



**HAL**  
open science

## A Lemma in complex function theory II

R Balasubramanian, K Ramachandra

► **To cite this version:**

R Balasubramanian, K Ramachandra. A Lemma in complex function theory II. Hardy-Ramanujan Journal, 1989, Volume 12 - 1989, pp.6 - 13. 10.46298/hrj.1989.109 . hal-01104341

**HAL Id: hal-01104341**

**<https://hal.science/hal-01104341v1>**

Submitted on 16 Jan 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Hardy-Ramanujan Journal  
Vol.12 (1989) 6-13

## A LEMMA IN COMPLEX FUNCTION THEORY-II

BY

R. BALASUBRAMANIAN AND K. RAMACHANDRA

§1. INTRODUCTION. This is a continuation of [1] and [2]. However the method here is different and is self-contained. In [2] we proved a general result which implied the following

**THEOREM 1.** *Let  $f(z)$  be analytic in  $|z| \leq r$  and there let  $|f(z)| \leq M$ , ( $M \geq 3$ ). Let  $A \geq 1$ . Then*

$$|f(0)| \leq (24A \log M) \left( \frac{1}{2r} \int_{-r}^r |f(iy)| dy \right) + M^{-A}. \quad (1)$$

We also proved a corresponding result with  $|f(x+iy)|$  in place of  $|f(iy)|$ , with suitable restrictions on  $x$  and also on the range of integration namely on  $y$ . These are statements about  $|f(z)|$  where  $f(z)$  is analytic. We now consider  $|f(z)|^k$  where  $k > 0$  is any real number independent of  $z$ . We prove

**THEOREM 2.** *Let  $k$  be any positive real number. Let  $f(z)$  be analytic in  $|z| \leq 2r$  and there  $|f(z)|^k \leq M$  ( $M \geq 9$ ). Let  $x = r(\log M)^{-1}$ , and let  $x_1$  be any real number with  $|x_1| \leq x$ . Put  $r_0 = \sqrt{4r^2 - x_1^2}$ . Then with  $A \geq 1$  we have*

$$|f(0)|^k \leq 2e^{84A} M^{-A} + \frac{24}{(2\pi)^2} e^{84A} \log M \left( \frac{1}{2r_0} \int_{-r_0}^{r_0} |f(x_1+iy)|^k dy \right). \quad (2)$$

**REMARK 1.** It is easy to remember a somewhat crude result namely

$$|f(0)|^k \leq e^{90A} \{M^{-A} + (\log M) \left( \frac{1}{2r_0} \int_{-r_0}^{r_0} |f(x_1 + iy)|^k dy \right)\}. \quad (2')$$

**REMARK 2.** In Theorem 1 the constants are reasonably small whereas in Theorem 2 they are big. We have not attempted to get optimal constants.

**REMARK 3.** Let  $k_1, k_2, \dots, k_m$  be any set of positive real numbers. Let  $f_1(z), f_2(z), \dots, f_m(z)$  be analytic in  $|z| \leq 2r$ , and there

$$|(f_1(z))^{k_1} \dots (f_m(z))^{k_m}| \leq M (M \geq 9).$$

Then Theorem 2 holds good with  $|f(z)|^k$  replaced by  $|(f_1(z))^{k_1} \dots (f_m(z))^{k_m}|$ .

**REMARK 4.** A corollary to our result mentioned in Remark 3 was pointed out to us by Professor J.P. Demailly. It is this : Theorem 2 holds good with  $|f(z)|^k$  replaced by  $Exp(u)$  where  $u$  is any subharmonic function. To prove

this it suffices to note that the set of functions of the form  $\sum_{j=1}^m k_j \log |f_j(z)|$

is dense in  $L^1_{loc}$  in the set of subharmonic functions. (This follows by using Green-Riesz representation formula for  $u$  and approximating the measure  $\Delta_u$  by finite sums of Dirac measures).

**REMARK 5.** Consider  $k = 1$  in Theorem 2. Put  $\varphi(z) = f^{(\ell)}(z)$  the  $\ell$ th derivative of  $f(z)$ . Then our method of proof gives

$$|\varphi(0)| \leq CM^{-A} + C(\log M)^{\ell+1} \left( \frac{1}{4r} \int_{-4r}^{4r} |f(iy)| dy \right),$$

where  $C$  depends only on  $A$  and  $\ell$ .

**REMARK 6.** (Due to J.-P. Demailly). In view of the example  $f(z) = \left(\frac{e^{nz}-1}{nz}\right)^2$ , where  $n$  is a large positive integer and  $r = 1$ , the result of Remark 5 is best possible.

**§ 2. PROOF OF THEOREM 2.** The proof consists of four steps.

**STEP 1.** First we consider the circle  $|z| = r$ . Let

$$0 < 2x \leq r \quad (3)$$

and let PQS denote respectively the points  $re^{i\theta}$  where  $\theta = -\cos^{-1}\left(\frac{2x}{r}\right), \cos^{-1}\left(\frac{2x}{r}\right)$  and  $\pi$ . By the consideration of Riemann mapping theorem and the zero cancellation factors we have for a suitable meromorphic function  $\phi(z)$  (in PQSP)

that (we can assume that  $f(z)$  has no zeros on the boundary)

$$F(z) = (\phi(z)f(z))^k \quad (4)$$

is analytic in the region enclosed by the straight line PQ and the circular arc QSP. Further  $\phi(z)$  satisfies

$$|\phi(z)| = 1 \quad (5)$$

on the boundary of PQSP and also

$$|\phi(0)| \geq 1. \quad (6)$$

Let

$$X = \text{Exp}(u_1 + u_2 + \dots + u_n) \quad (7)$$

where  $u_1, u_2, \dots, u_n$  vary over the box  $\mathcal{B}$  defined by

$$0 \leq u_j \leq B (j = 1, 2, \dots, n),$$

and  $B > 0$ .

We begin with

**LEMMA 1.** *The function  $F(z)$  defined above satisfies*

$$F(0) = I_1 + I_2 \quad (8)$$

where

$$I_1 = \frac{1}{2\pi i} \int_{PQ} F(z) X^z \frac{dz}{z} \quad (9)$$

and

$$I_2 = \frac{1}{2\pi i} \int_{QSP} F(z) X^z \frac{dz}{z} \quad (10)$$

where the lines of integration are the straight line PQ and the circular arc QSP.

**PROOF.** Follows by Cauchy's theorem.

**LEMMA 2.** *We have*

$$|I_1| \leq \frac{e^{2Bnx}}{2\pi} \int_{PQ} |(f(z))^k \frac{dz}{z}| \quad (11)$$

**PROOF.** Follows since  $|X^z| \leq e^{2Bnx}$  and also  $|\phi(z)| = 1$  on PQ.

**LEMMA 3.** We have,

$$\left| B^{-n} \int_{\mathbf{B}} I_2 du_1 \dots du_n \right| \leq e^{2Bnx} \left( \frac{2}{Br} \right)^n M. \quad (12)$$

**PROOF.** Follows since on QSP we have  $|\phi(z)| = 1$  (and so  $|F(z)| \leq M$ ) and also

$$\left| B^{-n} \int_{\mathbf{B}} \left( \int_{QSP} X^z \frac{dz}{2\pi iz} \right) du_1 \dots du_n \right| \leq \left( \frac{2}{Br} \right)^n.$$

**LEMMA 4.** We have,

$$|f(0)|^k \leq e^{2Bnx} \left( \frac{2}{Br} \right)^n M + \frac{e^{2Bnx}}{2\pi} \int_{PQ} \left| (f(z))^k \frac{dz}{z} \right|. \quad (13)$$

**PROOF.** Follows by Lemmas 1,2 and 3.

**STEP 2.** Next in (13), we replace  $|f(z)|^k$  by an integral over a chord  $P_1Q_1$  (parallel to PQ) of  $|w| = 2r$ , of slightly bigger length with a similar error. Let  $x_1$  be any real number with

$$|x_1| \leq x. \quad (14)$$

$$\left\{ \begin{array}{l} \text{Let } P_1Q_1R_1 \text{ be the points } 2re^{i\theta} \\ \text{where } \theta = -\cos^{-1}\left(\frac{x_1}{2r}\right), 0 \text{ and } \cos^{-1}\left(\frac{x_1}{2r}\right). \\ \text{(If } x_1 \text{ is negative we have to consider the points} \\ \theta = -\frac{\pi}{2} - \sin^{-1}\left(\frac{x_1}{2r}\right), 0 \text{ and } \frac{\pi}{2} + \sin^{-1}\left(\frac{x_1}{2r}\right)). \end{array} \right. \quad (15)$$

Let  $X$  be as in (7). As before let

$$G(w) = (\psi(w)f(w))^k \quad (16)$$

be analytic in the region enclosed by the circular arc  $P_1R_1Q_1$  and the straight line  $Q_1P_1$  (we can assume that  $f(z)$  has no zeros on the boundary  $P_1R_1Q_1P_1$ ). By the consideration of Riemann mapping theorem and the zero cancelling factors there exists such a meromorphic function  $\psi(w)$  (in  $P_1R_1Q_1P_1$ ) with the extra properties,

$$|\psi(w)| = 1 \text{ on the boundary of } P_1R_1Q_1P_1 \text{ and } |\psi(z)| \geq 1. \quad (17)$$

**LEMMA 5.** we have with  $z$  on PQ,

$$G(z) = I_3 + I_4$$

where

$$I_3 = \frac{1}{2\pi i} \int_{Q_1 P_1} G(w) X^{-(w-z)} \frac{dw}{w-z} \quad (18)$$

and

$$I_4 = \frac{1}{2\pi i} \int_{P_1 R_1 Q_1} G(w) X^{-(w-z)} \frac{dw}{w-z} \quad (19)$$

**PROOF.** Follows by Cauchy's theorem

**LEMMA 6.** We have with  $z$  on PQ

$$|I_3| \leq \frac{e^{3Bnz}}{2\pi} \int_{P_1 Q_1} |(f(w))^k| \frac{dw}{w-z} \quad (20)$$

**PROOF.** Follows since  $|X^{-(w-z)}| \leq e^{3Bnz}$  and  $|\psi(w)| = 1$  on  $P_1 Q_1$ .

**LEMMA 7.** We have with  $z$  on PQ,

$$|B^{-n} \int_{\mathbf{B}} I_4 du_1 \dots du_n| \leq e^{3Bnz} \left(\frac{2}{Br}\right)^n M. \quad (21)$$

**PROOF.** Follows since on  $P_1 R_1 Q_1$  we have  $|\psi(w)| = 1$  (and so  $|G(w)| \leq M$ ) and also

$$|B^{-n} \int_{\mathbf{B}} \int x^{-(w-z)} \frac{dw}{2\pi i(w-z)} du_1 \dots du_n| \leq \left(\frac{2}{Br}\right)^n.$$

**LEMMA 8.** We have with  $z$  on PQ,

$$|f(z)|^k \leq e^{3Bnz} \left(\frac{2}{Br}\right)^n M + \frac{e^{3Bnz}}{2\pi} \int_{P_1 Q_1} |f(w)|^k \frac{dw}{w-z}. \quad (22)$$

**PROOF.** Follows from Lemmas 5,6 and 7.

**STEP 3.** We now combine Lemmas 4 and 8.

**LEMMA 9.** We have

$$|f(0)|^k \leq e^{2Bnz} \left(\frac{2}{Br}\right)^n M + J_1 + J_2 \quad (23)$$

where

$$J_1 = \frac{e^{5Bnz}}{2\pi} \left(\frac{2}{Br}\right)^n M \int_{PQ} \left| \frac{dz}{z} \right|, \quad (24)$$

and

$$J_2 = \frac{e^{5Bnz}}{(2\pi)^2} \int_{P_1 Q_1} |f(w)|^k \left( \int_{PQ} \left| \frac{dz}{z(w-z)} \right| \right) |dw| \quad (25)$$

**LEMMA 10.** *We have*

$$\int_{PQ} \left| \frac{dz}{z} \right| \leq 2 + 2 \log\left(\frac{r}{2x}\right). \quad (26)$$

**PROOF.** On PQ we have  $z = 2x + iy$  with  $|y| \leq r$  and  $2x \leq r$ . We split the integral into  $|y| \leq 2x$  and  $2x \leq |y| \leq r$ . On these, we use respectively the lower bounds  $|z| \geq 2x$  and  $|z| \geq y$ . The lemma follows by these observations.

**LEMMA 11.** *We have for  $w$  on  $P_1Q_1$  and  $z$  on PQ,*

$$\int_{PQ} \left| \frac{dz}{z(w-z)} \right| \leq \frac{6}{x}. \quad (27)$$

**PROOF.** On PQ we have  $\operatorname{Re} z = 2x$  and on  $P_1Q_1$  we have  $|\operatorname{Re} w| \leq x$  and so  $|\operatorname{Re}(w-z)| \geq x$ . We have

$$\left| \frac{dz}{z(w-z)} \right| \leq \left| \frac{dz}{z^2} \right| + \left| \frac{dz}{(w-z)^2} \right|.$$

Writing  $z = 2x + iy$  we have

$$\begin{aligned} \int_{PQ} \left| \frac{dz}{z^2} \right| &\leq \frac{2}{(2x)^2} 2x + 2 \int_{2x}^{\infty} \frac{dy}{y^2} \\ &= \frac{2}{x}. \end{aligned}$$

Similarly

$$\begin{aligned} \int_{PQ} \left| \frac{dz}{(w-z)^2} \right| &\leq 2\left(\frac{1}{x} + \int_x^{\infty} \frac{dy}{y^2}\right) \\ &= \frac{4}{x}. \end{aligned}$$

This completes the proof of the lemma.

**STEP 4.** We collect together the results in Steps 3 and 4 and choose the parameters  $B$  and  $n$  and this will give Theorem 2. Combining Lemmas 9,10 and 11 we state the following lemma.

**LEMMA 12.** *We have*

$$\begin{aligned} |f(0)|^k &\leq e^{2Bnz} \left(\frac{2}{Br}\right)^n M + \frac{e^{5Bnz}}{\pi} \left(\frac{2}{Br}\right)^n (1 + \log \frac{r}{2x}) M \\ &\quad + \frac{e^{5Bnz}}{(2\pi)^2} \cdot \frac{6}{x} \int_{P_1Q_1} |(f(w))^k dw|, \end{aligned} \quad (28)$$

where  $0 < 2x \leq r$ ,  $x_1$  is any real number with  $|x_1| \leq x$ ,  $n$  any natural number and  $B$  is any positive real number and  $P_1Q_1$  is the straight line joining  $-r_0$  and  $r_0$  where  $r_0 = \sqrt{4r^2 - x_1^2}$ .

Next we note that  $1 + \log \frac{r}{2x} \leq \frac{r}{2x}$  and so by putting  $x = r(\log M)^{-1}$  the first two terms on the RHS of (28) together do not exceed

$$\left(\frac{2}{Br}\right)^n e^{5Bnz} \left(1 + \frac{1}{2\pi} \log M\right) M \leq 2\left(\frac{2}{Br}\right)^n e^{5Bnz} M \log M.$$

Also,

$$\frac{6}{x} = \frac{6 \log M}{r} = 6 \log M \left(\frac{2r_0}{r}\right) \frac{1}{2r_0} \leq (24 \log M) \left(\frac{1}{2r_0}\right).$$

Thus RHS of (28) does not exceed

$$2\left(\frac{2}{Br}\right)^n e^{5Bnz} M \log M + \left(\frac{24}{(2\pi)^2} e^{5Bnz} \log M\right) \left(\frac{1}{2r_0} \int_{P_1Q_1} |(f(w))^k dw|\right).$$

We have chosen  $x = r(\log M)^{-1}$ . We now choose  $B$  such that  $Br = 2e$  and  $n = [C \log M] + 1$ , where  $C \geq 1$  is any real number. We have  $5Bnz \leq \frac{5Bnr}{\log M} \leq 10e(C+1) \leq 28(C+1)$  and also

$$\left(\frac{2}{Br}\right)^n \leq e^{-C \log M} = M^{-C}.$$

With these choices of  $x, B, n$  we see that RHS of (28) does not exceed

$$2M^{-C} e^{28(C+1)} M \log M + \left(\frac{24}{(2\pi)^2} e^{28(C+1)} \log M\right) \left(\frac{1}{2r_0} \int_{P_1Q_1} |f(w)|^k dw\right).$$

Putting  $C = A + 2$  we obtain Theorem 2 since  $C + 1 \leq 3A$ . This completes the proof of Theorem 2.

**ACKNOWLEDGEMENT.** The authors are thankful to Dr. R.R. SIMHA for some discussions on Riemann mapping theorem. They are also thankful to Professor J.-P. DEMAILLY for his Remarks 4 and 6 below Theorem 2.



**REFERENCES**

- [1 ] R. BALASUBRAMANIAN AND K. RAMACHANDRA, *Some local convexity theorems for the zeta-function-like analytic functions*, Hardy-Ramanujan J., 11 (1988), 1-12.
- [2 ] R. BALASUBRAMANIAN AND K. RAMACHANDRA, *A lemma in complex function theory - I*, Hardy-Ramanujan J., Vol.12 (1989), 1-5.

**ADDRESS OF THE AUTHORS**

1) PROFESSOR R. BALASUBRAMANIAN,  
MATSCIENCE, (ON DEPUTATION FROM T.I.F.R.)  
THARAMANI P.O.,  
MADRAS - 600 113  
INDIA

2) PROFESSOR K. RAMACHANDRA,  
SCHOOL OF MATHEMATICS,  
TATA INSTITUTE OF FUNDAMENTAL RESEARCH,  
HOMI BHABHA ROAD,  
BOMBAY - 400 005  
INDIA

(MANUSCRIPT COMPLETED ON 24 AUGUST 1989)