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A TRIVIAL REMARK ON GOLDBACH CONJECTURE

BY
K. RAMACHANDRA

§ 1. **INTRODUCTION.** In [3] S. Srinivasan has proved the following conditional theorem. (We write $e(x) = \exp(2\pi iz)$).

THEOREM. *Let $B > 10$ be arbitrary. Suppose there is a function δ_B tending to zero as $B \rightarrow \infty$, such that*

$$\left| \int_{\mathcal{M}_3} S^2(\alpha) e(-N\alpha) d\alpha \right| < \delta_B \frac{N}{(\log N)^2} \left(\frac{N}{\varphi(N)} \right),$$

where $S(\alpha) = \sum_{p \leq X} e(p\alpha)$ (p : primes), $N = [X]$ and \mathcal{M}_3 is the part of the minor arcs corresponding to denominators q ,

$$Q(\log X)^{-2} < q < Q,$$

with $Q = X(\log X)^{-B}$. Then the number $r(N)$ of representations of N as a sum of two odd primes satisfies

$$r(N) = (1 + o(1)) S_0(N) T(N)$$

as $N \rightarrow \infty$, where $T(N) = \sum (\log m_1 \log m_2)^{-1}$ the summation being over integers $m_1 > 1, m_2 > 1$ and $m_1 + m_2 = N$, and

$$S_0(N) = \prod_{p|N} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p \nmid N} \left(1 + \frac{1}{p-1}\right).$$

The analogous statement holds good for the twin-primes conjecture also.

In this note we prove the theorem above with $Q(\log X)^{-2} < q < Q$ replaced by $Q(\log X)^{-1-\varepsilon} < q < Q$ where $\varepsilon > 0$ is any small constant. (In conversation with Srinivasan I came to know that he also knew this result. In fact it follows by his conditions on A, B, C . However my treatment is different.)

We now describe his basic and supplementary intervals. Let

$$L = \log X, Q = X L^{-B}, Q_1 = L^A, Q_0 = X L^{-(B+C)}$$

where the constants A, B, C satisfy

$$A > 3(B+C), B > 1, C > 1.$$

Let \mathcal{M}_1 , be the union of α -intervals defined by

$$\left| \alpha - \frac{h}{q} \right| < \frac{1}{Q}, 0 \leq h < q, (h, q) = 1, \text{ and } q \leq Q_1.$$

These intervals are disjoint and

$$\int_{\mathcal{M}_1} S^2(\alpha) e(-N\alpha) d\alpha = T(N) S_0(N) + O\left(\frac{1}{A} \frac{N}{Q(N)} \frac{N}{(\log N)^2}\right).$$

\mathcal{M}_2 is in union of the α -intervals defined by

$$\mathcal{M}_2 = \bigcup_{Q_1 \leq q \leq Q_0} \bigcup_{\substack{h=0 \\ (h,q)=1}}^{q-1} \mathcal{M}(h, q)$$

where $\mathcal{M}(h, q) = \left[\frac{h}{q} - (qQ)^{-1}, \frac{h}{q} + (qQ)^{-1}\right]$.

Let

$$I = [-Q^{-1}, 1 - Q^{-1}] \text{ and } \mathcal{M} = \bigcup_{1 \leq q \leq Q} \bigcup_{\substack{h=0 \\ (h,q)=1}}^{q-1} \mathcal{M}(h, q).$$

We know that $\mathcal{M} = I$ and we have to get an asymptotic formula for

$$\int_I S^2(\alpha) e(-N \alpha) d \alpha.$$

We put $\mathcal{M}_3 = I - (\mathcal{M}_1 \cup \mathcal{M}_2)$ and we get the result of Srinivasan. (To sum up, the basic intervals treatment of Srinivasan follows Prachar (see [2] p.182) and that of Gallagher (see [1], Lemma 7) and his supplementary intervals treatment involves a simple but new idea). We give in the next section our new treatment of supplementary intervals.

§ 2. NEW TREATMENT OF SUPPLEMENTARY INTERVALS.
In Lemmas 2 to 5 constants implied by the Vinogradov symbol \ll depend on the integer constant k .

LEMMA 1. *Let v and γ be any two real numbers and $f(x)$ a k times continuously differentiable function defined in $v \leq x \leq v + k\gamma$. Then*

$$\begin{aligned} & \int_0^\gamma \dots \int_0^\gamma f^k(v + u_1 + \dots + u_k) du_1 \dots du_k \\ &= f(v + k\gamma) - \binom{k}{1} f(v + (k-1)\gamma) + \dots + (-1)^k f(v). \end{aligned}$$

PROOF. The lemma follows by induction on k .

LEMMA 2. *Let Q_0, Q_1, Q and q be positive integers with*

$$Q_1 \leq q \leq Q_0 \leq Q.$$

Let a be an integer satisfying $1 \leq a \leq q, (a, q) = 1, \gamma$ a real number satisfying $|\gamma| \leq \frac{1}{q Q_0}, k$ a fixed natural number, $v = \frac{a}{q} + \beta$ with $0 \leq |\beta| \leq \frac{1}{q Q}$. Put $F(x) = \sum_{p \leq N} e(p x)$ where N is any natural number and $f(x) = |F(x)|^2$.

Then

$$f(v) \ll \frac{N^{k+2}}{(\log N)^2 (q Q_0)^k} + \sum_{j=1}^k f(v + j \gamma).$$

PROOF. The first term on the RHS comes from a trivial estimation of the multiple integral in Lemma 1 and the second term is obvious.

LEMMA 3. *We have,*

$$\begin{aligned} \frac{2f(v)}{q Q_0} &= \int_{|\gamma| \leq \frac{1}{q Q_0}} f(v) d\gamma \\ &\ll \frac{N^{k+2}}{(\log N)^2 (q Q_0)^{k+1}} + \int_{|w| \leq \frac{k+1}{q Q_0}} f\left(\frac{a}{q} + w\right) d w. \end{aligned}$$

PROOF. Follows from Lemma 2 on integration with respect to γ since for $1 \leq j \leq k$ we have

$$\int_{|\gamma| \leq \frac{1}{q Q_0}} f\left(\frac{a}{q} + \beta + j\gamma\right) d\gamma \ll \int_{|w| \leq \frac{k+1}{q Q_0}} f\left(\frac{a}{q} + w\right) d w$$

(on putting $\beta + j\gamma = w$).

LEMMA 4. *Any fixed interval*

$$\frac{a}{q} \pm \frac{k+1}{q Q_0} \quad (1 \leq a \leq q, (a, q) = 1, 1 \leq q \leq Q_0)$$

intersects at most $O(1)$ other intervals and so

$$\begin{aligned} &\sum_{q \leq Q_0} \sum_a \int_{|\beta| \leq \frac{1}{q}} \int_{|w| \leq \frac{k+1}{q Q_0}} f\left(\frac{a}{q} + w\right) d w \\ &\ll \left(\int_0^1 f(x) dx \right) \frac{Q_0}{Q} \ll \frac{N}{\log N} \left(\frac{Q_0}{Q} \right). \end{aligned}$$

PROOF. Follows by the remark that the innermost integral on the LHS is independent of β .

LEMMA 5. *We have*

$$\sum_{Q_1 \leq q \leq Q_0} \sum_a \int_{|\beta| \leq \frac{1}{q}} f\left(\frac{a}{q} + \beta\right) d\beta$$

$$\ll \frac{N^2}{Q(\log N)^2} \left(\frac{N}{Q_0 Q_1}\right)^k + \frac{N}{\log N} \left(\frac{Q_0}{Q}\right).$$

PROOF. By Lemma 3, we have,

$$\sum_a f\left(\frac{a}{q} + \beta\right) \ll \frac{N^{k+2} q Q_0}{(\log N)^2 (q Q_0)^{k+1}} + q Q_0 \sum_a \int_{|\gamma| \leq \frac{k+1}{q Q_0}} f\left(\frac{a}{q} + w\right) d w.$$

Integrating with respect to β in $|\beta| \leq \frac{1}{q Q}$ and summing over q in

$$Q_1 \leq q \leq Q_0$$

we obtain

$$\begin{aligned} & \sum_{Q_1 \leq q \leq Q_0} \sum_a \int_{|\beta| \leq \frac{1}{q Q}} f\left(\frac{a}{q} + \beta\right) d\beta \\ & \frac{N^{k+2}}{(\log N)^2} \sum_{q \geq Q_1} \frac{q Q_0}{(q Q_0)^{k+1}} \left(\frac{1}{q Q}\right) + \left(\frac{Q_0}{Q}\right) \left(\frac{N}{\log N}\right) \\ & \ll \frac{N^2}{Q(\log N)^2} \left(\frac{N}{Q_0 Q_1}\right)^k + \left(\frac{Q_0}{Q}\right) \left(\frac{N}{\log N}\right). \end{aligned}$$

This proves the lemma.

§ 3. REMARKS. The basic intervals are already investigated by S. Srinivasan [3]. The application of the results of §2 to estimation of a trivial portion of supplementary intervals is clear. For instance we can take

$$Q = \frac{N}{(\log N)^A}, Q_0 = N(\log N)^{-A-1-\epsilon}, Q_1 = (\log N)^{A+1+2\epsilon}, k \geq k_0(\epsilon, A).$$

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