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## LARGE VALUES OF SOME ZETA-FUNCTIONS NEAR THE LINE $\sigma = 1$

Aleksandar Ivić

### § 1. Introduction.

To determine the order of  $\zeta(1 + it)$  is one of the central problems in the theory of the Riemann zeta-function  $\zeta(s)$ . The best known upper bound at present is

$$(1.1) \quad \zeta(1 + it) \ll \log^{2/3} t.$$

It is obtained by an application of the estimate

$$\sum_{N < n \leq N'} n^{it} \ll N \exp\left(-\frac{C \log^3 N}{\log^2 t}\right) \quad (C = 10^{-5}, N < N' \leq 2N, 1 \ll N \ll t)$$

which is a consequence of Vinogradov's method (see [4], Chapter 6) for the estimation of exponential sums. Here and later  $f \ll g$  and  $f = O(g)$  both mean  $|f(x)| \leq Cg(x)$  for some  $C > 0$  and  $x \geq x_0$ . On the other hand, it is known (see [14], Chapter 9) that on the Riemann hypothesis

$$e^\gamma \leq \limsup_{t \rightarrow \infty} \frac{|\zeta(1 + it)|}{\log \log t} \leq 2e^\gamma,$$

where  $\gamma = 0,577\dots$  is Euler's constant. Thus it seems of interest to investigate the occurrence of large values of  $\zeta(1 + it)$ , where "large" means roughly of the order not less than  $\log \log t$ . An interesting result in this direction was obtained recently by K. Ramachandra [11]: Let  $X = \exp(\log \log T / \log \log \log T)$ , and cover  $[T, T + e^X]$  with intervals of length  $1/X$  (the last interval may be shorter). If  $0 < \varepsilon < 1$  is an arbitrary constant, then  $|\log \zeta(1 + it)| \geq \varepsilon \log \log T$  for  $t$  in all of these intervals, except in at most  $K$  of them, where

$K = K(\varepsilon)$  is a constant. Ramachandra obtained his result by shrewdly applying an elementary inequality for complex numbers, and using complex integration to evaluate a certain sum over primes. His theorem is "local" in nature, in the sense that  $X = o_\delta(\log^\delta T)$  as  $T \rightarrow \infty$  for any  $\delta > 0$ . It seems also interesting to consider the "global" problem of the estimation of  $R$ , the number of points  $t_r$  in  $[T, 2T]$  such that  $|t_r - t_s| \geq 1$  for  $r \neq s$ ,  $|\log \zeta(1 + it_r)| \geq \varepsilon \log \log T$  (or  $|\zeta(1 + it_r)| \geq (\log T)^\varepsilon$ , where the method of [11] furnishes the same bound as in the former case). Breaking the interval  $[T, 2T]$  into subintervals of length  $e^X$  and applying Ramachandra's estimate to each of these intervals one easily obtains

$$(1.2) \quad R \ll_\varepsilon T e^{-\frac{1}{2}X}.$$

We could also suppose that  $|t_r - t_s| \geq X^{-1}$  for  $r \neq s$ , but this would only have the effect that the bound in (1.2) is multiplied by  $X$ .

A non-trivial result on large values of  $\zeta(\sigma + it)$  when  $\sigma$  is close to 1 follows from Corollary 1 of K. Ramachandra [9]: Let  $R$  denote the number of points  $t_r$  in  $[1, T]$  such that  $|t_r - t_s| \geq 1$  for  $r \neq s$  and  $|\zeta(\sigma + it_r)| \geq V$ , where  $\frac{1}{2} \leq \sigma \leq 1$  and

$$(1.3) \quad \exp(C(\log T \log \log T)^{\frac{1}{2}}) \ll V \ll T^{100(1-\sigma)^{3/2}} \log^{2/3} T$$

for a suitable constant  $C > 0$ . Then uniformly

$$(1.4) \quad R \ll TV^{-A(1-\sigma)^{-3/2}} \quad (A = 1/300\sqrt{2} = 0,002357\dots).$$

The upper bound for  $V$  in (1.3) follows from the best known upper bound for  $\zeta(\sigma + it)$  (see Chapter 6 of [4]). From (1.3) it follows that

$$\sigma \leq 1 - C(\log \log T / \log T)^{1/3}$$

must hold with some suitable  $C > 0$ , hence (1.4) holds only if  $\sigma$  is not too close to 1. From (1.4) one deduces easily

$$(1.5) \quad m(\sigma) \geq A(1 - \sigma)^{-3/2} \quad (A = 1/300\sqrt{2})$$

for  $\frac{1}{2} < \sigma \leq \sigma_0$ , where  $\sigma_0 < 1$  is fixed, and  $m(\sigma)$  is the infimum of numbers  $m$  such that for a given  $\sigma$  and any fixed  $\varepsilon > 0$

$$\int_1^T |\zeta(\sigma + it)|^m dt \ll T^{1+\varepsilon}.$$

The bound (1.5) is mentioned in the Notes of Chapter 8 of [4], without any specific value of the constant  $A$ . It is superseded by Theorem 8.4 of [4] for  $\sigma \leq \sigma_1$ , where  $\sigma_1$  can be explicitly evaluated, but for  $\sigma$  close to 1 (1.5) remains the best existing lower bound for  $m(\sigma)$ . Some other relevant results on large values of  $\zeta(\sigma + it)$  and related topics may be found in K. Ramachandra [10].

The aim of this paper is to provide estimates which improve (1.2) and (1.4) when  $\sigma$  is sufficiently close to 1, and  $V$  lies in a certain range. This will be expounded in the next section. The method of approach is fairly general, and can be used to furnish analogous results for a class of zeta-functions that are similar to  $\zeta(s)$ . To this class belong the zeta-functions associated with the Fourier coefficients of cusp forms and the Dedekind zeta-function of algebraic number fields. These zeta-functions will be dealt with in § 3 and § 4, respectively.

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§ 2. Large values of  $\zeta(\sigma + it)$ .

The basic analytic principle of our approach is simple. In suitable horizontal strips, free of zeros of  $\zeta(s)$ ,  $\frac{\zeta'}{\zeta}(s)$  can be estimated and found to be small. Integration shows that  $\log \zeta(s)$  is then small, too. On the other hand, the number of well-spaced points  $t_r$  for which  $\zeta(\sigma + it_r)$  lies near a zero of  $\zeta(s)$  may be estimated satisfactorily by zero-density estimates.

We proceed now to give the details of this method. Henceforth suppose that  $t$  is given,  $w$  is a complex variable and

$$(2.1) \quad \zeta(w) \neq 0 \text{ for } \operatorname{Re} w > \alpha_0 \left(\frac{1}{2} \leq \alpha_0 < 1\right), | \operatorname{Im} w - t | \leq \log^2 T, T \leq t \leq 2T, T \geq T_0.$$

We shall bound  $|\zeta(\sigma + it)|$  for  $\alpha < \sigma \leq 1, \alpha > \alpha_0$ , where  $\alpha$  and  $\alpha_0$  will be specified later. The starting point is the inversion formula ( $s = \sigma + it$ )

$$(2.2) \quad \sum_{n=1}^{\infty} \Lambda(n) e^{-n/Y} n^{-s} = -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\zeta'}{\zeta}(s+w) \Gamma(w) Y^w dw.$$

This follows from the Mellin integral (see (A.7) of [4])

$$e^{-x} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(w) x^{-w} dw \quad (x > 0)$$

on setting  $x = n/Y$ , multiplying by  $\Lambda(n)n^{-s}$  and summing over  $n$ , since

$$\sum_{n=1}^{\infty} \Lambda(n)n^{-s} = -\frac{\zeta'(s)}{\zeta(s)} \quad (\sigma = \operatorname{Re} s > 1).$$

In (2.2)  $\Lambda(n)$  is the familiar von Mangoldt function ( $\Lambda(n) = \log p$  if  $n = p^m$ ,  $p$  prime, and  $\Lambda(n) = 0$  otherwise), and  $Y$  is a suitable parameter which satisfies  $1 \ll Y \ll \log^2 T$ . In (2.2) we replace the line of integration  $\operatorname{Re} w = 2$  by the contour consisting of  $[\alpha - \sigma - \frac{1}{2} i \log^2 T, \alpha - \sigma + \frac{1}{2} i \log^2 T]$ ,  $[\alpha - \sigma \pm \frac{1}{2} i \log^2 T, 2 \pm \frac{1}{2} i \log^2 T]$ ,  $[2 \pm \frac{1}{2} i \log^2 T, 2 \pm i\infty]$ . In view of (2.1) it is seen that  $s + w$  will stay in a region free of zeros of the zeta-function, hence  $\zeta'(s+w)$  will be regular as a function of  $w$ . The only pole of the integrand will be  $w = 0$ , which yields the residue  $-\zeta'(s)$ . We shall use the bound

$$(2.3) \quad \Gamma(w) \ll \frac{e^{-|\operatorname{Im} w|}}{|w|},$$

which is a weak form of Stirling's formula, to estimate the integrals in question. In this way we obtain from (2.2)

$$(2.4) \quad \sum_{n=1}^{\infty} \Lambda(n)e^{-n/Y} n^{-s} = -\frac{\zeta'(s)}{\zeta(s)} + o(1) - \frac{1}{2\pi i} \int_L \frac{\zeta'(s+w)}{\zeta(s+w)} \Gamma(w) Y^w dw \quad (T \rightarrow \infty),$$

where  $L$  denotes the segment  $[\alpha - \sigma - \frac{1}{2} i \log^2 T, \alpha - \sigma + \frac{1}{2} i \log^2 T]$ . To estimate the last integral we note that for  $z = s + w$ ,  $|\operatorname{Im} w| \leq \frac{1}{2} \log^2 T$ ,  $s = \sigma + it$  we have (see (1.52) of [4])

$$(2.5) \quad \frac{\zeta'(z)}{\zeta(z)} = \sum_{\rho: \zeta(\rho)=0, |\operatorname{Im} \rho - \operatorname{Im} z| \leq 1} \frac{1}{z - \rho} + O(\log T) \ll \frac{\log T}{\alpha - \alpha_0},$$

since (2.1) holds and there are  $\ll \log T$  zeros  $\rho$  in every horizontal strip of unit width. Using again (2.3) it follows from (2.4) that

$$(2.6) \quad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)e^{-n/Y} n^{-s} + O(1) + O\left(\frac{Y^{\alpha-\sigma} \log T}{(\alpha - \alpha_0)(\sigma - \alpha)}\right)$$

holds uniformly for  $s = \sigma + it$ ,  $\alpha < \sigma \leq 1$ , if (2.1) is true. Set now in (2.6)  $\sigma = \theta$  and integrate over  $\theta$  for  $\sigma \leq \theta \leq 2$ ,  $\alpha < \sigma \leq 1$ . If we define

$$\Lambda_1(n) = \begin{cases} 0, & n = 1, n \neq p^m, \\ \frac{1}{m}, & n = p^m, \end{cases}$$

where  $p$  denotes primes, we obtain that uniformly

$$(2.7) \log \zeta(s) = \sum_{n=1}^{\infty} \Lambda_1(n) e^{-n/Y} n^{-s} + O(1) + O\left(\frac{Y^{\alpha-\sigma}}{(\alpha-\alpha_0)} \log T \log \frac{1}{\sigma-\alpha}\right),$$

if  $s = \sigma + it, \alpha < \sigma \leq 1$ , and (2.1) holds.

We shall first examine some consequences of (2.7). Suppose  $\alpha_0$  in (2.1) is fixed, and take  $\sigma = 1, \alpha = \frac{1}{2}(1 + \alpha_0) < 1$ . Then from (2.7) we infer that

$$\log \zeta(1 + it) = \sum_{n \leq Y} \Lambda_1(n) e^{-n/Y} n^{-1-it} + O(1) + O_{\alpha_0}(Y^{\alpha-1} \log T),$$

where the subscript in the last  $O$ -term means that the constant in question depends on  $\alpha_0$ . With the choice  $Y = (\log T)^{1/(1-\alpha)}$  this gives

$$\begin{aligned} \log |\zeta(1 + it)| &\leq |\log \zeta(1 + it)| \leq \sum_{n \leq Y} \Lambda_1(n) n^{-1} + O_{\alpha_0}(1) \\ &= \sum_{p \leq Y} p^{-1} + O_{\alpha_0}(1) = \log \log Y + O_{\alpha_0}(1) = \log \log \log T + O_{\alpha_0}(1). \end{aligned}$$

Therefore

$$(2.8) \quad |\zeta(1 + it)| \leq C(\alpha_0) \log \log t \quad (t \geq t_0(\alpha_0))$$

for some constant  $C(\alpha_0) > 0$  if (2.1) holds for a fixed  $\alpha_0$ . In case the Riemann hypothesis that all complex zeros of  $\zeta(s)$  satisfy  $\text{Re } s = \frac{1}{2}$  is true, then (2.1) holds with  $\alpha_0 = \frac{1}{2}$ , and as mentioned in § 1 one may take  $C(\frac{1}{2}) = 2e^\gamma + \varepsilon$  in that case.

Suppose now again that  $\alpha_0$  in (2.1) is fixed, and take  $\alpha = \alpha_0 + \varepsilon, \alpha_0 + 2\varepsilon \leq \sigma \leq \sigma_0$ , where  $\sigma_0 < 1$  is fixed. For  $N, Y \gg 1$  note that

$$\sum_{p > N} p^{-\sigma} e^{-p/Y} \ll \int_N^{\infty} t^{-\sigma} e^{-t/Y} (\log t)^{-1} dt \ll \frac{N^{-\sigma}}{\log N} \int_N^{\infty} e^{-t/Y} dt \ll \frac{Y N^{-\sigma}}{\log N},$$

where  $p$  denotes primes. Also using the prime number theorem we have

$$\sum_{p \leq Y} p^{-\sigma} = \frac{Y^{1-\sigma}}{(1-\sigma) \log Y} + O\left(\frac{Y^{1-\sigma}}{\log^2 Y}\right).$$

Integrating (2.6) we obtain then

$$\log \zeta(s) = \sum_{n=1}^{\infty} \Lambda_1(n) e^{-n/Y} n^{-s} + O(1) + O_{\epsilon}(Y^{\alpha-\sigma} \frac{\log T}{\log Y}),$$

which gives

$$\begin{aligned} \log |\zeta(\sigma + it)| &\leq |\log \zeta(\sigma + it)| \leq \sum_{n \leq Y} \Lambda_1(n) e^{-n/Y} n^{-\sigma} + O\left(\frac{Y^{1-\sigma}}{\log Y}\right) \\ &+ O_{\epsilon}(Y^{\alpha-\sigma} \frac{\log T}{\log Y}) \ll_{\epsilon} \frac{Y^{1-\sigma}}{\log Y} + Y^{\alpha-\sigma} \frac{\log T}{\log Y} \ll_{\epsilon} \frac{(\log T)^{(1-\sigma)/(1-\alpha)}}{\log \log T} \end{aligned}$$

on choosing  $Y = (\log T)^{1/(1-\alpha)}$ . This means that

$$(2.9) \quad |\zeta(\sigma + it)| \leq \exp\left(\frac{D(\log t)^{\frac{1-\sigma}{1-\alpha}}}{\log \log t}\right) \quad (t \geq t_0(\epsilon), D > 0)$$

if  $\alpha = \alpha_0 + \epsilon$ ,  $\alpha_0 + 2\epsilon \leq \sigma \leq \sigma_0$ ,  $\alpha_0$  and  $\sigma_0$  are fixed,  $\epsilon > 0$  is a small, positive number,  $D = D(\epsilon)$  and (2.1) holds. If the Riemann hypothesis is true, then one has (2.9) with  $\alpha = \frac{1}{2}$  for  $t \geq t_0$  by Theorem 14.5 of [14], and with more care the foregoing proof could be adapted to give this result also.

Now we shall give an upper bound for  $|\zeta(\sigma + it)|$  in the whole range  $\alpha_0 < \alpha < \sigma \leq 1$ , choosing  $\alpha_0 = 1 - \frac{3A}{\log \log T}$ ,  $\alpha = 1 - \frac{2A}{\log \log T}$ ,  $1 - \frac{A}{\log \log T} \leq \sigma \leq 1$ , where  $A > 0$  is an absolute constant. In that case we obtain from (2.6)

$$\log \zeta(s) = \sum_{n \leq Y} \Lambda_1(n) e^{-n/Y} n^{-s} + O\left(\frac{Y^{1-\sigma}}{\log Y}\right) + O(Y^{-A/\log \log T} \frac{\log T}{\log Y} (\log \log T)^2),$$

$$\log |\zeta(\sigma + it)| \leq |\log \zeta(\sigma + it)| \leq \sum_{p \leq Y} p^{-\sigma} + O(1) + O\left(\frac{Y^{1-\sigma}}{\log Y}\right)$$

$$+ O(Y^{-A/\log \log T} \frac{\log T}{\log Y} (\log \log T)^2) \leq Y^{1-\sigma} (\log \log Y + O(1))$$

$$+ O(Y^{\frac{-A}{\log \log T}} \frac{\log T}{\log Y} (\log \log T)^2).$$

Choose now

$$Y = \exp\left(\frac{2}{A}(\log\log T)^2\right),$$

so that the last  $O$ -term above is bounded (but it depends on  $A$ ). We obtain

$$\log |\zeta(\sigma + it)| \leq \exp\left(\frac{2-2\sigma}{A}(\log\log T)^2\right) \log(C(A)(\log\log T)^2),$$

that is

$$(2.10) \quad |\zeta(\sigma + it)| \leq \exp\left\{\exp\left(\frac{2-2\sigma}{A}(\log\log T)^2\right) \cdot \log(C(A)(\log\log T)^2)\right\}.$$

This bound is valid for  $1 - \frac{A}{\log\log T} \leq \sigma \leq 1$ ,  $C(A) > 0$  a constant depending on  $A$  (whose value could be made explicit), if (2.1) holds with  $T \geq T_0(A)$ .

Consider now the region

$$\mathcal{D} = \left\{s \in \mathbb{C} : \operatorname{Re} s \geq 1 - \frac{A}{\log\log T}, T \leq \operatorname{Im} s \leq 2T\right\}$$

and divide it into subregions

$$\mathcal{D}_k = \{s \in \mathcal{D} : T + (k-1)\log^2 T \leq \operatorname{Im} s < T + k \log^2 T\},$$

where  $k = 1, 2, \dots, [T/\log^2 T]$ . Using the zero-density estimate

$$(2.11) \quad N(\sigma, T) = \sum_{\substack{\rho: \zeta(\rho)=0, \\ \operatorname{Re} \rho \geq \sigma, |\operatorname{Im} \rho| \leq T}} 1 \ll T^{1600(1-\sigma)^{3/2}} \log^{15} T$$

(see Theorem 11.3 of [4]), it is seen that there are

$$\ll \exp\left(\frac{1600A^{3/2}\log T}{(\log\log T)^{3/2}}\right) \log^{15} T \ll \exp\left(\frac{1700A^{3/2}\log T}{(\log\log T)^{3/2}}\right)$$

zeros of  $\zeta(s)$  in  $\mathcal{D}$ . Hence there are at most

$$\exp\left(\frac{1800A^{3/2}\log T}{(\log\log T)^{3/2}}\right)$$

values of  $k$  such that  $\mathcal{E}_k = \mathcal{D}_{k-1} \cup \mathcal{D}_k \cup \mathcal{D}_{k+1}$  contains a zero of  $\zeta(s)$ . Thus



if we construct  $R$  arbitrary points  $t_1, \dots, t_R$  belonging to  $[T, 2T]$  such that  $|t_r - t_s| \geq 1$  for  $r \neq s$  and (2.10) fails to hold for  $t = t_1, \dots, t_R$  and a suitable  $C(A)$ , then each point  $\sigma + it_1, \dots, \sigma + it_R$  must fall into some  $\mathcal{E}_k$  which contains a zero of  $\zeta(s)$ . This provides us with an upper bound for  $R$ , contained in

**THEOREM 1.** Let  $1 - \frac{A}{\log \log T} \leq \sigma \leq 1$  for a constant  $A > 0$ , and let  $R$  be the number of points  $t_r \in [T, 2T]$  such that  $|t_r - t_s| \geq 1$  for  $r \neq s$  and

$$|\zeta(\sigma + it_r)| \geq \exp\left\{\exp\left(\frac{2-2\sigma}{A}(\log \log T)^2\right) \cdot \log(C(A)(\log \log T)^2)\right\}$$

for some suitable  $C(A) > 0$ . Then for  $T \geq T_0(A)$

$$(2.12) \quad R \leq \exp\left(\frac{2000A^{3/2} \log T}{(\log \log T)^{3/2}}\right).$$

Theorem 1 thus provides a large values estimate in the region

$$1 - \frac{A}{\log \log T} \leq \sigma \leq 1,$$

which in a sense complements Ramachandra's bound (1.4). Of course it is possible to obtain a similar type of result for a somewhat different region, but this one is of a particularly simple shape. Moreover in this region we have the bound (2.12), which is much stronger than just  $R \ll_\epsilon T^\epsilon$ . Note also that  $\exp\left(\frac{2-2\sigma}{A}(\log \log T)^2\right) \leq e$  for  $\sigma \geq 1 - \frac{A}{2(\log \log T)^2}$ , so that in the last region  $|\zeta(\sigma + it)| \leq B(A)(\log \log t)^{2e}$  except for a relatively few points.

For  $\sigma = 1$  we have that (2.12) holds for the number of points  $t_r$  for which  $|\zeta(1 + it_r)| \geq C(A)(\log \log T)^2$  for some  $C(A) > 0$ . The same method of proof gives also that  $R \ll_\epsilon T^\epsilon$  holds for any fixed  $\epsilon > 0$  if  $|\zeta(1 + it_r)| \geq D(\epsilon) \log \log T$  for a suitable constant  $D(\epsilon) > 0$ . No information seems obtainable by our method in the case when  $|\zeta(1 + it_r)| \geq f(T)$  and  $f(T) > 0$  is a function which satisfies  $f(T) = o(\log \log T)$  as  $T \rightarrow \infty$ .

If  $\sigma_0 \leq \sigma \leq \sigma_1 < 1$  is fixed, then it follows by the method of proof of Theorem 1 that

$$R \ll_\epsilon T^{1600(1-\sigma)^{3/2} + 2\epsilon}$$

for the number of points  $t_r \in [T, 2T]$  such that  $|t_r - t_s| \geq 1$  for  $r \neq s$  and  $\zeta(\sigma + it_r) \gg T^\epsilon$ . Hence using the bound (see Chapter 6 of [4])

$$\zeta(\sigma + it) \ll t^{100(1-\sigma)^{3/2}} \log^{2/3} t \quad \left(\frac{1}{2} \leq \sigma \leq 1\right)$$

it follows that

$$(2.13) \quad m(\sigma) \geq \frac{1}{100}(1-\sigma)^{-3/2} - 16 \quad (\sigma_0 \leq \sigma \leq \sigma_1 < 1)$$

for any fixed  $\sigma$ ,  $\frac{1}{2} < \sigma_0 < \sigma < 1$ . Since, for  $1 - 3200^{-2/3} \leq \sigma \leq 1$ , we have

$$\frac{1}{100}(1-\sigma)^{-3/2} - 16 \geq \frac{1}{200}(1-\sigma)^{-3/2},$$

it is seen that we obtain an alternative proof of (1.5), with a better value of  $A$ .

### § 3. Large values of zeta-functions of cusp forms near $\sigma = 1$ .

There are several classes of zeta-functions besides  $\zeta(s)$  to which the result of Theorem 1 can be generalized, with appropriate modifications. A property, essential that such a generalization may be made, is the existence of a simple Euler product representation for the zeta-function in question in the region  $\sigma = \text{Re } s > 1$ . One such class is given by Dirichlet functions  $L(s, \chi)$ , where the generalization is obvious and straightforward. More interesting examples appear to be the Dedekind zeta-functions, and the zeta-functions associated with Fourier coefficients of cusp forms, which will be treated in this section. A classical representative is

$$T(s) = \sum_{n=1}^{\infty} \tau(n)n^{-11/2-s} \quad (\text{Re } s > 1),$$

where  $\tau(n)$  is Ramanujan's function, defined by

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\{(1-x)(1-x^2)(1-x^3)\dots\}^{24} \quad (|x| < 1).$$

More generally, let  $a(n)$  be the Fourier coefficients (see e.g. T.M. Apostol

[1]) of a normalized Hecke eigenform (cusp form) of weight  $\kappa$  for the full modular group. Let  $\bar{a}(n) = a(n)n^{-\frac{1}{2}(\kappa-1)}$ , and let

$$(3.1) \quad F(s) = \sum_{n=1}^{\infty} \bar{a}(n)n^{-s} = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \bar{\alpha}_p p^{-s})^{-1} \quad (\operatorname{Re} s > 1)$$

be the zeta-function associated with  $a(n)$ . The zeta-function  $F(s)$  seems more natural than the zeta-function associated with  $a(n)$  directly (i.e. if we had in (3.1)  $a(n)$  and not  $\bar{a}(n)$ ), whose "critical strip" is  $\frac{1}{2}(\kappa - 1) < \operatorname{Re} s < \frac{1}{2}(\kappa + 1)$ . On the other hand, the critical strip for  $F(s)$  is  $0 < \operatorname{Re} s < 1$  as in the case of  $\zeta(s)$ , and the Riemann hypothesis for  $F(s)$  is that all complex zeros  $\rho$  of  $F(s)$  satisfy  $\operatorname{Re} \rho = \frac{1}{2}$ . The zeta-function  $F(s)$  is in many ways similar to  $\zeta^2(s)$ , which is an analogy that is often exploited (see M. Jutila [5]). It is known that the numbers  $\alpha_p$  in (3.1) are of the form  $\alpha_p = e^{i\theta(p)}$ , and  $\theta(p)$  is real by a deep result of P. Deligne [2]. It is precisely the Euler product representation which is important in our problem, namely the investigation of values of  $F(\sigma + it)$  for  $\sigma$  close to 1. Taking the logarithmic derivative in (3.1) we obtain

$$-\frac{F'(s)}{F(s)} = \sum_{n=1}^{\infty} \Lambda_F(n)n^{-s} = \sum_p \sum_{m=1}^{\infty} (\alpha_p^m + \bar{\alpha}_p^m) \frac{\log p}{p^{ms}} \quad (\operatorname{Re} s > 1).$$

Hence equating coefficients it follows that

$$\Lambda_F(n) = \begin{cases} 0, & n = 1, n \neq p^m, \\ (\alpha_p^m + \bar{\alpha}_p^m) \log p, & n = p^m, \end{cases}$$

so that  $\Lambda_F(n)$  is the analogue of the von Mangoldt function  $\Lambda(n)$  for  $F(s)$ . Proceeding as in the case of  $\zeta(s)$  we have

$$(3.2) \quad \log F(s) = \sum_{n=1}^{\infty} \Lambda_{1,F}(n) e^{-n/Y} n^{-s} + O(1) + O\left(\frac{Y^{\alpha-\sigma} \log T}{(\alpha - \alpha_0)} \log \frac{1}{\sigma - \alpha}\right)$$

uniformly for  $s = \sigma + it$ ,  $\alpha_0 < \alpha < \sigma \leq 1$ , provided that (2.1) holds with  $F(s)$  in place of  $\zeta(s)$ , where

$$\Lambda_{1,F}(n) = \begin{cases} 0, & n = 1, n \neq p^m, \\ (\alpha_p^m + \bar{\alpha}_p^m) m^1, & n = p^m. \end{cases}$$

In the course of the proof one needs the fact that the analogue of (2.5) holds

for  $F(s)$ , which follows e.g. from Lemma 3.4 of C.J. Moreno [7]. In the special case when  $\sigma = 1$ ,  $\alpha_0$  is fixed  $\alpha = \frac{1}{2}(1 + \alpha_0)$ , we obtain from (3.2)

$$\log F(1 + it) = \sum_{n=1}^{\infty} \Lambda_{1,F}(n) e^{-n/Y} n^{-1-it} + O(1) + O_{\alpha_0}(Y^{\alpha-1} \frac{\log T}{\log Y}).$$

Hence for  $Y = (\log T)^{1/(1-\alpha)}$

$$\begin{aligned} \log |F(1 + it)| &\leq |\log F(1 + it)| \leq \sum_{n \leq Y} |\Lambda_{1,F}(n)| n^{-1} + O_{\alpha_0}(1) \\ &= \sum_{p \leq Y} |\tilde{a}(p)| p^{-1} + O_{\alpha_0}(1), \end{aligned}$$

since by (3.1)

$$\tilde{a}(p) = \alpha_p + \bar{\alpha}_p = \Lambda_{1,F}(p)$$

and clearly

$$\sum_{n \leq Y, n \neq p} |\Lambda_{1,F}(n)| n^{-1} \ll 1.$$

Now we use the asymptotic formula

$$\sum_{n \leq x} |a(n)|^2 \Lambda(n) = x^{\kappa} + O(x^{\kappa} \exp(-c\sqrt{\log x})) \quad (c > 0),$$

proved by A. Perelli [8] (we could also use e.g. Lemma 2 of M. Ram Murty [12]; this would give (3.3) with  $\log \log t$  replaced by  $(\log \log t)^{1+\epsilon}$ ). This is the analogue of the prime number theorem for modular forms, and gives by partial summation

$$\sum_{p \leq Y} |\tilde{a}(p)|^2 p^{-1} = \sum_{p \leq Y} |a(p)|^2 p^{-\kappa} = \log \log Y + O(1).$$

Hence by the Cauchy-Schwarz inequality

$$\sum_{p \leq Y} |\tilde{a}(p)| p^{-1} \leq \left( \sum_{p \leq Y} |\tilde{a}(p)|^2 p^{-1} \right)^{\frac{1}{2}} \left( \sum_{p \leq Y} p^{-1} \right)^{\frac{1}{2}} = \log \log Y + O(1),$$

since

$$\sum_{p \leq Y} p^{-1} = \log \log Y + O(1).$$

Therefore

$$\log |F(1+it)| \leq \log \log \log T + O_{\alpha_0}(1) \quad (T \leq t \leq 2T),$$

$$(3.3) \quad F(1+it) \ll_{\alpha_0} \log \log t,$$

provided that (2.1) holds for  $F(s)$ . In particular, (3.3) is then true if the Riemann hypothesis for  $F(s)$  holds.

The arguments that yield Theorem 1 will work also in the case of  $F(s)$ . The analogue of (2.11) can be obtained for  $F(s)$ , but the sketch of proof of this result would lead us too much astray. Thus the only noteworthy change in the proof is that we shall use the zero-density estimate

$$(3.4) \quad N_F(\sigma, T) = \sum_{\rho: F(\rho)=0, \operatorname{Re} \rho \geq \sigma, |\operatorname{Im} \rho| \leq T} 1 \ll T^{3-3\sigma} \log^C T$$

which certainly holds for  $\frac{19}{20} \leq \sigma \leq 1$  and some  $C > 0$ . One can easily obtain (3.4) by using the techniques developed for  $\zeta(s)$  in Chapter 11 of [4] and the estimate

$$\int_1^T |F(\frac{1}{2} + it)|^2 dt \ll T \log T.$$

This bound follows on representing  $F(\frac{1}{2} + it)$  as a sum of Dirichlet polynomials of length  $\ll t$ , and then using the mean value theorem for Dirichlet polynomials (see Chapter 5 of [4]). A sharp asymptotic formula for the integral in question is established by A. Good [3]. In this way we obtain

**THEOREM 2.** Let  $1 - \frac{A}{\log \log T} \leq \sigma \leq 1$  for a constant  $A > 0$ , and let  $R$  be the number of points  $t_r \in [T, 2T]$  such that  $|t_r - t_s| \geq 1$  for  $r \neq s$  and

$$(3.5) \quad |F(\sigma + it_r)| \geq \exp\left\{\exp\left(\frac{2-2\sigma}{A}(\log \log T)^2\right) \cdot \log(C(A)(\log \log T)^2)\right\}$$

for a suitable constant  $C(A) > 0$ . Then for  $T \geq T_0(A)$

$$(3.6) \quad R \leq \exp\left(\frac{3A \log T}{\log \log T}\right).$$

Actually (3.6) can be replaced by the same type of bound as (2.12). To obtain this, it is necessary to show how the analogue of (2.11) holds for the zero-counting function of  $F(s)$ . We hope to return to this question elsewhere.

§ 4. Large values of the Dedekind zeta-function near  $\sigma = 1$ .

We shall sketch now how the analogue of Theorem 2 may be established for the Dedekind zeta-function

$$\zeta_K(s) = \sum_{n=1}^{\infty} H(n)n^{-s} \quad (\sigma = \operatorname{Re} s > 1)$$

of an algebraic number field  $K$  such that  $[K : Q] = N$ . Here  $H(n)$  denotes the number of non-zero integral ideals of  $K$  with norm equal to  $n$ . From the theory of algebraic number fields (see e.g. D.A. Marcus [6], Theorem 21 and Theorem 24) it is known that

$$(p) = \prod_i \mathfrak{p}_i^{e_i}, N\mathfrak{p}_i = p^{f_i}, \sum_i e_i f_i = N,$$

and  $e_i = 1$  for almost all primes  $p$ . Let  $P_0$  be the finite set of primes which have some  $e_i > 1$ . Factorising the polynomials  $X^{f_i} - 1$  we obtain, for  $\operatorname{Re} s > 1$ ,

$$\begin{aligned} \zeta_K(s) &= \prod_{\mathfrak{p}} (1 - (N\mathfrak{p})^{-s})^{-1} = \prod_p \prod_{\mathfrak{p}|(p)} (1 - (N\mathfrak{p})^{-s})^{-1} \\ &= \prod_{p \notin P_0} \prod_{j=1}^N (1 - \chi_j(p)p^{-s})^{-1} \prod_{p \in P_0} \prod_{j=1}^{N_p} (1 - \chi_j(p)p^{-s})^{-1} \end{aligned}$$

where  $|\chi_j(p)| = 1$  and  $N_p < N$ . If  $p \in P_0$ , then for  $N_p < j \leq N$  we define  $\chi_j(p) = 0$ . With this notation it follows that

$$(4.1) \quad \zeta_K(s) = \prod_{j=1}^N \prod_p (1 - \chi_j(p)p^{-s})^{-1} \quad (|\chi_j(p)| \leq 1, \operatorname{Re} s > 1).$$

Hence if we define  $\Lambda_K(n)$ , the analogue of  $\Lambda(n)$  for  $\zeta_K(s)$ , by

$$(4.2) \quad -\frac{\zeta'_K(s)}{\zeta_K(s)} = \sum_{n=1}^{\infty} \Lambda_K(n) n^{-s} \quad (\operatorname{Re} s > 1),$$

then taking the logarithmic derivative of (4.1) and comparing with (4.2) we obtain

$$\Lambda_K(n) = \begin{cases} 0, & n = 1, n \neq p^m, \\ N \sum_{j=1}^m \chi_j^m(p) \log p, & n = p^m. \end{cases}$$

Thus  $0 \leq |\Lambda_K(n)| \leq N\Lambda(n)$ , and in several ways  $\zeta_K(s)$  is analogous to  $\zeta^N(s)$ . The analysis made for  $F(s)$  in § 3 can be carried over to  $\zeta_K(s)$  with obvious modifications in the proof. For example, (3.3) will become

$$(4.3) \quad \zeta_K(1+it) \ll_{N, \alpha_0} (\log \log t)^N$$

provided that (2.1) holds with  $\zeta_K(s)$  in place of  $\zeta(s)$ . However, using the prime ideal theorem for algebraic number fields in the form

$$\sum_{Np \leq x} 1 = \int_2^x \frac{dt}{\log t} + O(x \exp(-c\sqrt{\log x})) \quad (c > 0),$$

one obtains by partial summation

$$\sum_{Np \leq x} (Np)^{-1} = \log \log x + O(1).$$

This in turn gives a sharpening of (4.3), namely

$$\zeta_K(1+it) \ll_{\alpha_0} \log \log t.$$

In this case instead of Theorem 2 we shall obtain

**THEOREM 3.** Let  $1 - \frac{A}{\log \log T} \leq \sigma \leq 1$  for a constant  $A > 0$ , and let  $R$  be the number of points  $t_r \in [T, 2T]$  such that  $|t_r - t_s| \geq 1$  for  $r \neq s$  and

$$|\zeta_K(\sigma + it_r)| \geq \exp\left\{\exp\left(\frac{2-2\sigma}{A} (\log \log T)^2\right) \cdot \log(B(\log \log T)^2)\right\}$$

for some constant  $B = B(A, N, K) > 0$ . Then for  $T \geq T_0(A, N, K)$  and a suitable constant  $D = D(A, N, K) > 0$  we have

$$(4.4) \quad R \leq \exp\left(\frac{D \log T}{\log \log T}\right).$$

A similar type of result could be obtained if, instead of the analogue of (3.4) which yields (4.4), we use the zero-density estimate of W. Staś [13] for  $\zeta_K(s)$ . As remarked in the Introduction, the whole approach is fairly general and can be used to deal with many other zeta-functions. For example, analogous results may be readily obtained for  $L(s, \chi)$  and  $L_{K, \mathcal{F}}(s, \chi)$ , where  $\chi$  is a character mod  $q$ , if we do not insist on uniformity in  $q$  etc.



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