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## A REMARK ON $\zeta(1 + it)$

K. RAMACHANDRA

### §1. INTRODUCTION.

It is well-known that  $\zeta(\sigma + it) \neq 0$  for  $\sigma \geq 1$ . Let us consider  $\log \zeta(1 + it)$  for  $t \geq 1000$ . The object of this note is to prove the following theorem.

**THEOREM.** *Let  $T \geq 1000$ . Put  $X = \text{Exp}(\frac{\log \log T}{\log \log \log T})$ . Consider any set of disjoint open intervals  $I$  each of length  $\frac{1}{X}$  all contained in the interval  $T \leq t \leq T + e^X$ . Let  $\varepsilon$  be any positive constant not exceeding 1. Then with the exception of  $K$  intervals  $I$  (where  $K$  depends only on  $\varepsilon$ ) we have*

$$|\log \zeta(1 + it)| \leq \varepsilon \log \log T.$$

**REMARK 1.** This note has its origin in the concluding result in the Appendix to [1].

**REMARK 2.** The proof depends on the inequalities  $|\alpha + \sum_{\alpha} p^{-i\alpha}|^2 \geq 0$  and  $|a i + \sum_{\alpha} p^{-i\alpha}|^2 \geq 0$ , where  $a$  is any real number and  $\alpha$  runs over a finite set of distinct real numbers. By replacing  $\sum_{\alpha} p^{-i\alpha}$  by  $\chi(p) \sum_{\alpha} p^{-i\alpha}$  where  $\chi$  is a residue class character we can work out the analogues of the theorem stated above for  $\log L(1 + it, \chi)$ , where  $L(s, \chi)$  denotes the  $L$ -function defined with respect to  $\chi$ . In this case we can (instead of  $T \leq t \leq T + e^X$ ) consider for example the interval  $0 \leq t \leq 1$ . However we reserve these analogues and other generalisations to number fields and so on for another paper.

**REMARK 3.** Letting  $\alpha$  run over complex numbers  $\alpha + i\beta$  with  $\beta \leq \beta_0 = (\log T)^{-\mu}$  where  $\mu$  is a constant  $> \frac{2}{3}$ , we can prove the following result. Let  $\varepsilon$  be a constant satisfying  $0 < \varepsilon < 1$ . Let  $T \geq T_0(\mu, \varepsilon)$ . Let  $X$  be as in the theorem and  $I$  as in the theorem. Let  $J = [1 - \beta_0, \infty) \times I$  be the Cartesian product of the  $\sigma$  interval  $[1 - \beta_0, \infty)$  and the  $t$  interval  $I$ . Then with the exception of  $K$  rectangles  $J$  (where  $K$  depends only on  $\mu$  and  $\varepsilon$ ) we have, for  $s$  in  $J$

$$|\log \zeta(s)| \leq \varepsilon \log \log T.$$

We postpone details of this result and refinements to a later paper. \* We can take  $\beta_0 = A(\log T)^{-\mu}(\log \log T)^{-2\mu}$  where  $\mu = \frac{2}{3}$  and  $A$  is any positive constant.

§ 2. **PROOF OF THE THEOREM.** We begin by remarking that in the course of the proof we give in some ways better results for  $|\log |\zeta(1+it)||$  and  $|\arg \zeta(1+it)|$ . We combine these two results to get an upper bound for  $|\log \zeta(1+it)|$ . Throughout this section  $k$  will be a fixed positive integer;  $\alpha$ 's will denote  $k$  distinct real numbers satisfying  $T \leq \alpha \leq T + e^X$ . We write  $Y = \text{Exp}(10^{10}(\log T)^\lambda(\log \log T)^{10})$ . Here  $\lambda = 1$  if we assume the trivial zero-free region  $\sigma \geq 1 - \frac{c}{\log t}$  and  $\lambda = \frac{2}{3}$  if we assume the Vinogradov zero-free region  $\sigma \geq 1 - \frac{c}{(\log t)^{2/3}(\log \log t)^{1/3}}$ , ( $t \geq 1000$ );  $p$  will denote a prime and  $a$  a real constant to be chosen later. We prove the theorem by a series of lemmas.

**LEMMA 1.** *We have,*

$$0 \leq |a + \sum_{\alpha} p^{-i\alpha}|^2 = a^2 + k + 2a \sum_{\alpha} \text{Re}(p^{-i\alpha}) + \sum_{\alpha \neq \alpha'} p^{-i(\alpha-\alpha')} \quad (1)$$

and

$$0 \leq |ai + \sum_{\alpha} p^{-i\alpha}|^2 = a^2 + k + 2a \sum_{\alpha} \text{Im}(p^{-i\alpha}) + \sum_{\alpha \neq \alpha'} p^{-i(\alpha-\alpha')} \quad (2)$$

**PROOF.** Trivial.

**LEMMA 2.** *Let*

$$S_1 = \sum_p \frac{1}{p} e^{-\frac{p}{T}}, S_2 = \sum_p p^{-i\alpha-1} e^{-\frac{p}{T}}$$

and

$$S_3 = \sum_p p^{-i(\alpha-\alpha')-1} e^{-\frac{p}{T}}. \text{ Then}$$

$$0 \leq (a^2 + k)S_1 + 2a \sum_{\alpha} \text{Re}(S_2) + \sum_{\alpha \neq \alpha'} S_3. \quad (3)$$

**PROOF.** Follows from the first part of Lemma 1.

**LEMMA 3.** *We have*

$$S_1 = \log \log Y + O(1). \quad (4)$$

**PROOF.** Using

$$e^{-\frac{Y}{p}} \begin{cases} \leq \frac{Y}{p} & \text{for } p \geq Y \\ = 1 + O(\frac{Y}{p}) & \text{for } p \leq Y \end{cases}$$

the lemma follows on using the well-known result  $\sum_{p \leq Y} \frac{1}{p} = \log \log Y + O(1)$ .

**LEMMA 4.** *We have,*

$$S_2 = \log \zeta(1 + i\alpha) + O(1). \quad (5)$$

**PROOF.** Since,  $\sigma \geq \frac{3}{4}$ , the prime powers contribute  $O(1)$  to the series for  $\log \zeta(s)$ , we have

$$S_2 = \frac{1}{2\pi i} \int_{Re(w)=2} \log \zeta(1 + i\alpha + w) Y^w \Gamma(w) dw + O(1). \quad (6)$$

Put  $L = 2(\log T)c^{-1}$  or  $2(\log T)^{2/3}(\log \log T)^{1/3}c^{-1}$ . We assume the zero free-region  $\sigma \geq 1 - \frac{1}{L}$ ,  $|t| \leq 2T$  (which comes from the two zero-free regions already referred to). By a simple application of the Borel-Carathéodory theorem (see p.282 of [2]; see also p. 174 of [3]) we have  $\log \zeta(s) = O((\log T)^2)$  in  $\sigma \geq 1 - \frac{1}{2L}$ ,  $T - (\log T)^2 \leq t \leq T + e^X + (\log T)^2$ . (In fact we can get better estimates, but we do not need these). We deform the Contour  $Re(w) = 2$  as follows. Let  $\delta = \frac{1}{2L}$ ,  $Q_1 = 2 - i\infty$ ,  $Q_2 = 2 - i(T - (\log T)^2)$ ,  $Q_3 = -\delta - i(T - (\log T)^2)$ ,  $Q_4 = -\delta + i(T + e^X + (\log T)^2)$ ,  $Q_5 = 2 + i(T + e^X + (\log T)^2)$ ,  $Q_6 = 2 + i\infty$ . We deform the Contour  $Re(w) = 2$  to be the path formed by the straight line segments joining the points  $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$  in this order. The pole  $w = 0$  gives the residue  $\log \zeta(1 + i\alpha)$ . Since  $\Gamma(w) = O(e^{-|Imw|})$  and  $Y^\delta > (\log T)^{30}$ , Lemma 4 follows by taking the upper bound for the integrals along the new path, obtained by replacing the integrand by its absolute value.

**LEMMA 5.** *Let  $\tau = \alpha - \alpha'$ . Then, we have,*

$$(a) S_3 = \log \zeta(1 + i\tau) + O(1), \text{ if } |\tau| \geq (\log T)^{10}; \quad (7)$$

$$(b) |S_3| \leq \log \log Z + O(1) + O\left(\frac{1}{|\tau| \log Z}\right) \text{ otherwise, where } Z \text{ is any number satisfying } \text{Exp}(100(\log \log T)^2) \leq Z \leq Y. \quad (8)$$

**PROOF.** The part (a) follows by an argument similar to that of Lemma 4. We prove (b) as follows. By using the inequalities used in the proof of Lemma 3 we have

$$S_3 = \sum_{p \leq Y} p^{-i\tau-1} + O(1) = \sum_{p \leq Z} p^{-i\tau-1} + \sum_{Z < p \leq Y} p^{-i\tau-1} + O(1). = S_4 + S_5 + O(1).$$

say.

Clearly  $|S_4| \leq \log \log Z + O(1)$ , and

$$S_5 = \int_Z^Y u^{-i\tau-1} d\pi(u) = I_1 + I_2$$

where

$$I_1 = \int_Z^Y u^{-i\tau-1} \frac{du}{\log u} \quad \text{and} \quad I_2 = \int_Z^Y u^{-i\tau-1} dE(u)$$

where  $E(u) = \pi(u) - \int_2^u \frac{du}{\log u}$ . Using the mean value theorem for integrals we see that  $I_1 = O\left(\frac{1}{|\tau| \log Z}\right)$ . Now

$$\begin{aligned} I_2 &= u^{-i\tau-1} E(u) \Big|_Z^Y + \int_Z^Y (i\tau + 1) u^{-i\tau-2} E(u) du \\ &= O(1) + O((\log T)^{10} \int_Z^Y u^{-1} e^{-2\sqrt{\log u}} du) \end{aligned}$$

by using the fact that  $E(u) = O(u e^{-2\sqrt{\log u}})$ . This completes the proof of (b).

**LEMMA 6.** *Let  $100(\log \log T)^2 \leq X \leq \log Y$ ,  $Z = e^X$ , and  $X^{-1} \leq |\tau| \leq e^X$ . Then*

$$0 \leq (a^2 + k) \log \log Y + 2a \sum_{\alpha} \operatorname{Re}(\log \zeta(1 + i\alpha)) + O(\log X). \quad (9)$$

**PROOF.** Follows from Lemmas 2 to 5 on observing that  $\log \zeta(1 + it) = O(\log \log t)$  for  $t \geq 100$ .

We now come to the proof of the theorem.

**LEMMA 7.** *In the theorem consider the set of alternate intervals. Out of these fix  $k - 1$  intervals  $I$  for which minimum of  $\operatorname{Re}(\log \zeta(1 + i\alpha))$  are successively as small as possible. If  $J_1$  denotes  $\operatorname{Re}(\log \zeta(1 + i\alpha))$  for  $\alpha$  in any one of the remaining intervals then, we have with  $a > 0$ ,*

$$0 \leq (a^2 + k) \log \log Y + 2akJ_1 + O(\log X) \quad (10)$$

and so

$$J_1 \geq -\frac{1}{2k} \left(a + \frac{k}{a}\right) \log \log Y + O(\log X). \quad (11)$$

**COROLLARY.** *With the exception of  $2(k - 1)$  intervals, we have,*

$$J_1 \geq (-k^{-1/2} - \epsilon) \log \log Y \quad (12)$$

for  $T \geq T_0(k, \epsilon)$ .

**PROOF.** Lemma 7 follows from Lemma 6 and the Corollary follows by taking  $a = k^{1/2}$  and considering the other set of alternate intervals also.

**LEMMA 8.** *In the theorem consider one set of alternate intervals. Out of these fix  $k - 1$  intervals  $I$  for which the maximum of  $\operatorname{Re}(\log \zeta(1 + i\alpha))$  are successively as large as possible. If  $J_2$  denotes  $\operatorname{Re}(\log \zeta(1 + i\alpha))$  for  $\alpha$  in any one of the remaining intervals then, we have, with  $a < 0$ ,*

$$-2akJ_2 \leq (a^2 + k)\log \log Y + O(\log X) \quad (13)$$

and so

$$J_2 \leq \frac{1}{2k} \left(-a - \frac{k}{a}\right) \log \log Y + O(\log X). \quad (14)$$

**COROLLARY.** *With the exception of  $2(k - 1)$  intervals, we have,*

$$J_2 \leq (k^{-1/2} + \epsilon) \log \log Y \quad (15)$$

for  $T \geq T_0(k, \epsilon)$ .

**PROOF.** Lemma 8 follows from Lemma 6, and the Corollary follows by taking  $a = -k^{1/2}$  and considering the other set of alternate intervals also.

**LEMMA 9.** *With the exception of  $4(k - 1)$  intervals  $I$  we have,*

$$|\operatorname{Re}(\log \zeta(1 + it))| \leq (k^{-1/2} + \epsilon) \log \log Y \quad (16)$$

for  $T \geq T_0(k, \epsilon)$ .

**PROOF.** Follows from the Corollaries to Lemmas 7 and 8.

**LEMMA 10.** *With the exception of  $4(k - 1)$  intervals  $I$  we have,*

$$|\operatorname{Im}(\log \zeta(1 + it))| \leq (k^{-1/2} + \epsilon) \log \log Y \quad (17)$$

for  $T \geq T_0(k, \epsilon)$ .

**PROOF.** Just as we deduced Lemma 9 starting from the inequality (1) by a sequence of lemmas, we can deduce Lemma 10 from the inequality (2).

**LEMMA 11.** *With the exception of  $8(k - 1)$  intervals  $I$  we have,*

$$|\log \zeta(1 + it)| \leq \left(\sqrt{\frac{2}{k}} + \epsilon\right) \log \log Y \quad (18)$$

for  $T \geq T_0(k, \epsilon)$ .

**PROOF.** Follows from Lemmas 9 and 10.

Since  $k$  is an arbitrary positive integer constant and  $\varepsilon$  any positive constant, Lemma 11 completes the proof of the theorem stated in the introduction.

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## REFERENCES

- (1) K. Ramachandra, *Mean-value of the Riemann zeta-function and other Remarks-II*, International conference on analytic number theory (Vinogradov's 90th birthday celebrations), Steklov Institute, Moscow (14th to 19th September 1981); Trudy Mats Inst. Steklova 163 (1984), 200-204; Proceedings of the Steklov Institute of Mathematics (1985) Issue 4, p. 233-237.
- (2) E.C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford, Clarendon Press 1951.
- (3) E.C. Titchmarsh, *The theory of functions*, Oxford University Press, (1939).

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