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ON THE EQUATION

$$a(x^m - 1) / (x - 1) = b(y^n - 1) / (y - 1) \quad (II)$$

By

T. N. SHOREY

§ 1. Let $m > 1, n > 1, x > 1, y > 1$ and a, b with $(a, b) = 1$ be positive integers satisfying $a(y-1) \neq b(x-1)$. This equation of Goormaghtigh arose from the question whether an integer has all the digits identically equal in their expansions to two distinct bases. It follows from Baker's effective version [1] of Thue's theorem [6] that the equation

$$(1) \quad a \frac{x^m - 1}{x - 1} = b \frac{y^n - 1}{y - 1}$$

implies that $\max(m, n)$ is bounded by an effectively computable number depending only on a, b, x and y . Further Balasubramanian and the author [3] applied the theory of linear forms in logarithms to generalise this result by showing that equation (1) implies that $\max(a, b, x, y, m, n)$ is bounded by an effectively computable number depending only on the greatest prime factor of $abxy$. In this paper, we apply the theory of linear forms in logarithms to obtain the following generalisations. We shall always write $z = \max(x, y)$.

Theorem 1.

Let $0 < \theta < 1$ and $F > 1$. If positive integers $m > n > 1, x > 1, y > 1, a$ and b with $a < x, b < y, a(y-1) \neq b(x-1)$ and

$$(2) \quad |x - y| \leq \max(F, z^\theta)$$

satisfy (1), then m is bounded by an effectively computable number depending only on θ and F .

For an account of earlier results in the direction of equation (1), see [3]. We combine theorem 1 with an elementary argument to obtain the following result.

Theorem 2.

Let $F_1 > 1$. There exists an effectively computable absolute constant $C > 0$ and an effectively computable number $C_1 > 0$ depending only on F_1 such that equation (1) in positive integers $m > n > 1$, $m > 2$, $x > 1$, $y > 1$, a and b with $(a, b) = 1$, $a < x$, $b < y$, $a(y - 1) \neq b(x - 1)$ and

$$(3) \quad |x - y| < \max(F_1, (\log z)^G)$$

implies that

$$\max(m, n, x, y, a, b) < C_1.$$

Combining theorem 1 with lemma 1 and an estimate on p -adic linear forms in logarithms, we have

Theorem 3.

If $m > 1$, $n > 1$, $x > 1$ and $y > 1$ with $(x, y) = 1$ satisfy

$$(4) \quad \frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1},$$

then

$$\max(m, n, x, y) < C_2$$

where $C_2 > 0$ is an effectively computable number depending only on the greatest prime factor of $x(y - x)$.

If a and b are fixed, the restriction (3) in theorem 2 can be relaxed considerably.

Theorem 4.

Let a and b be positive integers. Let $F_2 > 1$. There exists an effectively computable number $C_3 > 0$ depending only on a, b and F_2 such that equation (1) in positive integers $m > n > 1$, $x > 1, y > 1$ with $a(y-1) \neq b(x-1)$ and

$$(5) \quad |x-y| < F_2 z / (\log z)^2 (\log \log z)^3$$

implies that

$$\max(m, n, x, y) < C_3.$$

§ 2. The proof of these results depend on the following application of a theorem of Baker [2] on linear forms in logarithms.

Lemma 1.

Let $F_3 > 1$. Let $m, n, x > 1, y > 1, a < F_3 x$, and $b < F_3 y$ with $a(y-1) \neq b(x-1)$ satisfy (1). Put $z = \max(x, y)$ and $z_1 = \min(x, y)$.

Then

$$\max(m, n) < C_4 (\log z)^3 (\log \log z)^2 / (\log z_1)$$

where $C_4 > 0$ is an effectively computable number depending only on F_3 .

Proof of Lemma 1.

We may assume that $m > n$. Denote by $C_1 > 0$ and $C_2 > 0$ effectively computable numbers depending only on F_3 .

By equation (1), we have

$$(6) \quad 0 \neq \left| \frac{ax^m}{x-1} - \frac{by^n}{y-1} \right| < C_1$$

By an estimate of Baker [2], the left hand side of inequality (6) exceeds $x^m \exp(-C_2 (\log m) (\log z)^3 (\log \log z))$.

Now the lemma follows immediately by combining these estimates.

Proof of theorem 1.

Denote by C_3, C_4, \dots, C_9 effectively computable positive numbers depending only on θ and F . Suppose that the assumptions of theorem 1 are satisfied. Then, by lemma 1 and (2), we have

$$(7) \quad m \leq C_3 (\log x)^3.$$

In view of (7) and (2), we may assume that

$$(8) \quad \min(x, y) \geq C_4$$

with C_4 sufficiently large. Further, by (1), we have

$$(9) \quad \begin{aligned} ax^{m-1} &< by^{n-1} \left(1 + \frac{2}{y}\right), \\ by^{n-1} &< ax^{m-1} \left(1 + \frac{2}{x}\right) \end{aligned}$$

Now it follows from (9), (2), (7) and (8) that either $m = n$ or $m = n+1$.

Let $m = n$. It follows from (9), (2) and (7) that

$$\left| \log \frac{a}{b} \right| \leq C_5 x^{-1+\theta} (\log x)^3.$$

Further, by (2), we have

$$\max \left(\left| \log \frac{x}{y} \right|, \left| \log \left(\frac{x-1}{y-1} \right) \right| \right) < C_6 x^{-1+\theta}.$$

Consequently

$$\left| \log \frac{a(y-1)}{b(x-1)} \right| < (C_5 + C_6) x^{-1+\theta} (\log x)^3.$$

Now it follows from an estimate of Waldschmidt [7] or Ramachandra and the author [5] (in the latter reference, the arguments allow to prove the estimate without the restriction on the multiplicative independence of α_1 and α_2) on linear forms in logarithms that the left hand side of inequality (6), with $m = n$ and $F_3 \simeq 1$, exceeds x^{m-C_7} .

Thus, by (6), $x^{m-C_7} < C_1$ which implies that $m < C_8$, since $x > 1$.

Let $m = n + 1$. Re-write equation (1) as

$$ax \frac{x^n - 1}{x - 1} = b \frac{y^n - 1}{y - 1} - a.$$

Now argue as in the case $m = n$ to conclude that $m < C_9$. This completes the proof of theorem 1.

Proof of theorem 2.

We shall choose, later, an effectively computable absolute constant C satisfying $0 < C < 1$.

Case I:

$F_1 \geq (\log z)^C$. Then $|x - y| < F_1$. Hence, by theorem 1, we see that m is bounded by an effectively computable

number depending only on F_1 . Now the assertion of theorem 2 follows from (1), since $(a, b) = 1$.

Case II :

$F_1 < (\log z)^C$. Then, by (3), observe that

$$|x - y| < (\log z)^C < \log z < \max(27, z^{\frac{1}{2}}).$$

Now apply theorem 1 to conclude that m is bounded by an effectively computable absolute constant C_{10} . Let $2 < m < C_{10}$ be given. Denote by $C_{11}, C_{12}, \dots, C_{16}$ effectively computable positive numbers depending only on m . We may assume that $\min(x, y) > C_{11}$ with C_{11} sufficiently large, otherwise the assertion of theorem 2 follows from (1) and $(a, b) = 1$. Then equation (1) implies that $m = n$ or $m = n + 1$.

If $x = y$, then $a \neq b$ and equation (1) implies that

$$x^n (ax^{m-n} - b) = a - b$$

which, since $n > 1$, is not possible if C_{11} is sufficiently large.

Thus we may assume that $x \neq y$.

Re-write equation (1) as

$$a \left(\frac{P_m(x)}{d} \right) = b \left(\frac{P_n(y)}{d} \right)$$

where

$$P_m(X) = \frac{X^m - 1}{X - 1}, \quad P_n(Y) = \frac{Y^n - 1}{Y - 1}$$

and d is the greatest common divisor of $P_m(x)$ and $P_n(y)$. Thus

$$P_m(x) d^{-1} < b < 2x.$$

Put

$$\rho_m = e^{2\pi i/m}, \rho_n = e^{2\pi i/n}, K = Q(\rho_m, \rho_n).$$

For a prime p dividing d , let $\text{ord}_p(d) = \alpha_p$. Let \wp be a prime ideal in the ring of integers of K dividing p . Then \wp divides an ideal

$$(11) \quad [x - y - \rho_m^r + \rho_n^s]$$

for some positive integers $r < m$ and $s < n$.

Put

$$\sigma = e^{2\pi i/6}, T_1 = \{\sigma, \sigma^5\}, T_2 = \{\sigma^2, \sigma^4\}.$$

Suppose that (11) is a zero ideal. Then, since $x \neq y$ and $1 < r < m$, $1 < s < n$, we see that $|x - y| = 1$. Then

$$\cos\left(\frac{2\pi r}{m}\right) = \pm 1 + \cos\left(\frac{2\pi s}{n}\right), \sin\left(\frac{2\pi r}{m}\right) = \sin\left(\frac{2\pi s}{n}\right).$$

These equations imply that either $\rho_m^r \in T_1, \rho_n^s \in T_2$ or $\rho_m^r \in T_2, \rho_n^s \in T_1$. Thus $m \neq n+1$, since m and n are divisible by 3. Therefore $m=n$. Then m is even, since m is divisible by 6. If $x-y=1$, then equation (1) with $m=n$ implies that $x = y+1$ divides $a(x^{m-1} + \dots + 1)$. Therefore x divides a , which is not possible, since $a < x$. Similarly if $y-x=1$ and m even, equation (1) with $m=n$ has no solution. Thus we may assume that (11) is a non-zero ideal.

Put

$$c' = (2m^2)^{-1}.$$

Then, since \wp divides a non-zero ideal (11), we obtain by taking norms,

$$p < C_{12} (\log x)^{\frac{1}{2}}.$$

In fact either $a_p < C_{13}$ or $p^{\alpha} - C_{13}$ divides a non-zero ideal of the form (11). Therefore

$$p^{\alpha} < (\log x)^{C_{14}}.$$

Hence we conclude $q < x^{\frac{1}{2}}$ which, together with (10), implies that

$$x^{m-1} < P_m(x) < 2x^{3/2}.$$

Then, since $m > 2$, we conclude that $x < C_{15}$. Then $y < 2x < 2C_{15}$. Further, by (1) and $(a, b) = 1$, we see that $\max(a, b) < C_{16}$. This completes the proof of theorem 2.

Proof of Theorem 3.

We may assume that $m \neq n$, otherwise equation (4) has no solution, since $x \neq y$. Denote by $C_{17}, C_{18}, \dots, C_{22}$ effectively computable positive numbers depending only on the greatest prime factor of $x(y-x)$. Put $y-x = k$. Then it follows from equation (4) and $(x, y) = 1$ that k divides

$$(x^{|m-n|} - 1) / (x-1).$$

Thus, for a prime p dividing k , we have

$$\text{ord}_p(k) \leq \text{ord}_p(x^{|m-n|} - 1).$$

Now we apply an estimate of van der Poorten [4] on p -adic linear forms in logarithms to obtain

$$\text{ord}_p(k) \leq C_{17} (\log |m+n| + \log \log x)^2$$

Thus

$$\log |k| \leq C_{18} (\log |m+n| + \log \log x)^2.$$

By lemma 1, we have

$$\max(m, n) \leq C_{19} (\log z)^4.$$

Therefore

$$(12) \quad \log |k| \leq C_{20} (\log \log z)^2.$$

Now we apply theorem 1 to conclude that $\max(m, n) \leq C_{21}$.

If $\max(x, y) > C_{22}$ with C_{22} sufficiently large, then it follows from (12) and (4) that $m=n$. This is not possible, since $x \neq y$. This completes the proof of theorem 3.

Proof of theorem 4.

Suppose that the assumptions of theorem 4 are satisfied. Denote by C_{23}, C_{24}, \dots effectively computable positive numbers depending only on a, b and F_2 . By lemma 1 and (5) we have

$$(13) \quad m \leq C_{23} ((\log x)(\log \log x))^2.$$

In view of (13) and (5), we may assume that $\max(x, y) > C_{24}$ with C_{24} sufficiently large. Then we use inequality (9) to conclude that equation (1) implies that $m = n$. Therefore we may assume that $a \neq b$, otherwise $x \neq y$ and equation (1) with $m = n$ has no solution. Then, by again applying (9), we see from (5) and (13) that

$$C_{25} < \left| \log \frac{a}{b} \right| < C_{26} (\log \log x)^{-1}$$

which implies that $x \leq C_{27}$. Hence, by (5) and (13), we conclude that $\max(m, x, y) \leq C_{28}$. This completes the proof of theorem 4.

References

1. **A. Baker**, *Contributions to the theory of Diophantine equations I: On the representation of integers by binary forms*, *Phil. Trans. Roy. Soc. London, A* 263 (1968), 173-208.
2. **A. Baker**, *The theory of linear forms in logarithms, Transcendence theory: Advances and applications*, Academic Press, London and New York (1977), 1-27.
3. **R. Balasubramanian and T. N. Shorey**, *On the equation*

$$a(x^m - 1) / (x - 1) = b(y^n - 1) / (y - 1)$$
, *Math. Scand.* 46 (1980), 177-182.
4. **A. J. Van der Poorten**, *Linear forms in logarithms in the p-adic case*, *Transcendence theory: Advances and applications*, Academic Press, London and New York (1977), 29-57.
5. **K. Ramachandra and T. N. Shorey**, *On gaps between numbers with a large prime factor*, *Acta Arith.* 24 (1973), 99-111.
6. **A. Thue**, *Über Annäherungswerte algebraischer Zahlen*, *J. reine angew. Math.* 135 (1909), 284-305.
7. **M. Waldschmidt**, *A lower bound for linear forms in logarithms*, *Acta Arith.* 37 (1980), 257-283.

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