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R Balasubramanian, K Ramachandra

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A LEMMA IN COMPLEX FUNCTION THEORY-II

BY

R. BALASUBRAMANIAN AND K. RAMACHANDRA

§1. **INTRODUCTION.** This is a continuation of [1] and [2]. However the method here is different and is self-contained. In [2] we proved a general result which implied the following

THEOREM 1. *Let $f(z)$ be analytic in $|z| \leq r$ and there let $|f(z)| \leq M, (M \geq 3)$. Let $A \geq 1$. Then*

$$|f(0)| \leq (24A \log M) \left(\frac{1}{2r} \int_{-r}^r |f(iy)| dy \right) + M^{-A}. \quad (1)$$

We also proved a corresponding result with $|f(x+iy)|$ in place of $|f(iy)|$, with suitable restrictions on x and also on the range of integration namely on y . These are statements about $|f(z)|$ where $f(z)$ is analytic. We now consider $|f(z)|^k$ where $k > 0$ is any real number independent of z . We prove

THEOREM 2. *Let k be any positive real number. Let $f(z)$ be analytic in $|z| \leq 2r$ and there $|f(z)|^k \leq M (M \geq 9)$. Let $x = r(\log M)^{-1}$, and let x_1 be any real number with $|x_1| \leq x$. Put $r_0 = \sqrt{4r^2 - x_1^2}$. Then with $A \geq 1$ we have*

$$|f(0)|^k \leq 2e^{84A} M^{-A} + \frac{24}{(2\pi)^2} e^{84A} \log M \left(\frac{1}{2r_0} \int_{-r_0}^{r_0} |f(x_1+iy)|^k dy \right). \quad (2)$$

REMARK 1. It is easy to remember a somewhat crude result namely

$$|f(0)|^k \leq e^{90A} \{M^{-A} + (\log M) \left(\frac{1}{2r_0} \int_{-r_0}^{r_0} |f(x_1 + iy)|^k dy \right)\}. \quad (2')$$

REMARK 2. In Theorem 1 the constants are reasonably small whereas in Theorem 2 they are big. We have not attempted to get optimal constants.

REMARK 3. Let k_1, k_2, \dots, k_m be any set of positive real numbers. Let $f_1(z), f_2(z), \dots, f_m(z)$ be analytic in $|z| \leq 2r$, and there

$$|(f_1(z))^{k_1} \dots (f_m(z))^{k_m}| \leq M (M \geq 9).$$

Then Theorem 2 holds good with $|f(z)|^k$ replaced by $|(f_1(z))^{k_1} \dots (f_m(z))^{k_m}|$.

REMARK 4. A corollary to our result mentioned in Remark 3 was pointed out to us by Professor J.P. Demailly. It is this : Theorem 2 holds good with $|f(z)|^k$ replaced by $\text{Exp}(u)$ where u is any subharmonic function. To prove

this it suffices to note that the set of functions of the form $\sum_{j=1}^m k_j \log |f_j(z)|$ is dense in L_{loc}^1 in the set of subharmonic functions. (This follows by using Green-Riesz representation formula for u and approximating the measure Δ_u by finite sums of Dirac measures).

REMARK 5. Consider $k = 1$ in Theorem 2. Put $\varphi(z) = f^{(\ell)}(z)$ the ℓ th derivative of $f(z)$. Then our method of proof gives

$$|\varphi(0)| \leq CM^{-A} + C(\log M)^{\ell+1} \left(\frac{1}{4r} \int_{-4r}^{4r} |f(iy)| dy \right),$$

where C depends only on A and ℓ .

REMARK 6. (Due to J.-P. Demailly). In view of the example $f(z) = \left(\frac{e^{nz}-1}{nz}\right)^2$, where n is a large positive integer and $r = 1$, the result of Remark 5 is best possible.

§ 2. PROOF OF THEOREM 2. The proof consists of four steps.

STEP 1. First we consider the circle $|z| = r$. Let

$$0 < 2x \leq r \quad (3)$$

and let PQS denote respectively the points $re^{i\theta}$ where $\theta = -\cos^{-1}\left(\frac{2x}{r}\right), \cos^{-1}\left(\frac{2x}{r}\right)$ and π . By the consideration of Riemann mapping theorem and the zero cancellation factors we have for a suitable meromorphic function $\phi(z)$ (in PQSP)

that (we can assume that $f(z)$ has no zeros on the boundary)

$$F(z) = (\phi(z)f(z))^k \quad (4)$$

is analytic in the region enclosed by the straight line PQ and the circular arc QSP. Further $\phi(z)$ satisfies

$$|\phi(z)| = 1 \quad (5)$$

on the boundary of PQSP and also

$$|\phi(0)| \geq 1. \quad (6)$$

Let

$$X = \text{Exp}(u_1 + u_2 + \dots + u_n) \quad (7)$$

where u_1, u_2, \dots, u_n vary over the box \mathcal{B} defined by

$$0 \leq u_j \leq B (j = 1, 2, \dots, n),$$

and $B > 0$.

We begin with

LEMMA 1. *The function $F(z)$ defined above satisfies*

$$F(0) = I_1 + I_2 \quad (8)$$

where

$$I_1 = \frac{1}{2\pi i} \int_{PQ} F(z) X^z \frac{dz}{z} \quad (9)$$

and

$$I_2 = \frac{1}{2\pi i} \int_{QSP} F(z) X^z \frac{dz}{z} \quad (10)$$

where the lines of integration are the straight line PQ and the circular arc QSP.

PROOF. Follows by Cauchy's theorem.

LEMMA 2. *We have*

$$|I_1| \leq \frac{e^{2Bnx}}{2\pi} \int_{PQ} |(f(z))^k \frac{dz}{z}| \quad (11)$$

PROOF. Follows since $|X^z| \leq e^{2Bnx}$ and also $|\phi(z)| = 1$ on PQ.

LEMMA 3. We have,

$$| B^{-n} \int_{\mathbf{B}} I_2 du_1 \dots du_n | \leq e^{2Bnx} \left(\frac{2}{Br} \right)^n M. \quad (12)$$

PROOF. Follows since on QSP we have $|\phi(z)| = 1$ (and so $|F(z)| \leq M$) and also

$$| B^{-n} \int_{\mathbf{B}} \left(\int_{QSP} X^z \frac{dz}{2\pi iz} \right) du_1 \dots du_n | \leq \left(\frac{2}{Br} \right)^n.$$

LEMMA 4. We have,

$$| f(0) |^k \leq e^{2Bnx} \left(\frac{2}{Br} \right)^n M + \frac{e^{2Bnx}}{2\pi} \int_{PQ} | (f(z))^k \frac{dz}{z} |. \quad (13)$$

PROOF. Follows by Lemmas 1,2 and 3.

STEP 2. Next in (13), we replace $|f(z)|^k$ by an integral over a chord P_1Q_1 (parallel to PQ) of $|w| = 2r$, of slightly bigger length with a similar error. Let x_1 be any real number with

$$|x_1| \leq x. \quad (14)$$

$$\left\{ \begin{array}{l} \text{Let } P_1Q_1R_1 \text{ be the points } 2re^{i\theta} \\ \text{where } \theta = -\cos^{-1}\left(\frac{x_1}{2r}\right), 0 \text{ and } \cos^{-1}\left(\frac{x_1}{2r}\right). \\ \text{(If } x_1 \text{ is negative we have to consider the points} \\ \theta = -\frac{\pi}{2} - \sin^{-1}\left(\frac{x_1}{2r}\right), 0 \text{ and } \frac{\pi}{2} + \sin^{-1}\left(\frac{x_1}{2r}\right)). \end{array} \right. \quad (15)$$

Let X be as in (7). As before let

$$G(w) = (\psi(w)f(w))^k \quad (16)$$

be analytic in the region enclosed by the circular arc $P_1R_1Q_1$ and the straight line Q_1P_1 (we can assume that $f(z)$ has no zeros on the boundary $P_1R_1Q_1P_1$). By the consideration of Riemann mapping theorem and the zero cancelling factors there exists such a meromorphic function $\psi(w)$ (in $P_1R_1Q_1P_1$) with the extra properties,

$$|\psi(w)| = 1 \text{ on the boundary of } P_1R_1Q_1P_1 \text{ and } |\psi(z)| \geq 1. \quad (17)$$

LEMMA 5. we have with z on PQ,

$$G(z) = I_3 + I_4$$

where

$$I_3 = \frac{1}{2\pi i} \int_{Q_1 P_1} G(w) X^{-(w-z)} \frac{dw}{w-z} \quad (18)$$

and

$$I_4 = \frac{1}{2\pi i} \int_{P_1 R_1 Q_1} G(w) X^{-(w-z)} \frac{dw}{w-z} \quad (19)$$

PROOF. Follows by Cauchy's theorem

LEMMA 6. We have with z on PQ

$$|I_3| \leq \frac{e^{3Bnz}}{2\pi} \int_{P_1 Q_1} |(f(w))^k| \frac{dw}{w-z} \quad (20)$$

PROOF. Follows since $|X^{-(w-z)}| \leq e^{3Bnz}$ and $|\psi(w)| = 1$ on $P_1 Q_1$.

LEMMA 7. We have with z on PQ,

$$|B^{-n} \int_{\mathbf{B}} I_4 du_1 \dots du_n| \leq e^{3Bnz} \left(\frac{2}{Br}\right)^n M. \quad (21)$$

PROOF. Follows since on $P_1 R_1 Q_1$ we have $|\psi(w)| = 1$ (and so $|G(w)| \leq M$) and also

$$|B^{-n} \int_{\mathbf{B}} \int x^{-(w-z)} \frac{dw}{2\pi i(w-z)} du_1 \dots du_n| \leq \left(\frac{2}{Br}\right)^n.$$

LEMMA 8. We have with z on PQ,

$$|f(z)|^k \leq e^{3Bnz} \left(\frac{2}{Br}\right)^n M + \frac{e^{3Bnz}}{2\pi} \int_{P_1 Q_1} |f(w)|^k \frac{dw}{w-z}. \quad (22)$$

PROOF. Follows from Lemmas 5,6 and 7.

STEP 3. We now combine Lemmas 4 and 8.

LEMMA 9. We have

$$|f(0)|^k \leq e^{2Bnz} \left(\frac{2}{Br}\right)^n M + J_1 + J_2 \quad (23)$$

where

$$J_1 = \frac{e^{5Bnz}}{2\pi} \left(\frac{2}{Br}\right)^n M \int_{PQ} \left| \frac{dz}{z} \right|, \quad (24)$$

and

$$J_2 = \frac{e^{5Bnz}}{(2\pi)^2} \int_{P_1 Q_1} |f(w)|^k \left(\int_{PQ} \left| \frac{dz}{z(w-z)} \right| \right) |dw| \quad (25)$$

LEMMA 10. *We have*

$$\int_{PQ} \left| \frac{dz}{z} \right| \leq 2 + 2 \log\left(\frac{r}{2x}\right). \quad (26)$$

PROOF. On PQ we have $z = 2x + iy$ with $|y| \leq r$ and $2x \leq r$. We split the integral into $|y| \leq 2x$ and $2x \leq |y| \leq r$. On these, we use respectively the lower bounds $|z| \geq 2x$ and $|z| \geq y$. The lemma follows by these observations.

LEMMA 11. *We have for w on P_1Q_1 and z on PQ,*

$$\int_{PQ} \left| \frac{dz}{z(w-z)} \right| \leq \frac{6}{x}. \quad (27)$$

PROOF. On PQ we have $\operatorname{Re} z = 2x$ and on P_1Q_1 we have $|\operatorname{Re} w| \leq x$ and so $|\operatorname{Re}(w-z)| \geq x$. We have

$$\left| \frac{dz}{z(w-z)} \right| \leq \left| \frac{dz}{z^2} \right| + \left| \frac{dz}{(w-z)^2} \right|.$$

Writing $z = 2x + iy$ we have

$$\begin{aligned} \int_{PQ} \left| \frac{dz}{z^2} \right| &\leq \frac{2}{(2x)^2} 2x + 2 \int_{2x}^{\infty} \frac{dy}{y^2} \\ &= \frac{2}{x}. \end{aligned}$$

Similarly

$$\begin{aligned} \int_{PQ} \left| \frac{dz}{(w-z)^2} \right| &\leq 2\left(\frac{1}{x} + \int_x^{\infty} \frac{dy}{y^2}\right) \\ &= \frac{4}{x}. \end{aligned}$$

This completes the proof of the lemma.

STEP 4. We collect together the results in Steps 3 and 4 and choose the parameters B and n and this will give Theorem 2. Combining Lemmas 9,10 and 11 we state the following lemma.

LEMMA 12. *We have*

$$\begin{aligned} |f(0)|^k &\leq e^{2Bnz} \left(\frac{2}{Br}\right)^n M + \frac{e^{5Bnz}}{\pi} \left(\frac{2}{Br}\right)^n (1 + \log \frac{r}{2x}) M \\ &\quad + \frac{e^{5Bnz}}{(2\pi)^2} \cdot \frac{6}{x} \int_{P_1Q_1} |(f(w))^k dw|, \end{aligned} \quad (28)$$

where $0 < 2x \leq r$, x_1 is any real number with $|x_1| \leq x$, n any natural number and B is any positive real number and P_1Q_1 is the straight line joining $-r_0$ and r_0 where $r_0 = \sqrt{4r^2 - x_1^2}$.

Next we note that $1 + \log \frac{r}{2x} \leq \frac{r}{2x}$ and so by putting $x = r(\log M)^{-1}$ the first two terms on the RHS of (28) together do not exceed

$$\left(\frac{2}{Br}\right)^n e^{5Bnx} \left(1 + \frac{1}{2\pi} \log M\right) M \leq 2\left(\frac{2}{Br}\right)^n e^{5Bnx} M \log M.$$

Also,

$$\frac{6}{x} = \frac{6 \log M}{r} = 6 \log M \left(\frac{2r_0}{r}\right) \frac{1}{2r_0} \leq (24 \log M) \left(\frac{1}{2r_0}\right).$$

Thus RHS of (28) does not exceed

$$2\left(\frac{2}{Br}\right)^n e^{5Bnx} M \log M + \left(\frac{24}{(2\pi)^2} e^{5Bnx} \log M\right) \left(\frac{1}{2r_0} \int_{P_1Q_1} |(f(w))^k dw|\right).$$

We have chosen $x = r(\log M)^{-1}$. We now choose B such that $Br = 2e$ and $n = [C \log M] + 1$, where $C \geq 1$ is any real number. We have $5Bnx \leq \frac{5Bnr}{\log M} \leq 10e(C+1) \leq 28(C+1)$ and also

$$\left(\frac{2}{Br}\right)^n \leq e^{-C \log M} = M^{-C}.$$

With these choices of x, B, n we see that RHS of (28) does not exceed

$$2M^{-C} e^{28(C+1)} M \log M + \left(\frac{24}{(2\pi)^2} e^{28(C+1)} \log M\right) \left(\frac{1}{2r_0} \int_{P_1Q_1} |f(w)|^k dw\right).$$

Putting $C = A + 2$ we obtain Theorem 2 since $C + 1 \leq 3A$. This completes the proof of Theorem 2.

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ADDRESS OF THE AUTHORS

1) PROFESSOR R. BALASUBRAMANIAN,
MATSCIENCE, (ON DEPUTATION FROM T.I.F.R.)
THARAMANI P.O.,
MADRAS - 600 113
INDIA

2) PROFESSOR K. RAMACHANDRA,
SCHOOL OF MATHEMATICS,
TATA INSTITUTE OF FUNDAMENTAL RESEARCH,
HOMI BHABHA ROAD,
BOMBAY - 400 005
INDIA

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