(b) Let  $d = \text{g.c.d.}(4, p-1)$ . Then for  $p \neq 2$ ,

$$\left| N_m(p, n) - p^{m-1} \right| \leq \left(1 - \frac{1}{p}\right) (d-1)^m p^{\frac{m}{2}}$$

**Proof :**

For the proof of Lemma 8 (a), we refer the reader to Hilfsatz 293 of LANDAU [5] or Prop. 4.3 of Thomas [6]. For the proof of lemma 8 (b), we refer to lemma 41 (page 91) of Thomas [6].

From Lemma 8, we have

**Lemma 9 :**

$$\text{Let } K > 100; \quad S_2 = \{ p > K; p \equiv 3 \pmod{4} \}$$

$$S_4 = \{ p > K; p \equiv 1 \pmod{4} \}. \quad \text{Then for } d = 2 \text{ or } 4$$

$$\begin{aligned} & \prod_{p \in S_d} (1 + \chi_p(n)) \\ & > \exp \left( -2 (d-1)^m K^{1 - \frac{m}{2}} \left( 1 + \frac{K}{2m-8} \right) \right) \end{aligned}$$

**Lemma 10 :**

$$\text{We have } \prod_{p > 114} (1 + \chi_p(n)) > 0.97$$

$$\text{Since } \prod_{p > 114} (1 + \chi_p(n))$$

$$\begin{aligned} &= \prod_{\substack{p > 127 \\ p \equiv 3 \pmod{4}}} (1 + \chi_p(n)) \prod_{\substack{p > 137 \\ p \equiv 1 \pmod{4}}} (1 + \chi_p(n)) \end{aligned}$$

the result follows from Lemma 9.

**Lemma 11 :**

$$\text{We have } \prod_{2 < p < 113} (1 + \chi_p(n)) > 0.297.$$



**Proof :**

This is easily verified using Lemma 8 (a). For details, we refer the reader to Thomas [6]

**Lemma 12 :**

If  $n \equiv \tilde{n} \pmod{16}$ ,  $2 < \tilde{n} < m-2$ , then

$$1 + X_2(n) = \binom{m}{\tilde{n}} 2^{-m+4} \text{ and hence}$$

$$1 + X_2(n) > \frac{45}{64}$$

Here  $\binom{m}{\tilde{n}}$  is the binomial coefficient

**Proof :**

This is straight forward. For details, we refer the reader to Theorem 4.2 of Thomas [6].

**Lemma 13 :**

We have  $S(n) > 0.2025$

provided  $n \equiv 2, 3, 4, 5, 6$  or  $7 \pmod{16}$

**Proof :**

This follows from lemmas 7, 10, 11 and 12.

**Lemma 14 :**

We have  $S(n) - S_1(n) \leq 0.002$

**Proof :**

Since  $|S(n) - S_1(n)| =$

$$\left| \sum_{q > p^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left( \frac{S_{a,q}}{q} \right)^m e\left(-\frac{an}{q}\right) \right|$$

$$< \sum_{q > p^{\frac{1}{2}}} \sum_{a=1}^q \left( 4.3 q^{-\frac{1}{4}} \right)^m,$$

we are through.

**Prop 3 :**

*There holds the inequality*

$$S_1(n) > 0.2$$

if  $n \equiv l \pmod{16}$ , and  $l \in \{2, 3, 4, 5, 6, 7\}$ .

§ 6 : A lower bound for  $W_0(N_0)$  :

Let us recall that  $W_0(N_0) =$

$$\int_m (T(\alpha))^m e(-\alpha N_0) d\alpha.$$

**Lemma 15 :**

If  $|\beta| < \frac{1}{8qP^3}$ , then

$$\begin{aligned} \left(T\left(\frac{a}{q} + \beta\right)\right)^m &= (\Psi(\beta) \frac{S_{a,q}}{q})^m \\ &+ \theta \cdot 4qm (4.3 B(\beta)q^{-\frac{1}{4}} + 4q)^{m-1} \end{aligned}$$

**Proof :**

We have,

$$\begin{aligned} T\left(\frac{a}{q} + \beta\right) &= \sum_{x=1}^P e\left(\left(\frac{a}{q} + \beta\right)x^4\right) \\ &= \sum_{y=0}^{q-1} \sum_{-\frac{y}{q} < t < \frac{P-y}{q}} e\left(\left(\frac{a}{q} + \beta\right)(qt + y)^4\right) \\ &= \sum_y \sum_t e\left(\frac{ay^4}{q} + \beta (qt + y)^4\right) \\ &= \sum_y e\left(\frac{ay^4}{q}\right) D_y(z), \text{ say} \end{aligned}$$

Now, using Lemma 2,

$$\begin{aligned}
 D_y(z) &= \int_{-\frac{y}{q}}^{\frac{P-y}{q}} e(\beta(qt+y)^4) dt + 4\theta \\
 &= \frac{1}{q} \int_0^P e(\beta x^4) dx + 4\theta \\
 &= \frac{1}{q} \Psi(\beta) + 4\theta.
 \end{aligned}$$

$$\text{Hence } T\left(\frac{a}{q} + \beta\right) = \frac{S_{a,q}}{q} \Psi(\beta) + 4\theta q$$

The result follows from the following inequalities.

$$\left| \frac{S_{a,q}}{q} \right| < (4.3) q^{-\frac{1}{4}}$$

$$|\Psi(\beta)| < B(\beta) \text{ and}$$

$$a^m - b^m < m(a-b) \max(|a|^{m-1}, |b|^{m-1})$$

**Lemma 16 :**

*We have*

$$W_0(N_0) = S_1(N_0) R(N_0) + \theta 10^{13} P^{m-\frac{9}{2}}$$

**Proof :**

Since the proof is similar to that of Lemma 4, we give only the sketch of the proof.

$$\begin{aligned}
 W_0(N_0) &= \int_m (T(\alpha))^m e(-\alpha N_0) d\alpha \\
 &= \sum_{q < P^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a, q)=1}}^q \int \frac{\frac{a}{q} + \frac{1}{8qP^3}}{\frac{a}{q} - \frac{1}{8qP^3}} (T(\alpha))^m e(-N_0 \alpha) d\alpha \\
 &= \sum_{q < P^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a, q)=1}}^q \int \frac{1}{8qP^3} \left( T\left(\frac{a}{q} + \beta\right) \right)^m e\left(-N_0\left(\frac{a}{q} + \beta\right)\right) d\beta - \frac{1}{8qP^3}
 \end{aligned}$$

First we replace  $\left(T\left(\frac{a}{q} + \beta\right)\right)^m$  by  $\left(\psi(\beta) \frac{S_{a, q}}{q}\right)^m$

and the error is, by Lemma 15, at most  $10^{12} P^{m-9/2}$ . Now we extend the range of integration of the integral to  $[-\infty, \infty]$  which gives an error at most  $10^{12} P^{m-9/2}$ . Hence

$$\begin{aligned}
 W_0(N_0) &= \sum_{q < P^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a, q)=1}}^q \int_{-\infty}^{\infty} (\psi(\beta))^m \left(\frac{S_{a, q}}{q}\right)^m \\
 &\quad e\left(-N_0\left(\frac{a}{q} + \beta\right)\right) d\beta + \theta 10^{13} P^{m-9/2} \\
 &= S_1(N_0) R(N_0) + \theta 10^{13} P^{m-9/2}
 \end{aligned}$$

**Prop 4 :** *We have*

$$W_0(N_0) \geq 0.05 N_0^{\frac{m}{4} - 1}$$

*provided*  $N_0 \equiv 2, 3, 4, 5, 6$  or  $7 \pmod{16}$

**Proof :**

This follows from Prop. 2, Prop. 3 and Lemma 16.

§ 7 : *The estimate on the minor arc :*

We define  $f(n) = n^{-a} \prod_{p|n} (1 - p^{-a})^{-1}$  where  $a (>0)$

is a constant. Ultimately we shall choose  $a = 0.1$

$$g(n) = \sqrt{f(n)}; \quad h(n) = \sum_{\substack{l|n \\ l \leq P; \frac{n}{l} \leq P}} \frac{1}{g(l)}$$

$$C_a = \prod_p \left( 1 + \frac{1}{p(P^a - 1)} \right)$$

$$D_a = \prod_p \left( 1 + \frac{(1 - P^{-a})^{-\frac{1}{2}} - 1}{P} \right)$$

$$k(m) = \sum_{d|m} \frac{1}{h(d)}$$

**Lemma 17**

Let  $\lambda(n)$  be a non negative multiplicative function with  $\prod_p \left( 1 + \frac{\lambda(p)}{p} \right)$  convergent. Then

$$\sum_{n \leq X} (n^{-\beta} \frac{\pi}{p/n} (1 + \lambda(p))) < 2 \frac{\pi}{p} \left( 1 + \frac{\lambda(p)}{p} \right) X^{1-\beta}$$

if  $|\beta| < \frac{1}{2}$

**Proof:**

$$\begin{aligned} \sum_{n \leq X} (n^{-\beta} \frac{\pi}{p/n} (1 + \lambda(p))) &= \sum_{n \leq X} n^{-\beta} \sum_{d/n} \mu^2(d) \lambda(d) \\ &= \sum_d \mu^2(d) \lambda(d) \sum_{\substack{n \leq X \\ n \equiv 0 \pmod{d}}} n^{-\beta} \\ &< 2 X^{1-\beta} \sum_d \frac{\mu^2(d) \lambda(d)}{d} \\ &< 2 X^{1-\beta} \frac{\pi}{p} \left( 1 + \frac{\lambda(p)}{p} \right) \end{aligned}$$

**Lemma 18**

*We have*

$$(a) \quad \sum_{n \leq X} f(n) < 2 C_a X^{1-a}$$

$$(b) \quad \sum_{n \leq X} g(n) < 2 D_a X^{1-\frac{a}{2}}$$

$$(c) \quad \sum_{n \leq X} g(n) n^{a/2} < 2 D_a X$$

$$(d) \quad \sum_{n \leq X} \frac{1}{g(n)} < X^{1+\frac{a}{2}}$$

$$(d) \quad \sum_{n \leq X} \frac{1}{f(n)} < X^{1+a}$$

**Proof:**

(a), (b), and (c) follow from Lemma 17 by the proper choice of  $\lambda(n)$  and  $\beta$ ; since  $\frac{1}{g(n)} \leq n^{a/2}$  and  $\frac{1}{f(n)} \leq n^a$ , (d) and (e) follow.

**Lemma 19:** *We have*

$$\sum_{n \leq P^2} f(n) (h(n))^2 \leq 20P^{2-a} C_a D_a^2 (\log P)$$

**Proof :**

$$\begin{aligned} \sum_{n \leq P^2} f(n) h^2(n) &= \sum_{n \leq P^2} f(n) \left( \sum_{\substack{l \leq P, \\ n/l \leq P}} \frac{1}{g(l)} \right)^2 \\ &< 2 \sum_{l_1 \leq P} \frac{1}{g(l_1)} \sum_{\substack{l_2 \leq l_1 \\ n \leq P/2 \\ n \equiv 0 \pmod{[l_1, l_2]}}} \frac{1}{g(l_2)} \\ &\quad \sum_{m \leq \frac{P/2}{[l_1, l_2]}} f(m) f([l_1, l_2]) \end{aligned}$$



$$\leq 4 C_a \sum_{l_1} \frac{1}{g(l_1)} \sum_{l_2} \frac{f([l_1, l_2])}{g(l_2)} \left( \frac{P/l_2}{[l_1, l_2]} \right)^{1-a}$$

$$\leq 4 C_a P^{1-a} \sum_d \sum_{(l_1, l_2) = d} \frac{1}{g(l_1)} \frac{g^2 \frac{(l_1/l_2)}{d}}{g(l_2)} \left( \frac{d}{l_1} \right)^{1-a}$$

$$\leq 4 C_a P^{1-a} \sum_d \sum_{(l_1, l_2) = d} \frac{1}{g(l_1) g(l_2)} \left( \frac{g(l_1) g(l_2)}{g(d)} \right)^2 \frac{d^{1-a}}{l_1^{1-a}}$$

$$\leq 4 C_a P^{1-a} \sum_d \frac{d^{1-a}}{g^2(d)} \sum_{\substack{l_1 \equiv 0 \pmod{d} \\ l_2 \equiv 0 \pmod{d} \\ l_2 < l_1}} \frac{g(l_1)}{l_1^{1-a}} \sum_{l_2 \equiv 0 \pmod{d}} g(l_2)$$

$$\leq 8 D_a P^{1-a} \sum_d \frac{1}{d^{a/2} g(d)} \sum_{\substack{l_1 \leq P \\ l_1 \equiv 0 \pmod{d}}} g(l_1) l_1^{a/2}$$

$$\leq 16 C_a D_a^2 P^{2-a} \sum_d \frac{1}{d}$$

$$< 20 C_a D_a^2 P^{2-a} (\log P).$$

**Lemma 20 :**

$$\text{If } f(x) = \alpha_k x^k + \alpha_{k-1} x^{k-1} + \dots + \alpha_0 \text{ and}$$

$$g_l(x) = f(x+l) - f(x) = k \alpha_k x^{k-1} + \dots,$$

$$\text{then } \left| \sum_{x=1}^P e(f(x)) \right|^2 < \\ 2 \sum_{l=1}^P \left| \sum_{x=1}^{P-l} e(g_l(x)) \right| + P$$

**Proof :**

$$\left| \sum_{x=1}^P e(f(x)) \right|^2 = \sum_x \sum_y e(f(y) - f(x)) \\ < \left| \sum_{\substack{x \\ x=y}} \sum_y e(f(y) - f(x)) \right| \\ + 2 \left| \sum_{\substack{x \\ y > x}} \sum_y e(f(y) - f(x)) \right|$$

We write  $y = x + l$  in the second sum and simplify. This gives the result.

**Lemma 21 :**

$$\text{If } \left| \alpha - \frac{h}{q} \right| < \frac{1}{8qP^3} \text{ and } P < q < 8P^3, \text{ then}$$

$$|T(d)| \leq 4 C_a^{1/8} D_a^{1/2} P^{\frac{7+a}{8}} (\log P)^{\frac{1}{4}}$$

**Proof :**

By Lemma 20,

$$|T(d)|^2 \leq 2 \sum_{l=1}^P \left| \sum_{x=1}^{P-l} e(4d/x^3 + \dots) \right| + P$$

Hence

$$T_1(d) = |T(d)|^2 - P$$

$$\leq 2 \sum_{l=1}^P \left| \sum_{x=1}^{P-l} e(4d/x^3 + \dots) \right|$$

$$|T_1(d)|^2 \leq 4 \left( \sum_{l=1}^P g(l) \right) \left( \sum_{l=1}^P \frac{1}{g(l)} \left| \sum_x e(4d/x^3 + \dots) \right|^2 \right)$$

$$\leq 8 D_a P^{1 - \frac{a}{2}} \left( \sum_{l=1}^P \frac{1}{g(l)} \left| \sum_x e(4d/x^3 + \dots) \right|^2 \right)$$

By Lemma 20 and 19 (d),

$$T_2(\mathfrak{d}) = \frac{|T_1(\mathfrak{d})|^2}{8D_a P^{1-\frac{a}{2}}} < \sum_{l=1}^P \frac{1}{g(l)}$$

$$\left\{ 2 \sum_{l_1=1}^{P-l} \left| \sum_{x=1}^{P-l-l_1} e(12 \mathfrak{d} //_1 x^2 + \dots) \right| + P \right\}$$

$$< 2 \sum_{l=1}^P \frac{1}{g(l)} \sum_{l_1=1}^{P-l} \left| \sum_{x=1}^{P-l-l_1} e(12 \mathfrak{d} //_1 x^2 + \dots) \right| + P^{2+a}$$

$$T_3(\mathfrak{d}) = T_2(\mathfrak{d}) - P^{2+a} < 2 \sum_{l, l_1} \frac{1}{g(l)}$$

$$\left| \sum_{x=1}^{P-l-l_1} e(12 \mathfrak{d} //_1 x^2 + \dots) \right|$$

$$< 2 \sum h(n)$$

$$n < \frac{P^2}{4}$$

$$P-l-l_1$$

$$\max_{l, l_1 = n} \left| \sum_{x=1} e(12 \mathfrak{d} //_1 x^2 + \dots) \right|$$

$$|T_3(\mathbf{d})|^2 < 4 \left( \sum_n f(n) h^2(n) \right) \left( \sum_n \frac{1}{f(n)} \right. \\ \left. \max_{l_1 = n} \left| \sum_x e(12 \mathbf{d} // l_1 x^2 + \dots) \right|^2 \right)$$

$$< \frac{(80 P^{2-a} C_a D_a^2 \log P)}{2^{2-a}} \left( \sum_n \frac{1}{f(n)} \right. \\ \left. \max_{l_1 = n} \left| \sum_x e(12 \mathbf{d} // l_1 x^2 + \dots) \right|^2 \right)$$

$$T_4(\mathbf{d}) = \frac{2^{2-a} T_3(\mathbf{d})^2}{80 P^{2-a} C_a D_a^2 \log P} < \sum_n \frac{1}{f(n)}$$

$$\max_{l_1 = n} \left| \sum_x e(12 \mathbf{d} // l_1 x^2 + \dots) \right|^2$$

$$< \sum_n \frac{1}{f(n)} \max_{l_1 = n} \left\{ 2 \sum_{l_2 = 1}^{P - l_1} \right.$$

$$\left. \left| \sum_{x=1}^{P - l_1 - l_2} e(24 \mathbf{d} // l_1 l_2 x + \dots) \right| + P \right\}$$

$$< 2 \sum_n \frac{1}{f(n)}$$

$$\max_{l_1 = n} \sum_{l_2 = 1}^{P - l_1} \min \left( P, \frac{1}{2 \lfloor 24 \mathbf{d} // l_1 l_2 \rfloor} \right) + P^{3+2a}$$

$$< 2 \sum_m k(m) \cdot \min \left( P, \frac{1}{2 \lfloor 24 \alpha m \rfloor} \right) + P^{3+2a}$$

One easily checks that  $k(m) < m^a$  and  $m = \lfloor 1/2 \rfloor < \frac{P^3}{27}$ ;

We also have  $a = 0.1$ . Hence

$$T_4(\alpha) < 2 \sum_{m < \frac{P^3}{27}} m^a \min \left( P, \frac{1}{2 \lfloor 24 \alpha m \rfloor} \right) + P^{3+2a}$$

$$< 2 \left( \frac{P^3}{27} \right)^a \sum_{m < \frac{P^3}{27}} \min \left( P, \frac{1}{2 \lfloor 24 \alpha m \rfloor} \right) + P^{3+2a}$$

For a given  $\alpha$ , we know that there exist  $h$  and  $q$ ,  $(h, q) = 1$  such that  $\left| \alpha - \frac{h}{q} \right| < \frac{1}{8qP^3}$ ,  $q < 8P^3$ .

Let  $q' = \frac{q}{(24, q)}$ . The  $m$ -sum is divided into at most  $\left( \frac{P^3}{27q'} + 1 \right)$  subintervals, each of length  $q'$ . In each subinterval, we check that the sum is at most  $2P + q' \log q' + q'$ .

Hence

$$\begin{aligned} |T_4(\alpha)| &< 2 \frac{P^{0.3}}{(27)^{0.3}} \left( \frac{P^3}{27q'} + 1 \right) \\ &\quad (2P + q' \log q' + q') + P^{3+2a} \\ &< 25 P^{3.3} \log P \end{aligned}$$

$$\text{Hence } \left| T_3(\alpha) \right|^2 < 10^3 P^{5.2} C_a D_a^2 (\log P)^2$$

$$\text{Hence } \left| T_2(\alpha) \right| < (31.63) P^{2.6} C_a^{\frac{1}{2}} D_a (\log P)$$

$$\text{Hence } \left| T_1(\alpha) \right| < (15.91) P^{\frac{7.1}{4}} C_a^{\frac{1}{4}} D_a (\log P)^{\frac{1}{4}}$$

which proves the result.

**Lemma 22** :

$$\text{If } \left| \alpha - \frac{h}{q} \right| < \frac{1}{8 q P^3} \text{ and } P^{\frac{1}{8}} < q \leq P,$$

$$\text{then } |T(\alpha)| < 100 P^{7/8}.$$

**Proof :**

Arguing as in Lemma 9.5 of Thomas [6] or Lemma 9 of Davenport [3], we have the result. (Lemma 9.5 of Thomas [6] is proved under the restriction  $\left| \alpha - \frac{h}{q} \right| < \frac{1}{64 q P^3}$  but the same proof holds in our case).

**Proof :**

$$\text{On the minor arc } |T(\alpha)| < 40 P^{\frac{71}{8}} (\log P)^{\frac{1}{4}}$$

**Proof :**

The proposition follows from Lemmas 21 and 22

§ 8 : A lower bound for the number of integers upto  $X$  representable as a sum of five fourth powers

**Lemma 23 :**

Let  $P \geq 10^{10}$ . Let  $M$  be the number of solutions of the equation  $x^4 + u_h = y^4 + u_j$ , subject to (i)  $P < x, y < 2P$  (ii)  $x \equiv y \pmod{2}$  (iii)  $u_h, u_j < P^{3+\mu}$  and (iv)  $u_h, u_j \in \mathcal{U}$ . Then

$$M < PU + P \frac{1 + 3\mu}{4} U^{5/4} B.$$

Here  $0 < \mu < \frac{1}{2}$ ;  $\mathcal{U}$  is a set of integers in  $[0, P^{3+\mu}]$ .  $U$  is the number of elements of  $\mathcal{U}$ .

$$B^4 = \frac{1}{4} \max d_4(m)$$

$$M < \frac{P^{3+\mu}}{192}$$

**Proof :**

Even though this has not been explicitly stated anywhere, this is essentially contained in Lemma 1 of Davenport [4] and Lemma 7.1 of Thomas [6].

**Lemma 24 :** The number of solutions of the equation

$$a^4 + b^4 = x^4 + y^4, \text{ with } a, b, x, y < A;$$

$a \equiv x \pmod{2}$ ;  $b \equiv y \pmod{2}$  is at most  $A^2 (\log A)^4$  provided  $A > 10^{10}$ .



**Proof :**

We write  $x + a = 2k$ ;  $x - a = 2l$ ;  $y + b = 2m$   
 $y - b = 2n$ .

Then we have to find the number of solutions of the equation  
 $k/(k^2 + l^2) = m(m^2 + n^2)$ , with  $k, m \leq A$ ;  $l, n \leq A/2$ .  
 The number of solutions are

$$\begin{aligned}
 &= \sum_{k, l} \sum_{m, n} 1 \\
 &\quad mn(m^2 + n^2) = k/(k^2 + l^2) \\
 &= \sum_{k, l} d(k/(k^2 + l^2)) \\
 &< \sum \frac{d^2(k) + d^2(l)}{2} d(k^2 + l^2) \\
 &< \sum_{k \leq A} d^2(k) d(k^2 + l^2) \\
 &< 2 \sum_{k \leq A} d^2(k) \sum_l \sum_{\substack{j/k^2 + l^2 \\ j \leq 2A}} 1 \\
 &< 2 \sum_{k \leq A} d^2(k) \sum_{j \leq 2A} \sum_{\substack{k^2 + l^2 \equiv 0 \pmod{j} \\ l \leq A}} 1 \\
 &< 2 \sum_{k \leq A} d^2(k) \sum_{j \leq 2A} \rho(j) \left( \frac{A}{j} + 1 \right)
 \end{aligned}$$

where  $P(j)$  is the number of solutions of  $k^2 + l^2 \equiv 0 \pmod{j}$ ,  $1 \leq l \leq j$ , for a fixed  $k$ . We observe that  $P(j) \leq \sum_{d|j} \chi(d)$ ,

where  $\chi$  is the Dirichlet character  $\pmod{4}$  and hence we easily check that

$$\sum_{j \leq 2A} \frac{P(j)}{j} \leq \log A$$

$$\sum_{j \leq 2A} P(j) \leq 4A$$

$$\text{and } \sum_{k \leq A} d^2(k) \leq \frac{A}{3} \log^3(A+2)$$

This proves the result.

**Lemma 25 :**

Let  $l$  be 0, 1 or 2. The number of integers upto  $X$ , representable as a sum of two biquadrates and belonging to  $l \pmod{16}$  is at least  $X^{0.45}$  provided  $X \geq 10^{188}$ .

**Proof :**

Choose  $f_0$  and  $f_1 \in (0, 1)$  such that  $f_0 + f_1 = l$ .

Let  $r(m)$  be the number of solutions of  $m = x^4 + y^4$ ,

$$x, y \leq \left(\frac{X}{2}\right)^{\frac{1}{4}}; \quad x \equiv f_0 \pmod{2}; \quad y \equiv f_1 \pmod{2}$$

$$\text{Then } \sum r(m) \geq \left(\left(\frac{X}{2}\right)^{\frac{1}{4}} - 1\right)^2$$

By lemma 24,  $\sum r^2(m) < \left(\frac{X}{2}\right)^{\frac{1}{4}} (\log X^{\frac{1}{4}})^4$   
 $< X^{0.5424}$ .

Hence the required estimate

$$\sum_{r(m) \neq 0} 1 \geq \frac{(\sum r(m))^2}{\sum r^2(m)}$$

and we are through.

**Lemma 26 :**

Let  $X > 0$ ;  $P = \left[\frac{X}{17}\right]^{\frac{1}{4}} + 1$ ; Let  $\mu, B$  and  $U_0$

satisfy the following conditions

(i)  $0 < \mu < \frac{1}{4}$

(ii)  $B^4 = \frac{1}{4} \cdot \max_{m < \frac{1}{192} P^{3+\mu}} d_4(m)$

(iii) The number of integers upto  $P^{3+\mu}$  which are sums of  $l$  biquadrates and belonging to  $f_0 \pmod{16}$  is atleast  $U_0$  for every  $f_0 \in (0, 1, 2, \dots, l)$

(iv)  $B^4 < U_0 P^3 - 3\mu$ .

Then the number of integers upto  $X$  which are sum of  $(l+1)$  biquadrates and belonging to  $f \pmod{16}$  is atleast  $\frac{PU_0}{8}$  for every  $f \in (0, 1, 2, \dots, (l+1))$ .

**Proof :**

Let  $f_0 \in (0, 1, 2, \dots, l)$  and  $f_1 \in (0, 1)$  such that  $f_0 + f_1 = f$ . Let  $\mathcal{U} = \{ n < P^{3+\mu} : n \text{ is sum of } l \text{ biquadrates and } n \equiv f_0 \pmod{16} \}$ ;  $U$  be the number of elements of  $\mathcal{U}$ ; By (iii),  $U > U_0$ .

Let  $r(m)$  be the number of solutions of  $m = x^4 + u_h$ ,  $P < x < 2P$ ;  $x \equiv f_1 \pmod{2}$  and  $u_h \in \mathcal{U}$ . The number  $N$  of integers upto  $X$ , which are representable as a sum of  $(l + 1)$  biquadrates and  $\equiv f \pmod{16}$  is atleast

$$\sum_m 1$$

$$r(m) \neq 0$$

$$> \left( \sum r(m) \right)^2 / \left( \sum r^2(m) \right)$$

Now, using lemma 23

$$N > \left( \frac{PU}{2} \right)^2 / \left( PU + P \frac{1+3\mu}{4} U^{5/4} B \right)$$

$$> \left( \frac{PU_0}{2} \right)^2 / \left( PU_0 + P \frac{1+3\mu}{4} U_0^{5/4} B \right)$$

By (iv),  $P \frac{1+3\mu}{4} U_0^{5/4} B < PU_0$ .

Hence  $N > \left( \frac{PU_0}{2} \right)^2 / 2PU_0 > \frac{PU_0}{8}$ .

**Lemma 27 :**

Let  $f$  be 0, 1, 2, or 3. Then the number of integers upto  $X$ , representable as a sum of three biquadrates belonging to  $f \pmod{16}$  is atleast  $X^{0.603}$  provided  $X > 10^{236}$ .

**Proof :**

We choose  $l = 2$ ;  $\mu = 0.20096$  and  $U_0 = P^{(3+\mu)}$  (0.45)

in lemma 26; since  $B^4 < 10^{12} P^{(3+\mu)}$  (0.235), we verify (iv) in lemma 26. This gives that the required estimate

$$> PU_0 > \frac{1}{8} \left( \frac{X}{17} \right)^{0.610108} > X^{0.603} .$$

**Lemma 28 :**

Let  $f$  be 0, 1, 2, 3 or 4. Then the number of integers upto  $X$ , representable as a sum of 4 biquadrates and belonging to  $f \pmod{16}$  is atleast  $X^{0.7095}$  provided  $X > 10^{307}$

**Proof :**

We choose  $l = 3$ ;  $\mu = 0.0869$ ;  $U_0 = P^{(3+\mu)}$  (0.603)

in lemma 26. We have to use  $B^4 < 10^{14} P^{(3+\mu)}$  (0.225)

This yields the result.

**Prop 6 :**

Let  $f$  be 0, 1, 2, 3, 4 or 5. Then the number of integers upto  $X$  representable as a sum of 5 biquadrates and belonging to  $f \pmod{16}$  is atleast  $X^{0.7795}$  provided  $X > 10^{410}$

**Proof :**

We take  $l = 4$  ;  $\mu = 0.01133$ ; and  $U_0 = P^{(3+\mu)} (0.7095)$ .

We use  $B^4 \leq (1.5) \times 10^{23}$  . This gives the result.

**§ 9. Conclusion of the proof :**

**Lemma 30 :**

*If  $N \geq 10^{412}$  , then  $N$  can be written as a sum of at most twenty biquadrates.*

**Proof :**

Choose  $f \in (0, 1, 2, 3, 4, 5)$  and

$l \in (2, 3, 4, 5, 6, 7)$  such that

$$N - 2f \equiv l \pmod{16}$$

$$P = [N^{\frac{1}{4}}]$$

Let  $\mathcal{U} = \left\{ n \leq \frac{N}{4} ; n \text{ is representable as a sum of} \right.$

five biquadrates :  $n \equiv f \pmod{16} \left. \right\}$

Let  $U(\mathcal{A}) = \sum_{u \in \mathcal{U}} e(u\mathcal{A})$

$$r(N) = \int_0^1 (T(\mathcal{A}))^{10} \cdot (u(\mathcal{A}))^2 e(-N\mathcal{A}) d\mathcal{A}$$

It suffices to prove that  $r(N) > 0$

$$\text{Now } r(N) = \int_m + \int_E$$

Now, using Proposition 4,

$$\begin{aligned} \int_m &= \sum_{u_1, u_2} \int_m (T(d))^{10} e^{-(N - u_1 - u_2)d} dd \\ &\geq \sum_{u_1, u_2} W_0(N - u_1 - u_2) \\ &> 0.01 U^2 P^6 \end{aligned}$$

on the other hand, using Proposition 5 and 6,

$$\begin{aligned} &\left| \int_m (T(d))^{10} (U(d))^2 e^{-Nd} dd \right| \\ &\leq \max_{d \in m} |T(d)|^{10} \int_0^1 |u(d)|^2 dd \\ &\leq (40)^{10} P^{\frac{71}{8}} \cdot (\log P)^{2.5} U \\ &\leq (40)^{10} P^{\frac{71}{8}} \cdot (\log P)^{2.5} U^2 / \left(\frac{N}{4}\right)^{0.7795} \\ &\leq 4^{11} 10^{10} P^{5.757} U^2 (\log P)^{2.5} \end{aligned}$$

It suffices to verify that, for  $P \leq 10^{102}$ ,

$$4^{11} 10^{10} P^{5.757} (\log P)^{2.5} < 0.01 P^6$$

and this is true.

**Lemma : 31**

*If  $N \leq 10^{412}$ , then  $N$  can be written as a sum of at most 20 biquadrates.*

**Proof :**

If  $N \leq 10^{412}$ , then  $N - [N^{\frac{1}{4}}]^4 < 10^{310}$  and hence representable as a sum of at most 19 biquadrates by Th. 3.3 of Thomas [7]

Thus the proof of the theorem is complete.

*Note :*

Some doubts have been raised about the correctness of lemma 31 and the author has not verified the calculations given in [7]. But even a weaker result, compared to lemma 31, will suffice for our purpose and we shall return to this subject at a later date.



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