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## ON INFINITUDE OF PRIMES

By

S. SRINIVASAN

Let  $x_m$  ( $m \geq 1$ ) be an increasing sequence of positive integers satisfying

$$(1) \quad x_m \mid x_{m+1}, \quad (x_m, x_{m+1} / x_m) = 1.$$

This immediately implies the infinitude of primes. For example,  $a(n) = n^2 + n + 1$  satisfies  $a(n^2) = a(n) a(-n)$  and  $(a(n), a(-n)) = 1$  giving (1) with, in particular,  $x_m = a(2^{2^m})$ .

Further, for given prime  $p$ , taking  $x_m = 2^{p^m} - 1$  we see that (1) is fulfilled, because then if a prime  $q$  divide  $x_m$  one has

$$(2) \quad x_{m+1} / x_m = 1 + (x_m + 1) + \dots + (x_m + 1)^{p-1} \\ \equiv p \pmod{q};$$

i. e.,  $(x_m, x_{m+1} / x_m)$  divides  $p$ ; but  $x_m \equiv 1 \pmod{p}$ .

Now, from (1), it easily follows that  $q \equiv 1 \pmod{p^{m+1}}$  for every prime  $q$  dividing  $x_{m+1} / x_m$ ;

in particular we have infinitude of primes  $\equiv 1 \pmod{p^r}$  for any given prime  $p$  and  $r > 1$ .

Actually, on the same principle, one can prove the infinitude of primes  $\equiv 1 \pmod{k}$  for any given integer  $k (> 1)$ . In fact, we can prove the following theorem.

**Theorem.**

Let  $K (> 1)$ ,  $k (> 1)$  be given integers. Then, for infinitely many primes  $q$ , we have

$$(3) \quad e_K(q) \equiv 0 \pmod{k^{[c_k \log \log q]}}$$

with a certain  $c_k > 0$ , where  $e_K(q)$  denotes the exponent of  $K$  modulo  $q$ . In particular,

$$(3') \quad q \equiv 1 \pmod{k} \text{ for an infinity of primes } q.$$

**Proof.**

Let  $k = \prod_{i=1}^s p_i^{a_i}$  ( $a_i > 1$ ; primes  $p_1 < \dots < p_s$ ) and set, for  $r \geq 1$ ,  $n = k^r$ . Next, define  $d_i = np_i^{-ra_i}$  and, for  $m > 1$ ,

$$(4) \quad y_m = K_1^{n^m} - (-1)^k;$$

$$y'_m = l c_m (y_m, y_{m,1}, \dots, y_{m,s}),$$

where  $y_{m,i} = K_1^{d_i^{m+1} p_i^{ra_i}} - (-1)^k$ , and

$$K_1 = K^{\phi(k)}.$$

Observe that

$$(5) \quad (y_j, K_1) = 1; y'_m \mid y_{m+1}$$

(set  $y'_m y''_m = y_{m+1}$ ).

Now consider  $m' = n^m + n^{m+1} (p_1^{-rma_1} + \dots + p_s^{-rma_s})$   
 $\leq c n^{m+1}$  with  $c = \frac{11}{12}$ . (For  $n = 2, 3$ , check  $c > \frac{3}{4}$  suffices;

and for  $n > 4$ ,  $c > \frac{\pi^2}{6} - 1 + \frac{1}{4}$  suffices.) Hence we have

$$y''_m > \left( \frac{1}{2} K_1^{n^{m+1}} \right) / (2^{s+1} K_1^{m'}) > K_1^{-(s+2) + n^{m+1}/12}$$

Because  $s < n-1$ , we obtain

$$(5') \quad y'_m > K_1^n \quad (m > 5).$$

As with (2), we get

$$(6) \quad (y'_m, y''_m) = 2^B$$

for some  $B > 0$ .

Case i.

*k* odd. Note that  $y_j \not\equiv 0 \pmod{4}$ , and so by (5') there is an odd prime  $q$  dividing  $y''_m$ . Now (since  $(K, q) = 1$  by (5))

$e_K(q)$  divides  $2\phi(K)n^{m+1}$  but does not divide  $\phi(k)n^{m+1}$ .

Denoting by  $b_i$  the exact power of  $p_i$  in  $e_K(q)$ , suppose for some  $i$ ,  $b_i < a_i r$ . This would mean that  $e_K(q)$  divides

$2\phi(k) d_i^{m+1} p_i^{ra_i}$  but does not divide  $\phi(k) d_i^{m+1} p_i^{ra_i}$ ,

Consequently  $q \mid y_{m,i} \mid y'_m$  in contradiction to (6)

So,  $b_i > ra_i$  ( $1 < i < s$ ), i. e.,

$$(7') \quad k^r \mid e_K(q) \mid 2\phi(k)n^{m+1}; \quad m > 5.$$

Taking here  $m = 5$ , say we get  $q < K_1^{k^{1+6r}} < K^{k^{8r}}$  giving (3).

This completes the proof in this case (on letting  $r \rightarrow \infty$ .)

**Case (ii)**

*k* even. Now we proceed to determine  $\alpha_j$ , the exact power of 2 in  $y_j$ . If  $K$  is even, we have  $\alpha_j = 0$ . If  $K = 2^\alpha K_0 + 1$ ,  $K_0 = 2^\beta K' - 1$  with  $\alpha > 1$ ,  $\beta > 1$  and  $K'$  odd, we see that  $\alpha_j = A + \beta_1 + rj\beta_2$ , where  $\beta_1, \beta_2$  denote the exact power of 2 in  $\phi(k), k$  (respectively) and  $A = \alpha$  or  $A = \beta + 1$  according as  $\alpha \neq 1$ , or  $\alpha = 1$ . Thus the exact power of 2 in  $y_{m+1} / y_m$  is  $\alpha_{m+1} - \alpha_m = r\beta_2$ . Since  $r\beta_2 < n$  (trivially), we again conclude that  $y_m^r$  has an odd prime divisor  $q$ . Proceeding, as in (i), with this  $q$  we can conclude that

$$(7'') \quad k^r \mid e_K(q) \mid \phi(k) n^{m+1}; m > 5.$$

The proof is completed again as before (in (i)).

**Remarks.**

(i) Taking  $r = 1$  above, with  $K = 2$  say, we obtain that for any given  $k (> 1)$  there is a prime  $q \equiv 1 \pmod{k}$ , with  $q < 2^{k^7}$ .

(ii) For given  $K (> 1), k (> 1)$  denoting by  $Q_K(k)$  the set of primes  $q$  (constructed as in the above proof, with  $r > 1$ ), we can conclude from (7'), (7'') that  $Q_K(k_1)$  and  $Q_K(k_2)$  are disjoint if  $k_j$  has a prime factor not dividing  $k_i \phi(k_i)$ .

In particular,

$$(8) \quad Q_K(p) \cap Q_K(p') = \phi, \text{ primes } p \neq p'.$$

(iii) For given  $k (> 2)$ , we can prove also the infinitude of primes  $\not\equiv 1 \pmod{k}$  via sequences  $x_m$  satisfying (1). To this

end, we see easily that it suffices if further  $x_{m+1}/x_m \not\equiv 1 \pmod{k}$  for sufficiently large  $m$ . These conditions are fulfilled by the choice  $x_m = q^m - (-1)^q$ , where (for example)  $q = 2$ , if  $k$  is not a power of 2 and  $q = 3$ , otherwise. (More can be similarly proved; like  $q \equiv 1 \pmod{k}$  for some  $l^2 \not\equiv 1 \pmod{k}$ , if  $k \times 24$ . However, these will appear elsewhere.)

(iv) Also, we have from (7'), (7'') that

$$(9) \quad P(\mathfrak{o}_K(q)) = P(k)$$

holds for infinitely many primes  $q \equiv 1 \pmod{k^{\lfloor c \log \log q \rfloor}}$  where  $P(m)$  denotes the greatest prime divisor of  $m$ .

(v) Perhaps the remarks in the current article are at least anticipated, as suggested by Professor H. Halberstam pointing out Ex. 5\* on p. 59 of [1]. However, it may be noted that, writing  $f_n(x)$  for the polynomial  $f(x)$  in the above exercise (which is close to the second paragraph of this article), the present article treats, in contrast,  $x$  as *fixed* and  $n$  as *varying*.

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