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## On the equation $a(x^m - 1)/(x - 1) = b(y^n - 1)/(y - 1)$ : II

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## ON THE EQUATION

$$a(x^m - 1) / (x - 1) = b(y^n - 1) / (y - 1) \quad (II)$$

By

T. N. SHOREY

§ 1. Let  $m > 1, n > 1, x > 1, y > 1$  and  $a, b$  with  $(a, b) = 1$  be positive integers satisfying  $a(y-1) \neq b(x-1)$ . This equation of Goormaghtigh arose from the question whether an integer has all the digits identically equal in their expansions to two distinct bases. It follows from Baker's effective version [1] of Thue's theorem [6] that the equation

$$(1) \quad a \frac{x^m - 1}{x - 1} = b \frac{y^n - 1}{y - 1}$$

implies that  $\max(m, n)$  is bounded by an effectively computable number depending only on  $a, b, x$  and  $y$ . Further Balasubramanian and the author [3] applied the theory of linear forms in logarithms to generalise this result by showing that equation (1) implies that  $\max(a, b, x, y, m, n)$  is bounded by an effectively computable number depending only on the greatest prime factor of  $abxy$ . In this paper, we apply the theory of linear forms in logarithms to obtain the following generalisations. We shall always write  $z = \max(x, y)$ .

### Theorem 1.

Let  $0 < \theta < 1$  and  $F > 1$ . If positive integers  $m > n > 1$ ,  $x > 1, y > 1, a$  and  $b$  with  $a < x, b < y, a(y-1) \neq b(x-1)$  and

$$(2) \quad |x - y| \leq \max(F, z^\theta)$$

satisfy (1), then  $m$  is bounded by an effectively computable number depending only on  $\theta$  and  $F$ .

For an account of earlier results in the direction of equation (1), see [3]. We combine theorem 1 with an elementary argument to obtain the following result.

**Theorem 2.**

Let  $F_1 > 1$ . There exists an effectively computable absolute constant  $C > 0$  and an effectively computable number  $C_1 > 0$  depending only on  $F_1$  such that equation (1) in positive integers  $m > n > 1$ ,  $m > 2$ ,  $x > 1$ ,  $y > 1$ ,  $a$  and  $b$  with  $(a, b) = 1$ ,  $a < x$ ,  $b < y$ ,  $a(y - 1) \neq b(x - 1)$  and

$$(3) \quad |x - y| < \max(F_1, (\log z)^G)$$

implies that

$$\max(m, n, x, y, a, b) < C_1.$$

Combining theorem 1 with lemma 1 and an estimate on  $p$ -adic linear forms in logarithms, we have

**Theorem 3.**

If  $m > 1$ ,  $n > 1$ ,  $x > 1$  and  $y > 1$  with  $(x, y) = 1$  satisfy

$$(4) \quad \frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1},$$

then

$$\max(m, n, x, y) < C_2$$

where  $C_2 > 0$  is an effectively computable number depending only on the greatest prime factor of  $x(y - x)$ .

If  $a$  and  $b$  are fixed, the restriction (3) in theorem 2 can be relaxed considerably.

**Theorem 4.**

Let  $a$  and  $b$  be positive integers. Let  $F_2 > 1$ . There exists an effectively computable number  $C_3 > 0$  depending only on  $a, b$  and  $F_2$  such that equation (1) in positive integers  $m > n > 1$ ,  $x > 1, y > 1$  with  $a(y-1) \neq b(x-1)$  and

$$(5) \quad |x-y| < F_2 z / (\log z)^2 (\log \log z)^3$$

implies that

$$\max(m, n, x, y) < C_3.$$

§ 2. The proof of these results depend on the following application of a theorem of Baker [2] on linear forms in logarithms.

**Lemma 1.**

Let  $F_3 > 1$ . Let  $m, n, x > 1, y > 1, a < F_3 x$ , and  $b < F_3 y$  with  $a(y-1) \neq b(x-1)$  satisfy (1). Put  $z = \max(x, y)$  and  $z_1 = \min(x, y)$ .

Then

$$\max(m, n) < C_4 (\log z)^3 (\log \log z)^2 / (\log z_1)$$

where  $C_4 > 0$  is an effectively computable number depending only on  $F_3$ .

**Proof of Lemma 1.**

We may assume that  $m > n$ . Denote by  $C_1 > 0$  and  $C_2 > 0$  effectively computable numbers depending only on  $F_3$ .

By equation (1), we have

$$(6) \quad 0 \neq \left| \frac{ax^m}{x-1} - \frac{by^n}{y-1} \right| < C_1$$

By an estimate of Baker [2], the left hand side of inequality (6) exceeds  $x^m \exp(-C_2 (\log m) (\log z)^3 (\log \log z))$ .

Now the lemma follows immediately by combining these estimates.

**Proof of theorem 1.**

Denote by  $C_3, C_4, \dots, C_9$  effectively computable positive numbers depending only on  $\theta$  and  $F$ . Suppose that the assumptions of theorem 1 are satisfied. Then, by lemma 1 and (2), we have

$$(7) \quad m \leq C_3 (\log x)^3.$$

In view of (7) and (2), we may assume that

$$(8) \quad \min(x, y) \geq C_4$$

with  $C_4$  sufficiently large. Further, by (1), we have

$$(9) \quad \begin{aligned} ax^{m-1} &< by^{n-1} \left(1 + \frac{2}{y}\right), \\ by^{n-1} &< ax^{m-1} \left(1 + \frac{2}{x}\right) \end{aligned}$$

Now it follows from (9), (2), (7) and (8) that either  $m = n$  or  $m = n+1$ .

Let  $m = n$ . It follows from (9), (2) and (7) that

$$\left| \log \frac{a}{b} \right| \leq C_5 x^{-1+\theta} (\log x)^3.$$

Further, by (2), we have

$$\max \left( \left| \log \frac{x}{y} \right|, \left| \log \left( \frac{x-1}{y-1} \right) \right| \right) < C_6 x^{-1+\theta}.$$

Consequently

$$\left| \log \frac{a(y-1)}{b(x-1)} \right| < (C_5 + C_6) x^{-1+\theta} (\log x)^3.$$

Now it follows from an estimate of Waldschmidt [7] or Ramachandra and the author [5] (in the latter reference, the arguments allow to prove the estimate without the restriction on the multiplicative independence of  $\alpha_1$  and  $\alpha_2$ ) on linear forms in logarithms that the left hand side of inequality (6), with  $m = n$  and  $F_3 \simeq 1$ , exceeds  $x^{m-C_7}$ .

Thus, by (6),  $x^{m-C_7} < C_1$  which implies that  $m < C_8$ , since  $x > 1$ .

Let  $m = n + 1$ . Re-write equation (1) as

$$ax \frac{x^n - 1}{x - 1} = b \frac{y^n - 1}{y - 1} - a.$$

Now argue as in the case  $m = n$  to conclude that  $m < C_9$ . This completes the proof of theorem 1.

### Proof of theorem 2.

We shall choose, later, an effectively computable absolute constant  $C$  satisfying  $0 < C < 1$ .

Case I:

$F_1 \geq (\log z)^C$ . Then  $|x - y| < F_1$ . Hence, by theorem 1, we see that  $m$  is bounded by an effectively computable

number depending only on  $F_1$ . Now the assertion of theorem 2 follows from (1), since  $(a, b) = 1$ .

Case II :

$F_1 < (\log z)^C$ . Then, by (3), observe that

$$|x - y| < (\log z)^C < \log z < \max(27, z^{\frac{1}{2}}).$$

Now apply theorem 1 to conclude that  $m$  is bounded by an effectively computable absolute constant  $C_{10}$ . Let  $2 < m < C_{10}$  be given. Denote by  $C_{11}, C_{12}, \dots, C_{16}$  effectively computable positive numbers depending only on  $m$ . We may assume that  $\min(x, y) > C_{11}$  with  $C_{11}$  sufficiently large, otherwise the assertion of theorem 2 follows from (1) and  $(a, b) = 1$ . Then equation (1) implies that  $m = n$  or  $m = n + 1$ .

If  $x = y$ , then  $a \neq b$  and equation (1) implies that

$$x^n (ax^{m-n} - b) = a - b$$

which, since  $n > 1$ , is not possible if  $C_{11}$  is sufficiently large. Thus we may assume that  $x \neq y$ .

Re-write equation (1) as

$$a \left( \frac{P_m(x)}{d} \right) = b \left( \frac{P_n(y)}{d} \right)$$

where

$$P_m(X) = \frac{X^m - 1}{X - 1}, \quad P_n(Y) = \frac{Y^n - 1}{Y - 1}$$

and  $d$  is the greatest common divisor of  $P_m(x)$  and  $P_n(y)$ . Thus

$$P_m(x) d^{-1} < b < 2x.$$

Put

$$\rho_m = e^{2\pi i/m}, \rho_n = e^{2\pi i/n}, K = Q(\rho_m, \rho_n).$$

For a prime  $p$  dividing  $d$ , let  $\text{ord}_p(d) = \alpha_p$ . Let  $\wp$  be a prime ideal in the ring of integers of  $K$  dividing  $p$ . Then  $\wp$  divides an ideal

$$(11) \quad [x - y - \rho_m^r + \rho_n^s]$$

for some positive integers  $r < m$  and  $s < n$ .

Put

$$\sigma = e^{2\pi i/6}, T_1 = \{\sigma, \sigma^5\}, T_2 = \{\sigma^2, \sigma^4\}.$$

Suppose that (11) is a zero ideal. Then, since  $x \neq y$  and  $1 < r < m$ ,  $1 < s < n$ , we see that  $|x - y| = 1$ . Then

$$\cos\left(\frac{2\pi r}{m}\right) = \pm 1 + \cos\left(\frac{2\pi s}{n}\right), \sin\left(\frac{2\pi r}{m}\right) = \sin\left(\frac{2\pi s}{n}\right).$$

These equations imply that either  $\rho_m^r \in T_1, \rho_n^s \in T_2$  or  $\rho_m^r \in T_2, \rho_n^s \in T_1$ . Thus  $m \neq n+1$ , since  $m$  and  $n$  are divisible by 3. Therefore  $m=n$ . Then  $m$  is even, since  $m$  is divisible by 6. If  $x-y=1$ , then equation (1) with  $m=n$  implies that  $x = y+1$  divides  $a(x^{m-1} + \dots + 1)$ . Therefore  $x$  divides  $a$ , which is not possible, since  $a < x$ . Similarly if  $y-x=1$  and  $m$  even, equation (1) with  $m=n$  has no solution. Thus we may assume that (11) is a non-zero ideal.

Put

$$c' = (2m^2)^{-1}.$$

Then, since  $\wp$  divides a non-zero ideal (11), we obtain by taking norms,



$$p \leq C_{12} (\log x)^{\frac{1}{2}}.$$

In fact either  $a_p < C_{13}$  or  $p^{\frac{a}{p} - C_{13}}$  divides a non-zero ideal of the form (11). Therefore

$$p^{\frac{a}{p}} < (\log x)^{C_{14}}.$$

Hence we conclude  $q < x^{\frac{1}{2}}$  which, together with (10), implies that

$$x^{m-1} < P_m(x) < 2x^{3/2}.$$

Then, since  $m > 2$ , we conclude that  $x \leq C_{15}$ . Then  $y \leq 2x \leq 2C_{15}$ . Further, by (1) and  $(a, b) = 1$ , we see that  $\max(a, b) \leq C_{16}$ . This completes the proof of theorem 2.

### Proof of Theorem 3.

We may assume that  $m \neq n$ , otherwise equation (4) has no solution, since  $x \neq y$ . Denote by  $C_{17}, C_{18}, \dots, C_{22}$  effectively computable positive numbers depending only on the greatest prime factor of  $x(y-x)$ . Put  $y-x = k$ . Then it follows from equation (4) and  $(x, y) = 1$  that  $k$  divides

$$(x^{|m-n|} - 1) / (x-1).$$

Thus, for a prime  $p$  dividing  $k$ , we have

$$\text{ord}_p(k) \leq \text{ord}_p(x^{|m-n|} - 1).$$

Now we apply an estimate of van der Poorten [4] on  $p$ -adic linear forms in logarithms to obtain

$$\text{ord}_p(k) \leq C_{17} (\log |m+n| + \log \log x)^2$$

Thus

$$\log |k| \leq C_{18} (\log |m+n| + \log \log x)^2.$$

By lemma 1, we have

$$\max(m, n) \leq C_{19} (\log z)^4.$$

Therefore

$$(12) \quad \log |k| \leq C_{20} (\log \log z)^2.$$

Now we apply theorem 1 to conclude that  $\max(m, n) \leq C_{21}$ .

If  $\max(x, y) > C_{22}$  with  $C_{22}$  sufficiently large, then it follows from (12) and (4) that  $m=n$ . This is not possible, since  $x \neq y$ . This completes the proof of theorem 3.

**Proof of theorem 4.**

Suppose that the assumptions of theorem 4 are satisfied. Denote by  $C_{23}, C_{24}, \dots$  effectively computable positive numbers depending only on  $a, b$  and  $F_2$ . By lemma 1 and (5) we have

$$(13) \quad m \leq C_{23} ((\log x) (\log \log x))^2.$$

In view of (13) and (5), we may assume that  $\max(x, y) > C_{24}$  with  $C_{24}$  sufficiently large. Then we use inequality (9) to conclude that equation (1) implies that  $m = n$ . Therefore we may assume that  $a \neq b$ , otherwise  $x \neq y$  and equation (1) with  $m = n$  has no solution. Then, by again applying (9), we see from (5) and (13) that

$$C_{25} < \left| \log \frac{a}{b} \right| < C_{26} (\log \log x)^{-1}$$

which implies that  $x \leq C_{27}$ . Hence, by (5) and (13), we conclude that  $\max(m, x, y) \leq C_{28}$ . This completes the proof of theorem 4.

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