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## ON THE ALGEBRAIC DIFFERENTIAL EQUATIONS SATISFIED BY SOME ELLIPTIC FUNCTIONS (I)

By

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### Summary

When  $a$  is an odd positive integer it is implicit in the work of Jacobi that the functions

$$Y = \sum_{n=1}^{\infty} \sigma_a(n) X^n$$

where

$$\sigma_a(n) = \sum_{d|n} d^a$$

(the sum of the  $a$  th powers of the divisors of  $n$ ) satisfy an algebraic differential equation i. e. there is a polynomial  $T$ , not identically 0, such that  $T(X, Y, Y_1, \dots, Y_m) = 0$ . When  $a=1$  we give a new argument based on Ramanujan that we may take  $m = 3$  here.

§ 1. With Ramanujan (Coll. Papers, pp 138-142) we write

$$P = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) X^n$$

$$Q = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) X^n$$

$$R = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) X^n$$

Then we have

$$(a) \quad X \frac{dP}{dX} = \frac{P^2 - Q}{12}$$

$$(\beta) \quad X \frac{dQ}{dX} = \frac{PQ - R}{3}$$

$$(\gamma) \quad X \frac{dR}{dX} = \frac{PR - Q^2}{2}$$

Now differentiate  $(\alpha)$  twice w.r.t. "X". The first step gives us

$$(\delta) \quad \frac{d^2P}{dX^2} \text{ in terms of } P, Q, R, X.$$

Another differentiation gives us

$$(\epsilon) \quad \frac{d^3P}{dX^3} \text{ in terms of } P, Q, R, X.$$

Now eliminate Q,R from the 3 equations  $(\alpha)$ ,  $(\delta)$ ,  $(\epsilon)$ .

This relation (which can be converted into a polynomial relation) gives us

$$T\left(X, P, \frac{dP}{dX}, \frac{d^2P}{dX^2}, \frac{d^3P}{dX^3}\right) = 0$$

Here T is a polynomial (not identically 0) in the 5 quantities

$X, P, P_m$  ( $1 \leq m \leq 3$ ) where  $P_m$  stands for  $\frac{d^m P}{dX^m}$ .

Thus we have a third order alg. diff. eqn. satisfied by P, and

therefore by  $\sum_1^{\infty} \sigma_1(n) X^n$ . We also risk a *conjecture*.

The function

$$Y = \sum_1^{\infty} \sigma_a(n) X^n$$

does *not* satisfy an alg. differential equation when  $a$  is an *even* positive integer.

We have been unable to prove anything in this direction.