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Sukumar Das Adhikari, R Balasubramanian, A Sankaranarayanan. An Ω -result related to $r_4(n)$. Hardy-Ramanujan Journal, 1989, Volume 12 - 1989, pp.29 - 30. 10.46298/hrj.1989.113. hal-01104372

HAL Id: hal-01104372

<https://hal.science/hal-01104372>

Submitted on 16 Jan 2015

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Hardy-Ramanujan Journal
Vol.12 (1989) 29-30

AN Ω -RESULT RELATED TO $r_4(n)$

SUKUMAR DAS ADHIKARI*, R. BALASUBRAMANIAN*

and

A. SANKARANARAYANAN**

§ 1. INTRODUCTION.

Let

$$g(n) = \sum_{d|n} h(d) \quad (1)$$

where h is a multiplicative function such that $\sum_{d=1}^{\infty} h(d)$ is convergent and $h(d) = O(\frac{1}{d})$. Let

$$M_0(x) = x \sum_{d=1}^{\infty} \frac{h(d)}{d}$$

$$M_1(x) = \frac{x^2}{2} \sum_{d=1}^{\infty} \frac{h(d)}{d}$$

$$R_0(x) = \sum_{n \leq x} g(n) - M_0(x)$$

and

$$R_1(x) = \sum_{n \leq x} ng(n) - M_1(x) \quad (2)$$

Let $r_k(n)$ denote the number of representations of the positive integer n as a sum of k squares. If we put $h(d) = \frac{\alpha(d)}{d}$ where

$$\begin{aligned}\alpha(d) &= -3 \text{ if } 4 \mid d \\ &= +1 \text{ if } 4 \nmid d, \text{ then we have} \\ g(n) &= \frac{r_4(n)}{8n}.\end{aligned}$$

(See page 205, Hua [5] for an equivalent expression).

Let $P_k(x)$ be the error term defined by

$$\sum_{n \leq x} r_k(n) = \frac{(\pi x)^{\frac{k}{2}}}{\Gamma(\frac{k+1}{2})} + P_k(x). \quad (3)$$

Szego [7] showed that, if $k \equiv 2, 3, 4 \pmod{8}$, then

$$P_k(x) = \Omega_-\left((x \log x)^{\frac{(k-1)}{4}}\right) \quad (4)$$

and if $k \equiv 6, 7, 8 \pmod{8}$, then

$$P_k(x) = \Omega_+\left((x \log x)^{\frac{(k-1)}{4}}\right) \quad (5)$$

For the particular case $k = 2$, the result had been proved by Hardy [4] and the best Ω_- result to date is due to Hafner [3] which is

$$P_2(x) = \Omega_-\left((x \log x)^{\frac{1}{4}} (\log \log x)^{\frac{\log 2}{4}} \exp(-B(\log \log \log x)^{\frac{1}{2}})\right) \quad (6)$$

The best Ω_+ result to date is due to Corradi and Katai [1], namely

$$P_2(x) = \Omega_+\left(x^{\frac{1}{4}} \exp(c(\log \log x)^{\frac{1}{4}} (\log \log \log x)^{-\frac{3}{4}})\right) \quad (7)$$

(for eg. see Grosswald page 21, [2]).

Our object of this paper is to consider the case $k = 4$ and we prove the following theorems.

THEOREM 1. *We have*

$$P_4(x) = \Omega_+(x \log \log x).$$

REMARK 1. Our treatment is inspired by a paper of Montgomery [6]. We also observe that an elementary proof of a theorem of Montgomery (Theorem

1 of [6]) follows from our treatment which we state as

THEOREM 2. If $g(n) = \frac{\varphi(n)}{n}$ where $\varphi(n)$ is the Euler's totient function, then we have

$$\frac{R_1(x)}{x} - R_0(x) \ll \exp(-c\sqrt{\log x})$$

for some $c > 0$.

REMARK 2. We feel that the treatment can be applied to some other interesting arithmetic functions. We intend to take them up in a further work.

§ 2. NOTATION.

- 1) $\{x\}$ denotes the fractional part of x .
- 2) $[x]$ denotes the integral part of x
- 3) (a, b) denotes the greatest common divisor of a and b .
- 4) $f(x) = O(g(x))$ or $f(x) \ll g(x)$ denotes that there exists a positive constant A such that $|f(x)| < A g(x)$, where $g(x)$ is real.

§ 3. SOME LEMMAS.

LEMMA 3.1. We have

$$R_G(x) = -x \sum_{d>x} \frac{h(d)}{d} - \sum_{d \leq x} h(d) \left\{ \frac{x}{d} \right\}$$

PROOF. We have,

$$\begin{aligned} R_0(x) &= \sum_{n \leq x} g(n) - M_0(x) \\ &= \sum_{n \leq x} \sum_{d|n} h(d) - M_0(x) \\ &= \sum_{d \leq x} h(d) \left(\frac{x}{d} - \left\{ \frac{x}{d} \right\} \right) - M_0(x) \\ &= -x \sum_{d>x} \frac{h(d)}{d} - \sum_{d \leq x} h(d) \left\{ \frac{x}{d} \right\} \\ &\quad \text{(by the definition of } M_0(x)) \end{aligned}$$

which proves the lemma.

LEMMA 3.2. *If $b, r (> 0)$ are integers such that $(b, r) = 1$ and β is a real number, then we have*

$$\sum_{n=1}^r \left\{ \frac{bn}{r} + \beta \right\} = \frac{r-1}{2} + \{r\beta\} \quad (3.2.1)$$

PROOF. We note that both sides are periodic in β with period $\frac{1}{r}$. So we assume that $0 \leq \beta < \frac{1}{r}$. We can also assume that $b = 1$. If $\beta = 0$, then the left hand side of (3.2.1) = $\sum_{n=1}^{r-1} \frac{n}{r} = \frac{r-1}{2}$. If $0 < \beta < \frac{1}{r}$, then we have

$$\begin{aligned} \sum_{n=1}^r \left\{ \frac{n}{r} + \beta \right\} &= \frac{1}{r} \cdot \frac{r(r-1)}{2} + r\beta \\ &= \frac{r-1}{2} + \{r\beta\} \end{aligned}$$

which proves the lemma.

LEMMA 3.3. *With notation as in Lemma 3.2, for any integer N , we have*

$$\sum_{n=1}^N \left\{ \frac{nb}{r} + \beta \right\} = \frac{N}{r} \{r\beta\} + \frac{N}{2} \left(\frac{r-1}{r} \right) + O(r).$$

PROOF. We have $N = Qr + R$ for some $0 \leq R < r$. Therefore from Lemma 3.2, we have

$$\begin{aligned} \sum_{n=1}^N \left\{ \frac{nb}{r} + \beta \right\} &= Q \{r\beta\} + Q \left(\frac{r-1}{2} \right) + \sum_{n=1}^R \left\{ \frac{nb}{r} + \beta \right\} \\ &= \frac{N}{r} \{r\beta\} + \frac{N(r-1)}{2r} + O(r). \end{aligned}$$

which proves the lemma.

LEMMA 3.4. *We have*

$$\frac{R_1(x)}{x} - R_0(x) = \frac{x}{2} \sum_{d>x} \frac{h(d)}{d} + \frac{1}{2} \sum_{d \leq x} h(d) - \frac{1}{2x} \sum_{d \leq x} h(d) \cdot d \left(\left\{ \frac{x}{d} \right\} - \left\{ \frac{x}{d} \right\}^2 \right).$$

PROOF. We have

$$\begin{aligned}
R_1(x) &= \sum_{n \leq x} ng(n) - M_1(x) \\
&= \sum_{\substack{n \leq x \\ d_1 d_2 \leq x}} h(d_1) \cdot d_1 d_2 - M_1(x) \\
&= \sum_{d_1 \leq x} d_1 h(d_1) \cdot \sum_{d_2 \leq \lfloor \frac{x}{d_1} \rfloor} d_2 - M_1(x) \\
&= \frac{1}{2} \sum_{d_1 \leq x} h(d_1) \cdot d_1 \left(\frac{x}{d_1} - \left\{ \frac{x}{d_1} \right\} \right) \left(\frac{x}{d_1} + 1 - \left\{ \frac{x}{d_1} \right\} \right) - M_1(x) \\
&= \frac{1}{2} \sum_{d \leq x} h(d) \cdot d \left(\frac{x^2}{d^2} + \frac{x}{d} - 2 \frac{x}{d} \left\{ \frac{x}{d} \right\} - \left\{ \frac{x}{d} \right\} + \left\{ \frac{x}{d} \right\}^2 \right) - M_1(x) \\
&= -\frac{x^2}{2} \sum_{\substack{d > x \\ d \leq x}} \frac{h(d)}{d} + \frac{x}{2} \sum_{\substack{d \leq x \\ d \leq x}} h(d) - x \sum_{\substack{d \leq x \\ d \leq x}} h(d) \left\{ \frac{x}{d} \right\} \\
&\quad - \frac{1}{2} \sum_{d \leq x} h(d) \cdot d \left\{ \frac{x}{d} \right\} + \frac{1}{2} \sum_{d \leq x} h(d) d \left\{ \frac{x}{d} \right\}^2.
\end{aligned}$$

From Lemma 3.1, we have

$$-\sum_{d \leq x} h(d) \left\{ \frac{x}{d} \right\} = R_0(x) + x \sum_{d > x} \frac{h(d)}{d}$$

and so we have

$$\begin{aligned}
\frac{R_1(x)}{x} - R_0(x) &= \frac{x}{2} \sum_{d > x} \frac{h(d)}{d} + \frac{1}{2} \sum_{d \leq x} h(d) \\
&\quad - \frac{1}{2x} \sum_{d \leq x} h(d) d \left(\left\{ \frac{x}{d} \right\} - \left\{ \frac{x}{d} \right\}^2 \right)
\end{aligned}$$

which completes the proof.

LEMMA 3.5. *If there exists a function $G(x)$ such that $G(x)$ and $\frac{x}{G(x)}$ are increasing functions of x , then we have*

$$R_0(x) = -\sum_{d \leq y} h(d) \left\{ \frac{x}{d} \right\} + O(1) \text{ for } y \geq \frac{x}{G(x)}.$$

PROOF. Since $\sum_{d > y} h(d) \left\{ \frac{x}{d} \right\} = O(1)$ for $y \geq \frac{x}{G(x)}$, the lemma follows from Lemma 3.1.

LEMMA 3.6. *For integers $q \approx G(N)$ with $G(N)$ as in Lemma 3.5,*

$\beta = q$ and $y = \frac{(N+1)q}{G(N)}$, we have

$$\sum_{n=1}^N R_0(nq + \beta) = N \sum_{\substack{e \leq y \\ p|e \Rightarrow p|q}} h(e) \frac{(e, q)}{e} \left(\frac{1}{2} - \left\{ \frac{\beta}{(e, q)} \right\} \right) \sum_{\substack{f \leq \frac{y}{e} \\ (f, q) = 1}} \frac{h(f)}{f} + O(N)$$

PROOF. We have

$$\begin{aligned} & \sum_{n=1}^N R_0(nq + \beta) \\ &= - \sum_{n=1}^N \sum_{d \leq y} h(d) \left\{ \frac{nq + \beta}{d} \right\} + O(N) \\ & \quad (\text{Since } y = \frac{(N+1)q}{G(N)} > \frac{nq + \beta}{G(nq)}) \\ &= - \sum_{d \leq y} h(d) \sum_{n=1}^N \left\{ \frac{nq + \beta}{d} \right\} + O(N) \\ &= - \sum_{d \leq y} h(d) \cdot \sum_{n=1}^N \left\{ \frac{q/(d, q)}{d/(d, q)} \cdot n + \beta \right\} + O(N) \\ &= - \sum_{d \leq y} h(d) \left(\frac{N}{d} (d, q) \left\{ \frac{\beta}{(d, q)} \right\} + \frac{N(d, q)}{d} \left(\frac{d}{2(d, q)} - 1 \right) + O\left(\frac{d}{(d, q)} \right) \right) + O(N) \\ & \quad (\text{by Lemma 3.3}) \\ &= - \sum_{d \leq y} h(d) \frac{N(d, q)}{d} \left(\left\{ \frac{\beta}{(d, q)} \right\} - \frac{1}{2} \right) + O(N) \\ &= -N \sum_{\substack{e \leq y \\ p|e \Rightarrow p|q}} h(e) \frac{(e, q)}{e} \left(\left\{ \frac{\beta}{(e, q)} \right\} - \frac{1}{2} \right) \sum_{\substack{f \leq \frac{y}{e} \\ (f, q) = 1}} \frac{h(f)}{f} + O(N) \\ & \quad (\text{by writing } d = ef \text{ where } p | e \Rightarrow p | q \text{ and } (f, q) = 1) \end{aligned}$$

which proves the lemma.

LEMMA 3.7. We have

$$\sum_{n \leq x} \frac{\alpha(n)}{n} = 2 \log 2 + O\left(\frac{1}{x}\right)$$

where $\alpha(n)$ is as defined in § 1.

PROOF. We have

$$\begin{aligned} \sum_{n \leq x} \frac{\alpha(n)}{n} &= - \sum_{4n \leq x} \frac{3}{4n} + \sum_{n \leq x} \frac{1}{n} - \sum_{4n \leq x} \frac{1}{4n} \\ &= \sum_{n \leq x} \frac{1}{n} - \sum_{n \leq [\frac{x}{4}]} \frac{1}{n} \\ &= 2 \log 2 + O\left(\frac{1}{x}\right) \end{aligned}$$

which proves the lemma.

LEMMA 3.8. If $h(n) = \frac{\alpha(n)}{n}$ and so $g(n) = \frac{r_4(n)}{8n}$, then we have

$$\frac{R_1(x)}{x} - R_0(x) = O(1).$$

PROOF. Since $\sum_{d=1}^{\infty} \frac{\alpha(d)}{d}$ is convergent and $|\sum_{d \leq x} \alpha(d)| \leq 3$ for all x , the lemma follows from Lemmas 3.4 and 3.7.

LEMMA 3.9. If $h(n) = \frac{\alpha(n)}{n}$ and so $g(n) = \frac{r_4(n)}{n}$, then we have

$$R_0(x) = - \sum_{d \leq y} \frac{\alpha(d)}{d} \left\{ \frac{x}{d} \right\} + O(1)$$

uniformly for $x \geq 2, y \geq \sqrt{x}$.

PROOF. From Lemma 3.1, we have

$$\begin{aligned} R_0(x) &= -x \sum_{d > x} \frac{\alpha(d)}{d^2} - \sum_{d \leq x} \frac{\alpha(d)}{d} \left\{ \frac{x}{d} \right\} \\ &= - \sum_{d \leq x} \frac{\alpha(d)}{d} \left\{ \frac{x}{d} \right\} + O(1) \\ &\quad \left(\text{Since } \sum_{d > x} \frac{\alpha(d)}{d^2} = O\left(\frac{1}{x}\right) \right). \end{aligned}$$

So it is enough to show that

$$\sum_{y \leq d \leq x} \frac{\alpha(d)}{d} \left\{ \frac{x}{d} \right\} = O(1) \text{ for } \sqrt{x} \leq y \leq x.$$

We choose k such that $1 \leq k \leq \frac{x}{y}$ and in $\frac{x}{k+1} < d \leq \frac{x}{k}$, the function $\left\{ \frac{x}{d} \right\}$ is monotone. Therefore we have

$$\sum_{\frac{x}{k+1} < d \leq \frac{x}{k}} \frac{\alpha(d)}{d} \left\{ \frac{x}{d} \right\} = O\left(\frac{k}{x}\right) \text{ (by Lemma 3.7)}$$

Now summing up for k in $1 \leq k \leq \frac{x}{y}$, we get

$$\begin{aligned} \sum_{y \leq d \leq x} \frac{\alpha(d)}{d} \left\{ \frac{x}{d} \right\} &= O\left(\frac{1}{x} \sum_{1 \leq k \leq \frac{x}{y}} k\right) \\ &= O\left(\frac{1}{x} \cdot \frac{x^2}{y^2}\right) \\ &= O(1) \text{ (since } y \geq \sqrt{x}\text{)} \end{aligned}$$

which proves the lemma.

§ 4. PROOF OF THEOREM 1.

We take $h(n) = \frac{\alpha(n)}{n}$. Therefore from Lemma 3.6 and 3.9 we have

$$\sum_{n=1}^N R_0(nq + \beta) = N \sum_{\substack{e \leq y \\ p|e \Rightarrow p|q}} \frac{\alpha(e)}{e} \frac{(e, q)}{e} \left(\frac{1}{2} - \left\{ \frac{\beta}{(e, q)} \right\}\right) \sum_{\substack{f \leq y/e \\ (f, q)=1}} \frac{\alpha(f)}{f^2} + O(N) \quad (4.1)$$

for $q \simeq \sqrt{N}$, $\beta \leq q$ and $y = \frac{(N+1)q}{\sqrt{N}} (= O(N))$. Since

$$\sum_{f \geq \frac{1}{e}} \frac{\alpha(f)}{f^2} = O\left(\sum_{f \geq \frac{1}{e}} \frac{1}{f^2}\right) = O\left(\frac{e}{y}\right),$$

we have

$$\begin{aligned} &N \sum_{e \leq y} \frac{\alpha(e)}{e} \frac{(e, q)}{e} \left(\frac{1}{2} - \left\{ \frac{\beta}{(e, q)} \right\}\right) \sum_{f \geq \frac{1}{e}} \frac{\alpha(f)}{f^2} \\ &\ll N \sum_{e \leq y} \frac{|\alpha(e)|}{e} \frac{(e, q)}{e} \left| \frac{e}{y} \right| \\ &\ll \frac{N}{y} \sum_{e \leq y} \frac{(e, q)}{e} \\ &= O(N). \end{aligned} \quad (4.2)$$

Therefore from (4.1) and (4.2) we have

$$\sum_{n=1}^N R_0(nq + \beta) = N \left(\prod_{p|q} \left(1 + \frac{\alpha(p)}{p^2} + \frac{\alpha(p^2)}{p^4} + \dots\right) \right) \sum_{\substack{e \leq y \\ p|e \Rightarrow p|q}} \frac{\alpha(e)}{e} (e, q) \left(\frac{1}{2} - \left\{ \frac{\beta}{(e, q)} \right\}\right) + O(N) \quad (4.3)$$

Now, we assume $q = \prod_{p \leq z} p$, $\beta = \prod_{2 < p \leq z} p = \frac{q}{2}$ where $z = \left[\frac{1}{2} \log N\right]$. Hence

$q \simeq \sqrt{N}$. From (4.3), we have

$$\sum_{n=1}^N R_0(nq + \beta) = N \left(\prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{-1} \right) \zeta(2) \sum_{\substack{e \leq y \\ p|e \Rightarrow p|q}} \frac{\alpha(e)}{e^2} (e, q) \left(\frac{1}{2} - \left\{ \frac{\beta}{(e, q)} \right\}\right) + O(N). \quad (4.4)$$

We note that, if $2 \mid e$, then $\frac{1}{2} - \{\frac{\beta}{(e,q)}\} = 0$. If $2 \nmid e$, then $\alpha(e) = 1$ and $\frac{1}{2} - \{\frac{\beta}{(e,q)}\} = \frac{1}{2}$. Therefore we have,

$$\begin{aligned} \sum_{\substack{e \leq y \\ p|e \Rightarrow p|q}} \frac{\alpha(e)}{e^2} (e, q) \left(\frac{1}{2} - \left\{ \frac{\beta}{(e, q)} \right\} \right) &\geq \sum_{\substack{e \leq y \\ p|e \Rightarrow p|q, 2 \nmid e}} \frac{1}{e^2} (e, q) \cdot \frac{1}{2} \\ &\geq \frac{1}{2} \sum_{\substack{e|q \\ 2 \nmid e}} \frac{1}{e} \\ &\approx \log z \\ &\approx \log \log N \quad (4.5) \end{aligned}$$

From (4.4) and (4.5), we have

$$\sum_{n=1}^N R_0(nq + \beta) \geq N \zeta(2) \cdot \frac{1}{2} \log \log N + O(N)$$

which implies

$$R_0(x) = \Omega_+(\log \log x)$$

and hence from Lemma 3.8, we have

$$R_1(x) = \Omega_+(x \log \log x)$$

which completes the proof of Theorem 1.

§ 5. PROOF OF THEOREM 2.

We take $h(n) = \frac{\mu(n)}{n}$. From Lemma 3.4, we have

$$\frac{R_1(x)}{x} - R_0(x) = \frac{x}{2} \sum_{d > x} \frac{\mu(d)}{d^2} + \frac{1}{2} \sum_{d \leq x} \frac{\mu(d)}{d} - \frac{1}{2x} \sum_{d \leq x} \mu(d) \left(\left\{ \frac{x}{d} \right\} - \left\{ \frac{x}{d} \right\}^2 \right)$$

We consider the sum $\sum_{d \leq x} \mu(d) \left\{ \frac{x}{d} \right\}$. For the range $d \leq x e^{-c\sqrt{\log x}}$, we use trivial estimate and get

$$\sum_{d \leq x e^{-c\sqrt{\log x}}} \mu(d) \left\{ \frac{x}{d} \right\} \ll x e^{-c\sqrt{\log x}}$$

For $x \geq d \geq xe^{-c\sqrt{\log x}}$, we notice that $\{\frac{x}{d}\}$ is monotonic in $\frac{x}{k+1} < d \leq \frac{x}{k}$ for any k such that $1 \leq k \leq e^{c\sqrt{\log x}}$. Therefore we get

$$\begin{aligned} \sum_{\frac{x}{k+1} < d \leq \frac{x}{k}} \mu(d)\left\{\frac{x}{d}\right\} &\ll \frac{x}{k} \exp(-c\sqrt{\log \frac{x}{k}}) \\ &\ll \frac{x}{k} \exp(-c_1\sqrt{\log x}) \end{aligned}$$

Now summing up over all k 's in $1 \leq k \leq \exp(c\sqrt{\log x})$, we get

$$\sum_{x \exp(-c\sqrt{\log x}) \leq d \leq x} \mu(d)\left\{\frac{x}{d}\right\} \ll \exp(-c_2\sqrt{\log x}).$$

The sum $\sum_{d \leq x} \mu(d)\left\{\frac{x}{d}\right\}^2$ can be treated similarly. Other sums are easy to deal with. Hence the Theorem 2 follows.

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ADDRESS OF THE AUTHORS

* PROFESSOR R. BALASUBRAMANIAN AND MR. S.D. ADHIKARI

THE INSTITUTE OF MATHEMATICAL SCIENCES,
MADRAS - 600113
INDIA

** MR. A. SANKARANARAYANAN
SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
BOMBAY 400 005
INDIA