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## ON $n$ NUMBERS ON A CIRCLE

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The following problem has been considered by B. Freeman [2], M. Lotan [3], L. Carlitz and R. Scoville [1].

A positive integer  $n$  is fixed. There is given a vector

$$\mathbf{a} = \mathbf{a}(0) = [a_1(0), \dots, a_n(0)] \in \mathbb{R}^n$$

and an infinite sequence

$$\mathbf{a}(t) = [a_1(t), \dots, a_n(t)]$$

is formed by means of the formulae

$$a_i(t+1) = |a_i(t) - a_{i+1}(t)|,$$

where the addition of indices is mod  $n$ . A convenient method to think about the numbers  $a_i(t)$  is to place them on the unit circle, at the  $n$ -th roots of 1. The problem is to describe the behaviour of the sequence  $\mathbf{a}(t)$  at infinity.

If  $\mathbf{a} \in \mathbb{Z}^n$  and  $n = 2^k$  then we have  $\mathbf{a}(t) = 0$  for all large  $t$ . This was proved in [2]. If  $\mathbf{a} \in \mathbb{Z}^n$  and  $n \neq 2^k$  then there exists an integer  $c$  such that  $\mathbf{a}(t) \in \{0, c\}^n$  for all large  $t$ . This has been deduced in [1] from a statement equivalent to Lemma 1 below, however the proof of this statement has been in our opinion insufficient. If  $\mathbf{a} \in \mathbb{R}^n$  then it may happen that  $\lim_{t \rightarrow \infty} \mathbf{a}(t) = 0$ , but  $\mathbf{a}(t) \neq 0$  for all  $t$ . This was shown in [1] by an example and for  $n = 4$  all such cases have been determined in [3].

O. Radić has suggested (unpublished) that if  $\mathbf{a} \in \mathbb{R}^n$  and  $n = 2^k$  then  $\lim_{t \rightarrow \infty} \mathbf{a}(t) = 0$ . This will be proved below; moreover all possible limit points of

the sequence  $\mathbf{a}(t)$  will be determined for every  $n$ . The problem that remains open is the following.

Is it true that for every  $n$  and every  $\mathbf{a} \in \mathbb{R}^n$  either  $\lim_{t \rightarrow \infty} \mathbf{a}(t) = 0$  or there exists  $c \in \mathbb{R}$  such that  $\mathbf{a}(t) \in \{0, c\}^n$  for all sufficiently large  $t$ ?

We can look at the whole situation in the following way. There is a continuous map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  given by a formula

$$T([b_1, \dots, b_n]) = [|b_1 - b_2|, \dots, |b_{n-1} - b_n|, |b_n - b_1|].$$

Then  $\mathbf{a}(t) = T^t(\mathbf{a})$ , where  $T^t$  denotes the  $t$ -th iterate of  $T$ . The set of all limit points of the sequence  $\mathbf{a}(t)$  (i.e. of the trajectory of  $\mathbf{a}$ ) is the  $\omega$ -limit set of the point  $\mathbf{a}$  (see e.g. [4]).

Clearly,  $T(\mathbb{R}^n) \subset \mathbb{R}_+^n$ , so we may restrict our attention to  $\mathbb{R}_+^n$ . For a vector  $\mathbf{b} = [b_1, \dots, b_n]$  we set  $|\mathbf{b}| = \max_{1 \leq i \leq n} |b_i|$ . Notice that

$$(1) \quad |T(\mathbf{b})| \leq |\mathbf{b}| \quad \text{for all } \mathbf{b} \in \mathbb{R}_+^n.$$

This implies in particular that for each  $\mathbf{a} \in \mathbb{R}_+^n$  the whole sequence  $\mathbf{a}(t)$  is contained in the compact set

$$\{\mathbf{b} \in \mathbb{R}_+^n : |\mathbf{b}| \leq |\mathbf{a}|\}.$$

Therefore this sequence has at least one limit point.

Using this approach we may state our main results in the following way. There is a finite set of periodic orbits such that every trajectory of  $T$  is attracted to one of them; moreover we find all these periodic orbits.

More precisely, we shall prove:

**THEOREM 1.** *For every vector  $\mathbf{a} \in \mathbb{R}^n$  and a limit point  $\mathbf{p}$  of the sequence  $\mathbf{a}(t)$  we have*

$$(2) \quad \mathbf{p} = c\mathbf{e}, \text{ where } c = \lim_{t \rightarrow \infty} |\mathbf{a}(t)| \text{ and } \mathbf{e} = [\varepsilon_1, \dots, \varepsilon_n] \in \{0, 1\}^n.$$

$$(3) \quad 1 + x^{2^\nu} \mid \sum_{i=1}^n \varepsilon_i x^{i-1} \pmod{2}, \text{ where } 2^\nu \parallel n.$$

**COROLLARY.** If  $n = 2^v$  then for all vectors  $a \in \mathbb{R}^n$  we have

$$\lim_{t \rightarrow \infty} a(t) = 0.$$

**THEOREM 2.** If  $a = ce$ , where  $c \in \mathbb{R}_+$  and  $e = [\varepsilon_1, \dots, \varepsilon_n] \in \{0, 1\}^n$  satisfies (3) then  $a(N) = a$  for some  $N > 0$ , i.e. the orbit of  $a$  is periodic.

The proof of Theorem 1 is based on three lemmata.

**LEMMA 1.** Assume that  $p = [p_1, \dots, p_n] \in \mathbb{R}_+^n$ ,  $1 \leq k \leq n$  and

$$0 < p_k < |p|.$$

Then

$$|p(n-1)| < |p|.$$

**PROOF.** We claim that for every  $i \geq 0$  with  $|p(i)| = |p|$  there exist integers  $j_i^\pm$  with the following properties:

$$(4) \quad j_i^+ \geq 0, j_i^- \leq -i,$$

$$(5) \quad 0 \leq p_{k+j}(i) < |p| \quad \text{for } j \in (j_i^-, j_i^+),$$

$$(6) \quad 0 < p_{k+j_i^\pm}(i) < |p|.$$

We use the induction on  $i$ . For  $i = 0$  we take  $j_0^\pm = 0$ . Assume that the claim is true for a certain  $i$ . If  $|p(i+1)| = |p|$  then also  $|p(i)| = |p|$  and we may assume without loss of generality that  $j_i^+, j_i^-$  are the greatest and the least integer respectively satisfying the conditions (4) - (6). (Note that  $(j_i^-, j_i^+) = (j_i^-, 0] \cup [0, j_i^+)$ , so that the assumption is logically correct). Hence

$$(7) \quad p_{k+j_i^\pm \pm 1}(i) \in \{0, |p|\}.$$

We now take

$$j_{i+1}^+ = j_i^+, \quad j_{i+1}^- = j_i^- - 1.$$

It follows from (4) that

$$j_{i+1}^+ \geq 0, \quad j_{i+1}^- \leq -i - 1.$$

Moreover, if  $j \in (j_{i+1}^-, j_{i+1}^+) = [j_i^-, j_i^+)$  then we have by (5) and (6),

$$p_{k+j}(i+1) = |p_{k+j+1}(i) - p_{k+j}(i)| < |p|.$$

Finally, by (6) and (7),

$$p_{k+j_{i+1}^\pm}(i+1) = |p_{k+j_{i+1}^\pm \pm 1}(i) - p_{k+j_{i+1}^\pm}(i)| \in (0, |p|).$$

This completes the induction step and the proof of the claim.

Assume now that  $|p(n-1)| = |p|$ . By (4) - (6) for  $i = n-1$ , there exist integers  $j_{n-1}^+ \geq 0$  and  $j_{n-1}^- \leq 1-n$  such that

$$0 \leq p_{k+j}(n-1) < |p| \quad \text{for all } j \in [j_{n-1}^-, j_{n-1}^+].$$

However, the interval  $[j_{n-1}^-, j_{n-1}^+]$  contains all residue classes mod  $n$ , and hence  $|p(n-1)| < |p|$ , contrary to the assumption.  $\square$

**LEMMA 2.** If  $p = [p_1, \dots, p_n]$  is a limit point of the sequence  $a(t)$  then

$$\text{Card}(\{p_1, \dots, p_n\} \setminus \{0\}) \leq 1.$$

**PROOF.** We may assume that  $a \in \mathbb{R}_+^n$ . Then, by (1), the sequence  $|a(t)|$  is non-increasing and therefore the limit

$$(8) \quad c = \lim_{t \rightarrow \infty} |a(t)|$$

exists. Suppose that

$$(9) \quad \text{Card}(\{p_1, \dots, p_n\} \setminus \{0\}) \geq 2$$

and that

$$(10) \quad p = \lim_{k \rightarrow \infty} a(t_k)$$

for some sequence  $t_k \rightarrow \infty$ . Since  $p \in \mathbb{R}_+^n$  and by (9), we can use Lemma 1. We get, taking (8) into account,

$$(11) \quad |T^{n-1}(p)| < |p| = c.$$

The map  $T$  is continuous and therefore from (10) we obtain

$$\lim_{k \rightarrow \infty} a(t_{k+n-1}) = T^{n-1}(p).$$

Together with (8) and (11) this gives a contradiction. □

**LEMMA 3.** *If  $e = [\varepsilon_1, \dots, \varepsilon_n] \in \mathbb{Z}^n$  and  $2^\nu \parallel n$  then*

$$1 + x^{2^\nu} \mid \sum_{i=1}^n \varepsilon_i (2^\nu) x^{i-1} \pmod{2}.$$

**PROOF.** By the definition of  $a(t)$ ,

$$\begin{aligned} \sum_{i=1}^n \varepsilon_i(t+1)x^i &\equiv \sum_{i=1}^n (\varepsilon_i(t) + \varepsilon_{i+1}(t))x^i \equiv \\ (12) \quad &\sum_{i=1}^n \varepsilon_i(t)x^i + \sum_{i=1}^n \varepsilon_i(t)x^{i-1} + \varepsilon_1(t)x^n \equiv \\ &(1+x) \sum_{i=1}^n \varepsilon_i(t)x^{i-1} + a_1(t)(1+x^n) \pmod{2} \end{aligned}$$

(cf. [1, p. 298]). Hence,

$$x \sum_{i=1}^n \varepsilon_i(t+1)x^{i-1} \equiv (1+x) \sum_{i=1}^n \varepsilon_i(t)x^{i-1} \pmod{2, 1+x^{2^v}},$$

and by induction on  $j$

$$x^j \sum_{i=1}^n \varepsilon_i(j)x^{i-1} \equiv (1+x)^j \sum_{i=1}^n \varepsilon_i x^{i-1} \pmod{2, 1+x^{2^v}},$$

Taking  $j = 2^v$  and observing that

$$(1+x)^{2^v} \equiv 1+x^{2^v} \pmod{2}$$

we obtain the lemma.  $\square$

**PROOF OF THEOREM 1.** As we noticed already, the limit (8) exists. Now (2) follows immediately from Lemma 2 (if  $c = 0$  then we can take  $\mathbf{e} = 0$ ).

Denote

$$P = \{c\mathbf{e} : \mathbf{e} = [\varepsilon_1, \dots, \varepsilon_n] \in \{0, 1\}^n\},$$

$$Q = \{c\mathbf{e} : \mathbf{e} = [\varepsilon_1, \dots, \varepsilon_n] \in \{0, 1\}^n \text{ and (3) holds}\}.$$

From Lemma 3 and the fact that

$$(13) \quad T(c\mathbf{b}) = cT(\mathbf{b}) \quad \text{for all } \mathbf{b} \text{ in } \mathbb{R}^n$$

we obtain

$$(14) \quad T^{2^v}(P) \subset Q.$$

Since the whole sequence  $\mathbf{a}(t)$  lies in a compact set, we have by (2),

$$\lim_{t \rightarrow \infty} (\min_{\mathbf{p} \in P} |\mathbf{a}(t) - \mathbf{p}|) = 0.$$

By this, (14) and continuity of  $T$ ,

$$\lim_{t \rightarrow \infty} (\min_{p \in Q} |a(t) - p|) = 0.$$

Therefore all limit points of the sequence  $a(t)$  belong to  $Q$ .  $\square$

**PROOF OF COROLLARY.** If  $n = 2^\nu$  then the only vector  $e \in \{0, 1\}^n$  satisfying (3) is  $e = 0$ . *Box*

**PROOF OF THEOREM 2.** By (13) and since the set

$$S = \{e : e = [\varepsilon_1, \dots, \varepsilon_n] \in \{0, 1\}^n \text{ and (3) holds}\}$$

is finite, it is enough to prove that the map  $T$  maps  $S$  bijectively onto itself.

By (12),  $T(S) \subset S$ . Therefore it remains to prove that if  $e = [\varepsilon_1, \dots, \varepsilon_n]$  and  $d = [\delta_1, \dots, \delta_n]$  belong to  $S$  and  $e \neq d$  then  $T(e) \neq T(d)$ . Suppose that  $T(e) = T(d)$ . For each  $i$  we have

$$|\varepsilon_i - \varepsilon_{i+1}| = |\delta_i - \delta_{i+1}|,$$

so  $\varepsilon_i = \delta_i$  if and only if  $\varepsilon_{i+1} = \delta_{i+1}$ . Since  $e \neq d$ , for some  $j$  we have  $\varepsilon_j \neq \delta_j$ . Then by induction we obtain  $\varepsilon_i \neq \delta_i$  for all  $i$ . Since  $\varepsilon_i, \delta_i \in \{0, 1\}$ , this means that  $\varepsilon_i + \delta_i = 1$  for all  $i$ . Therefore

$$1 + x^{2^\nu} \mid \sum_{i=1}^n x^{i-1} \pmod{2},$$

so

$$(1 + x^{2^\nu})(1 + x) \mid 1 + x^n \pmod{2}.$$

Setting  $m = n2^{-\nu}$ , we get

$$1 + x \mid \sum_{j=0}^{m-1} x^{j2^\nu} \pmod{2},$$



a contradiction since for  $x = 1$  we have

$$1 + x \equiv 0 \pmod{2}$$

and

$$\sum_{j=0}^{m-1} x^{j2^r} \equiv m \equiv 1 \pmod{2}. \quad \square$$

One can generalize the problem considered here in the following way. Let  $X$  be a compact topological space,  $f : X \rightarrow X$  a continuous map and  $C(X)$  the space of all continuous real functions on  $X$  with the topology of uniform convergence. We can define a continuous map  $T : C(X) \rightarrow C(X)$  by the formula

$$(T\varphi)(x) = |\varphi(x) - \varphi(f(x))|.$$

Then the problem is again to investigate the asymptotic behaviour of the trajectories of the map  $T$ . We can also consider other types of convergence, not necessarily uniform.

The case investigated here is obtained from this general problem by taking  $X$  consisting of  $n$  points and  $f$  - a cyclic permutation of  $X$ . Another interesting case can be obtained by taking  $X = S^1$  (a circle) and  $f$  an irrational rotation (i.e. a rotation by an angle non-commensurable with  $\pi$ ). Then the conjecture of M. Skalba (unpublished) says that all trajectories converge to 0.

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Note added on April 12. Here are further references for the case of integers.

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