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ON n NUMBERS ON A CIRCLE

M. Misiurewicz and A. Schinzel

The following problem has been considered by B. Freeman [2], M. Lotan [3], L. Carlitz and R. Scoville [1].

A positive integer n is fixed. There is given a vector

$$\mathbf{a} = \mathbf{a}(0) = [a_1(0), \dots, a_n(0)] \in \mathbb{R}^n$$

and an infinite sequence

$$\mathbf{a}(t) = [a_1(t), \dots, a_n(t)]$$

is formed by means of the formulae

$$a_i(t+1) = | a_i(t) - a_{i+1}(t) |,$$

where the addition of indices is mod n . A convenient method to think about the numbers $a_i(t)$ is to place them on the unit circle, at the n -th roots of 1. The problem is to describe the behaviour of the sequence $\mathbf{a}(t)$ at infinity.

If $\mathbf{a} \in \mathbb{Z}^n$ and $n = 2^k$ then we have $\mathbf{a}(t) = 0$ for all large t . This was proved in [2]. If $\mathbf{a} \in \mathbb{Z}^n$ and $n \neq 2^k$ then there exists an integer c such that $\mathbf{a}(t) \in \{0, c\}^n$ for all large t . This has been deduced in [1] from a statement equivalent to Lemma 1 below, however the proof of this statement has been in our opinion insufficient. If $\mathbf{a} \in \mathbb{R}^n$ then it may happen that $\lim_{t \rightarrow \infty} \mathbf{a}(t) = 0$, but $\mathbf{a}(t) \neq 0$ for all t . This was shown in [1] by an example and for $n = 4$ all such cases have been determined in [3].

O. Radić has suggested (unpublished) that if $\mathbf{a} \in \mathbb{R}^n$ and $n = 2^k$ then $\lim_{t \rightarrow \infty} \mathbf{a}(t) = 0$. This will be proved below; moreover all possible limit points of

the sequence $\mathbf{a}(t)$ will be determined for every n . The problem that remains open is the following.

Is it true that for every n and every $\mathbf{a} \in \mathbb{R}^n$ either $\lim_{t \rightarrow \infty} \mathbf{a}(t) = 0$ or there exists $c \in \mathbb{R}$ such that $\mathbf{a}(t) \in \{0, c\}^n$ for all sufficiently large t ?

We can look at the whole situation in the following way. There is a continuous map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by a formula

$$T([b_1, \dots, b_n]) = [|b_1 - b_2|, \dots, |b_{n-1} - b_n|, |b_n - b_1|].$$

Then $\mathbf{a}(t) = T^t(\mathbf{a})$, where T^t denotes the t -th iterate of T . The set of all limit points of the sequence $\mathbf{a}(t)$ (i.e. of the trajectory of \mathbf{a}) is the ω -limit set of the point \mathbf{a} (see e.g. [4]).

Clearly, $T(\mathbb{R}^n) \subset \mathbb{R}_+^n$, so we may restrict our attention to \mathbb{R}_+^n . For a vector $\mathbf{b} = [b_1, \dots, b_n]$ we set $|\mathbf{b}| = \max_{1 \leq i \leq n} |b_i|$. Notice that

$$(1) \quad |T(\mathbf{b})| \leq |\mathbf{b}| \quad \text{for all } \mathbf{b} \in \mathbb{R}_+^n.$$

This implies in particular that for each $\mathbf{a} \in \mathbb{R}_+^n$ the whole sequence $\mathbf{a}(t)$ is contained in the compact set

$$\{\mathbf{b} \in \mathbb{R}_+^n : |\mathbf{b}| \leq |\mathbf{a}|\}.$$

Therefore this sequence has at least one limit point.

Using this approach we may state our main results in the following way. There is a finite set of periodic orbits such that every trajectory of T is attracted to one of them; moreover we find all these periodic orbits.

More precisely, we shall prove:

THEOREM 1. *For every vector $\mathbf{a} \in \mathbb{R}^n$ and a limit point \mathbf{p} of the sequence $\mathbf{a}(t)$ we have*

$$(2) \quad \mathbf{p} = c\mathbf{e}, \text{ where } c = \lim_{t \rightarrow \infty} |\mathbf{a}(t)| \text{ and } \mathbf{e} = [\varepsilon_1, \dots, \varepsilon_n] \in \{0, 1\}^n.$$

$$(3) \quad 1 + x^{2^\nu} \mid \sum_{i=1}^n \varepsilon_i x^{i-1} \pmod{2}, \text{ where } 2^\nu \parallel n.$$

COROLLARY. If $n = 2^v$ then for all vectors $\mathbf{a} \in \mathbb{R}^n$ we have

$$\lim_{t \rightarrow \infty} \mathbf{a}(t) = \mathbf{0}.$$

THEOREM 2. If $\mathbf{a} = c\mathbf{e}$, where $c \in \mathbb{R}_+$ and $\mathbf{e} = [\varepsilon_1, \dots, \varepsilon_n] \in \{0, 1\}^n$ satisfies (3) then $\mathbf{a}(N) = \mathbf{a}$ for some $N > 0$, i.e. the orbit of \mathbf{a} is periodic.

The proof of Theorem 1 is based on three lemmata.

LEMMA 1. Assume that $\mathbf{p} = [p_1, \dots, p_n] \in \mathbb{R}_+^n$, $1 \leq k \leq n$ and

$$0 < p_k < |\mathbf{p}|.$$

Then

$$|\mathbf{p}(n-1)| < |\mathbf{p}|.$$

PROOF. We claim that for every $i \geq 0$ with $|\mathbf{p}(i)| = |\mathbf{p}|$ there exist integers j_i^\pm with the following properties:

$$(4) \quad j_i^+ \geq 0, j_i^- \leq -i,$$

$$(5) \quad 0 \leq p_{k+j}(i) < |\mathbf{p}| \quad \text{for } j \in (j_i^-, j_i^+),$$

$$(6) \quad 0 < p_{k+j_i^\pm}(i) < |\mathbf{p}|.$$

We use the induction on i . For $i = 0$ we take $j_0^\pm = 0$. Assume that the claim is true for a certain i . If $|\mathbf{p}(i+1)| = |\mathbf{p}|$ then also $|\mathbf{p}(i)| = |\mathbf{p}|$ and we may assume without loss of generality that j_i^+, j_i^- are the greatest and the least integer respectively satisfying the conditions (4) - (6). (Note that $(j_i^-, j_i^+) = (j_i^-, 0] \cup [0, j_i^+)$, so that the assumption is logically correct). Hence

$$(7) \quad p_{k+j_i^\pm \pm 1}(i) \in \{0, |\mathbf{p}|\}.$$

We now take

$$j_{i+1}^+ = j_i^+, \quad j_{i+1}^- = j_i^- - 1.$$

It follows from (4) that

$$j_{i+1}^+ \geq 0, \quad j_{i+1}^- \leq -i - 1.$$

Moreover, if $j \in (j_{i+1}^-, j_{i+1}^+) = [j_i^-, j_i^+)$ then we have by (5) and (6),

$$p_{k+j}(i+1) = |p_{k+j+1}(i) - p_{k+j}(i)| < |p|.$$

Finally, by (6) and (7),

$$p_{k+j_{i+1}^\pm}(i+1) = |p_{k+j_{i+1}^\pm \pm 1}(i) - p_{k+j_{i+1}^\pm}(i)| \in (0, |p|).$$

This completes the induction step and the proof of the claim.

Assume now that $|p(n-1)| = |p|$. By (4) - (6) for $i = n-1$, there exist integers $j_{n-1}^+ \geq 0$ and $j_{n-1}^- \leq 1-n$ such that

$$0 \leq p_{k+j}(n-1) < |p| \quad \text{for all } j \in [j_{n-1}^-, j_{n-1}^+].$$

However, the interval $[j_{n-1}^-, j_{n-1}^+]$ contains all residue classes mod n , and hence $|p(n-1)| < |p|$, contrary to the assumption. \square

LEMMA 2. *If $p = [p_1, \dots, p_n]$ is a limit point of the sequence $a(t)$ then*

$$\text{Card}(\{p_1, \dots, p_n\} \setminus \{0\}) \leq 1.$$

PROOF. We may assume that $a \in \mathbb{R}_+^n$. Then, by (1), the sequence $|a(t)|$ is non-increasing and therefore the limit

$$(8) \quad c = \lim_{t \rightarrow \infty} |a(t)|$$

exists. Suppose that

$$(9) \quad \text{Card}(\{p_1, \dots, p_n\} \setminus \{0\}) \geq 2$$

and that

$$(10) \quad \mathbf{p} = \lim_{k \rightarrow \infty} \mathbf{a}(t_k)$$

for some sequence $t_k \rightarrow \infty$. Since $\mathbf{p} \in \mathbb{R}_+^n$ and by (9), we can use Lemma 1. We get, taking (8) into account,

$$(11) \quad |T^{n-1}(\mathbf{p})| < |\mathbf{p}| = c.$$

The map T is continuous and therefore from (10) we obtain

$$\lim_{k \rightarrow \infty} \mathbf{a}(t_{k+n-1}) = T^{n-1}(\mathbf{p}).$$

Together with (8) and (11) this gives a contradiction. \square

LEMMA 3. If $\mathbf{e} = [\varepsilon_1, \dots, \varepsilon_n] \in \mathbb{Z}^n$ and $2^\nu \parallel n$ then

$$1 + x^{2^\nu} \mid \sum_{i=1}^n \varepsilon_i (2^\nu) x^{i-1} \pmod{2}.$$

PROOF. By the definition of $\mathbf{a}(t)$,

$$\begin{aligned} \sum_{i=1}^n \varepsilon_i(t+1)x^i &\equiv \sum_{i=1}^n (\varepsilon_i(t) + \varepsilon_{i+1}(t))x^i \equiv \\ (12) \quad &\sum_{i=1}^n \varepsilon_i(t)x^i + \sum_{i=1}^n \varepsilon_i(t)x^{i-1} + \varepsilon_1(t)x^n \equiv \\ &(1+x) \sum_{i=1}^n \varepsilon_i(t)x^{i-1} + \varepsilon_1(t)(1+x^n) \pmod{2} \end{aligned}$$

(cf. [1, p. 298]). Hence,

$$x \sum_{i=1}^n \varepsilon_i(t+1)x^{i-1} \equiv (1+x) \sum_{i=1}^n \varepsilon_i(t)x^{i-1} \pmod{(2, 1+x^{2^v})},$$

and by induction on j

$$x^j \sum_{i=1}^n \varepsilon_i(j)x^{i-1} \equiv (1+x)^j \sum_{i=1}^n \varepsilon_i x^{i-1} \pmod{(2, 1+x^{2^v})},$$

Taking $j = 2^v$ and observing that

$$(1+x)^{2^v} \equiv 1+x^{2^v} \pmod{2}$$

we obtain the lemma. □

PROOF OF THEOREM 1. As we noticed already, the limit (8) exists. Now (2) follows immediately from Lemma 2 (if $c = 0$ then we can take $\mathbf{e} = 0$).

Denote

$$P = \{c\mathbf{e} : \mathbf{e} = [\varepsilon_1, \dots, \varepsilon_n] \in \{0, 1\}^n\},$$

$$Q = \{c\mathbf{e} : \mathbf{e} = [\varepsilon_1, \dots, \varepsilon_n] \in \{0, 1\}^n \text{ and (3) holds}\}.$$

From Lemma 3 and the fact that

$$(13) \quad T(c\mathbf{b}) = cT(\mathbf{b}) \quad \text{for all } \mathbf{b} \text{ in } \mathbb{R}^n$$

we obtain

$$(14) \quad T^{2^v}(P) \subset Q.$$

Since the whole sequence $\mathbf{a}(t)$ lies in a compact set, we have by (2),

$$\lim_{t \rightarrow \infty} (\min_{\mathbf{p} \in P} |\mathbf{a}(t) - \mathbf{p}|) = 0.$$

By this, (14) and continuity of T ,

$$\lim_{t \rightarrow \infty} (\min_{p \in Q} |a(t) - p|) = 0.$$

Therefore all limit points of the sequence $a(t)$ belong to Q . \square

PROOF OF COROLLARY. If $n = 2^\nu$ then the only vector $e \in \{0, 1\}^n$ satisfying (3) is $e = 0$. *Box*

PROOF OF THEOREM 2. By (13) and since the set

$$S = \{e : e = [\varepsilon_1, \dots, \varepsilon_n] \in \{0, 1\}^n \text{ and (3) holds}\}$$

is finite, it is enough to prove that the map T maps S bijectively onto itself.

By (12), $T(S) \subset S$. Therefore it remains to prove that if $e = [\varepsilon_1, \dots, \varepsilon_n]$ and $d = [\delta_1, \dots, \delta_n]$ belong to S and $e \neq d$ then $T(e) \neq T(d)$. Suppose that $T(e) = T(d)$. For each i we have

$$|\varepsilon_i - \varepsilon_{i+1}| = |\delta_i - \delta_{i+1}|,$$

so $\varepsilon_i = \delta_i$ if and only if $\varepsilon_{i+1} = \delta_{i+1}$. Since $e \neq d$, for some j we have $\varepsilon_j \neq \delta_j$. Then by induction we obtain $\varepsilon_i \neq \delta_i$ for all i . Since $\varepsilon_i, \delta_i \in \{0, 1\}$, this means that $\varepsilon_i + \delta_i = 1$ for all i . Therefore

$$1 + x^{2^\nu} \mid \sum_{i=1}^n x^{i-1} \pmod{2},$$

so

$$(1 + x^{2^\nu})(1 + x) \mid 1 + x^n \pmod{2}.$$

Setting $m = n2^{-\nu}$, we get

$$1 + x \mid \sum_{j=0}^{m-1} x^{j2^\nu} \pmod{2},$$

a contradiction since for $x = 1$ we have

$$1 + x \equiv 0 \pmod{2}$$

and

$$\sum_{j=0}^{m-1} x^{j2^v} \equiv m \equiv 1 \pmod{2}. \quad \square$$

One can generalize the problem considered here in the following way. Let X be a compact topological space, $f : X \rightarrow X$ a continuous map and $C(X)$ the space of all continuous real functions on X with the topology of uniform convergence. We can define a continuous map $T : C(X) \rightarrow C(X)$ by the formula

$$(T\varphi)(x) = |\varphi(x) - \varphi(f(x))|.$$

Then the problem is again to investigate the asymptotic behaviour of the trajectories of the map T . We can also consider other types of convergence, not necessarily uniform.

The case investigated here is obtained from this general problem by taking X consisting of n points and f - a cyclic permutation of X . Another interesting case can be obtained by taking $X = S^1$ (a circle) and f an irrational rotation (i.e. a rotation by an angle non-commensurable with π). Then the conjecture of M. Skalba (unpublished) says that all trajectories converge to 0.

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Note added on April 12. Here are further references for the case of integers.

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