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A REMARK ON $\zeta(1 + it)$

K. RAMACHANDRA

§1. INTRODUCTION.

It is well-known that $\zeta(\sigma + it) \neq 0$ for $\sigma \geq 1$. Let us consider $\log \zeta(1 + it)$ for $t \geq 1000$. The object of this note is to prove the following theorem.

THEOREM. *Let $T \geq 1000$. Put $X = \text{Exp}(\frac{\log \log T}{\log \log \log T})$. Consider any set of disjoint open intervals I each of length $\frac{1}{X}$ all contained in the interval $T \leq t \leq T + e^X$. Let ε be any positive constant not exceeding 1. Then with the exception of K intervals I (where K depends only on ε) we have*

$$|\log \zeta(1 + it)| \leq \varepsilon \log \log T.$$

REMARK 1. This note has its origin in the concluding result in the Appendix to [1].

REMARK 2. The proof depends on the inequalities $|\alpha + \sum_{\alpha} p^{-i\alpha}|^2 \geq 0$ and $|a i + \sum_{\alpha} p^{-i\alpha}|^2 \geq 0$, where a is any real number and α runs over a finite set of distinct real numbers. By replacing $\sum_{\alpha} p^{-i\alpha}$ by $\chi(p) \sum_{\alpha} p^{-i\alpha}$ where χ is a residue class character we can work out the analogues of the theorem stated above for $\log L(1 + it, \chi)$, where $L(s, \chi)$ denotes the L -function defined with respect to χ . In this case we can (instead of $T \leq t \leq T + e^X$) consider for example the interval $0 \leq t \leq 1$. However we reserve these analogues and other generalisations to number fields and so on for another paper.

REMARK 3. Letting α run over complex numbers $\alpha + i\beta$ with $\beta \leq \beta_0 = (\log T)^{-\mu}$ where μ is a constant $> \frac{2}{3}$, we can prove the following result. Let ε be a constant satisfying $0 < \varepsilon < 1$. Let $T \geq T_0(\mu, \varepsilon)$. Let X be as in the theorem and I as in the theorem. Let $J = [1 - \beta_0, \infty) \times I$ be the Cartesian product of the σ interval $[1 - \beta_0, \infty)$ and the t interval I . Then with the exception of K rectangles J (where K depends only on μ and ε) we have, for s in J

$$|\log \zeta(s)| \leq \varepsilon \log \log T.$$

We postpone details of this result and refinements to a later paper. * We can take $\beta_0 = A(\log T)^{-\mu}(\log \log T)^{-2\mu}$ where $\mu = \frac{2}{3}$ and A is any positive constant.

§ 2. **PROOF OF THE THEOREM.** We begin by remarking that in the course of the proof we give in some ways better results for $|\log |\zeta(1+it)|$ and $|\arg \zeta(1+it)|$. We combine these two results to get an upper bound for $|\log \zeta(1+it)|$. Throughout this section k will be a fixed positive integer; α 's will denote k distinct real numbers satisfying $T \leq \alpha \leq T + e^X$. We write $Y = \text{Exp}(10^{10}(\log T)^\lambda(\log \log T)^{10})$. Here $\lambda = 1$ if we assume the trivial zero-free region $\sigma \geq 1 - \frac{c}{\log t}$ and $\lambda = \frac{2}{3}$ if we assume the Vinogradov zero-free region $\sigma \geq 1 - \frac{c}{(\log t)^{2/3}(\log \log t)^{1/3}}$, ($t \geq 1000$); p will denote a prime and a a real constant to be chosen later. We prove the theorem by a series of lemmas.

LEMMA 1. *We have,*

$$0 \leq |a + \sum_{\alpha} p^{-i\alpha}|^2 = a^2 + k + 2a \sum_{\alpha} \text{Re}(p^{-i\alpha}) + \sum_{\alpha \neq \alpha'} \sum_{\alpha'} p^{-i(\alpha-\alpha')} \quad (1)$$

and

$$0 \leq |ai + \sum_{\alpha} p^{-i\alpha}|^2 = a^2 + k + 2a \sum_{\alpha} \text{Im}(p^{-i\alpha}) + \sum_{\alpha \neq \alpha'} \sum_{\alpha'} p^{-i(\alpha-\alpha')} \quad (2)$$

PROOF. Trivial.

LEMMA 2. *Let*

$$S_1 = \sum_p \frac{1}{p} e^{-\frac{p}{T}}, S_2 = \sum_p p^{-i\alpha-1} e^{-\frac{p}{T}}$$

and

$$S_3 = \sum_p p^{-i(\alpha-\alpha')-1} e^{-\frac{p}{T}}. \text{ Then}$$

$$0 \leq (a^2 + k)S_1 + 2a \sum_{\alpha} \text{Re}(S_2) + \sum_{\alpha \neq \alpha'} \sum_{\alpha'} S_3. \quad (3)$$

PROOF. Follows from the first part of Lemma 1.

LEMMA 3. *We have*

$$S_1 = \log \log Y + O(1). \quad (4)$$

PROOF. Using

$$e^{-\frac{Y}{p}} \begin{cases} \leq \frac{Y}{p} & \text{for } p \geq Y \\ = 1 + O(\frac{Y}{p}) & \text{for } p \leq Y \end{cases}$$

the lemma follows on using the well-known result $\sum_{p \leq Y} \frac{1}{p} = \log \log Y + O(1)$.

LEMMA 4. We have,

$$S_2 = \log \zeta(1 + i\alpha) + O(1). \quad (5)$$

PROOF. Since, $\sigma \geq \frac{3}{4}$, the prime powers contribute $O(1)$ to the series for $\log \zeta(s)$, we have

$$S_2 = \frac{1}{2\pi i} \int_{Re(w)=2} \log \zeta(1 + i\alpha + w) Y^w \Gamma(w) dw + O(1). \quad (6)$$

Put $L = 2(\log T)c^{-1}$ or $2(\log T)^{2/3}(\log \log T)^{1/3}c^{-1}$. We assume the zero free-region $\sigma \geq 1 - \frac{1}{L}$, $|t| \leq 2T$ (which comes from the two zero-free regions already referred to). By a simple application of the Borel-Carathéodory theorem (see p.282 of [2]; see also p. 174 of [3]) we have $\log \zeta(s) = O((\log T)^2)$ in $\sigma \geq 1 - \frac{1}{2L}$, $T - (\log T)^2 \leq t \leq T + e^X + (\log T)^2$. (In fact we can get better estimates, but we do not need these). We deform the Contour $Re(w) = 2$ as follows. Let $\delta = \frac{1}{2L}$, $Q_1 = 2 - i\infty$, $Q_2 = 2 - i(T - (\log T)^2)$, $Q_3 = -\delta - i(T - (\log T)^2)$, $Q_4 = -\delta + i(T + e^X + (\log T)^2)$, $Q_5 = 2 + i(T + e^X + (\log T)^2)$, $Q_6 = 2 + i\infty$. We deform the Contour $Re(w) = 2$ to be the path formed by the straight line segments joining the points $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$ in this order. The pole $w = 0$ gives the residue $\log \zeta(1 + i\alpha)$. Since $\Gamma(w) = O(e^{-|Imw|})$ and $Y^\delta > (\log T)^{30}$, Lemma 4 follows by taking the upper bound for the integrals along the new path, obtained by replacing the integrand by its absolute value.

LEMMA 5. Let $\tau = \alpha - \alpha'$. Then, we have,

$$(a) S_3 = \log \zeta(1 + i\tau) + O(1), \text{ if } |\tau| \geq (\log T)^{10}; \quad (7)$$

$$(b) |S_3| \leq \log \log Z + O(1) + O\left(\frac{1}{|\tau| \log Z}\right) \text{ otherwise, where } Z \text{ is any number satisfying } \text{Exp}(100(\log \log T)^2) \leq Z \leq Y. \quad (8)$$

PROOF. The part (a) follows by an argument similar to that of Lemma 4. We prove (b) as follows. By using the inequalities used in the proof of Lemma 3 we have

$$S_3 = \sum_{p \leq Y} p^{-i\tau-1} + O(1) = \sum_{p \leq Z} p^{-i\tau-1} + \sum_{Z < p \leq Y} p^{-i\tau-1} + O(1). = S_4 + S_5 + O(1).$$

say.

Clearly $|S_4| \leq \log \log Z + O(1)$, and

$$S_5 = \int_Z^Y u^{-i\tau-1} d\pi(u) = I_1 + I_2$$

where

$$I_1 = \int_Z^Y u^{-i\tau-1} \frac{du}{\log u} \quad \text{and} \quad I_2 = \int_Z^Y u^{-i\tau-1} dE(u)$$

where $E(u) = \pi(u) - \int_2^u \frac{du}{\log u}$. Using the mean value theorem for integrals we see that $I_1 = O\left(\frac{1}{|\tau| \log Z}\right)$. Now

$$\begin{aligned} I_2 &= u^{-i\tau-1} E(u) \Big|_Z^Y + \int_Z^Y (i\tau + 1) u^{-i\tau-2} E(u) du \\ &= O(1) + O((\log T)^{10} \int_Z^Y u^{-1} e^{-2\sqrt{\log u}} du) \end{aligned}$$

by using the fact that $E(u) = O(u e^{-2\sqrt{\log u}})$. This completes the proof of (b).

LEMMA 6. *Let $100(\log \log T)^2 \leq X \leq \log Y$, $Z = e^X$, and $X^{-1} \leq |\tau| \leq e^X$. Then*

$$0 \leq (a^2 + k) \log \log Y + 2a \sum_{\alpha} \operatorname{Re}(\log \zeta(1 + i\alpha)) + O(\log X). \quad (9)$$

PROOF. Follows from Lemmas 2 to 5 on observing that $\log \zeta(1 + it) = O(\log \log t)$ for $t \geq 100$.

We now come to the proof of the theorem.

LEMMA 7. *In the theorem consider the set of alternate intervals. Out of these fix $k - 1$ intervals I for which minimum of $\operatorname{Re}(\log \zeta(1 + i\alpha))$ are successively as small as possible. If J_1 denotes $\operatorname{Re}(\log \zeta(1 + i\alpha))$ for α in any one of the remaining intervals then, we have with a $a > 0$,*

$$0 \leq (a^2 + k) \log \log Y + 2akJ_1 + O(\log X) \quad (10)$$

and so

$$J_1 \geq -\frac{1}{2k} \left(a + \frac{k}{a}\right) \log \log Y + O(\log X). \quad (11)$$

COROLLARY. *With the exception of $2(k - 1)$ intervals, we have,*

$$J_1 \geq (-k^{-1/2} - \epsilon) \log \log Y \quad (12)$$

for $T \geq T_0(k, \epsilon)$.

PROOF. Lemma 7 follows from Lemma 6 and the Corollary follows by taking $a = k^{1/2}$ and considering the other set of alternate intervals also.

LEMMA 8. *In the theorem consider one set of alternate intervals. Out of these fix $k - 1$ intervals I for which the maximum of $\text{Re}(\log \zeta(1 + i\alpha))$ are successively as large as possible. If J_2 denotes $\text{Re}(\log \zeta(1 + i\alpha))$ for α in any one of the remaining intervals then, we have, with $a < 0$,*

$$-2akJ_2 \leq (a^2 + k)\log \log Y + O(\log X) \quad (13)$$

and so

$$J_2 \leq \frac{1}{2k} \left(-a - \frac{k}{a}\right) \log \log Y + O(\log X). \quad (14)$$

COROLLARY. *With the exception of $2(k - 1)$ intervals, we have,*

$$J_2 \leq (k^{-1/2} + \epsilon) \log \log Y \quad (15)$$

for $T \geq T_0(k, \epsilon)$.

PROOF. Lemma 8 follows from Lemma 6, and the Corollary follows by taking $a = -k^{1/2}$ and considering the other set of alternate intervals also.

LEMMA 9. *With the exception of $4(k - 1)$ intervals I we have,*

$$|\text{Re}(\log \zeta(1 + it))| \leq (k^{-1/2} + \epsilon) \log \log Y \quad (16)$$

for $T \geq T_0(k, \epsilon)$.

PROOF. Follows from the Corollaries to Lemmas 7 and 8.

LEMMA 10. *With the exception of $4(k - 1)$ intervals I we have,*

$$|\text{Im}(\log \zeta(1 + it))| \leq (k^{-1/2} + \epsilon) \log \log Y \quad (17)$$

for $T \geq T_0(k, \epsilon)$.

PROOF. Just as we deduced Lemma 9 starting from the inequality (1) by a sequence of lemmas, we can deduce Lemma 10 from the inequality (2).

LEMMA 11. *With the exception of $8(k - 1)$ intervals I we have,*

$$|\log \zeta(1 + it)| \leq \left(\sqrt{\frac{2}{k}} + \epsilon\right) \log \log Y \quad (18)$$

for $T \geq T_0(k, \epsilon)$.

PROOF. Follows from Lemmas 9 and 10.

Since k is an arbitrary positive integer constant and ϵ any positive constant, Lemma 11 completes the proof of the theorem stated in the introduction.

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REFERENCES

- (1) K. Ramachandra, *Mean-value of the Riemann zeta-function and other Remarks-II*, International conference on analytic number theory (Vinogradov's 90th birthday celebrations), Steklov Institute, Moscow (14th to 19th September 1981); Trudy Mats Inst. Steklova 163 (1984), 200-204; Proceedings of the Steklov Institute of Mathematics (1985) Issue 4, p. 233-237.
- (2) E.C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford, Clarendon Press 1951.
- (3) E.C. Titchmarsh, *The theory of functions*, Oxford University Press, (1939).

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