Primes between $p\_n + 1$ and $p\_n + 1^2 - 1$.

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PRIMES BETWEEN $p_n + 1$ AND $p_{n+1}^2 - 1$

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§ 1. Introduction

We prove the following four theorems. We begin with some notation. Let $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ be the sequence of all prime numbers. Let Q be the product of the first n prime numbers. Let $Q_i = Qp_i^{-1}$ for $i = 1, 2, 3, \ldots, n$.

Let $K = n$. Let J stand for $\sum_{i=1}^{n} a_i Q_i - bQ$

Theorem 1

We have,

$$\sum_{a_1=1}^{p_1-1} \cdots \sum_{a_n=1}^{p_n-1} \sum_{0<b<K} x^J = x + \frac{1}{x} + f(x) + f\left(\frac{1}{x}\right)$$

where $f(x) = \sum^{*}_{p_{n+1}^2 + 1 \le m \le KQ, (m, Q) = 1} x^m$

$$+ \sum_{p_n + 1 \le m \le p_{n+1}^2 - 1, (m, Q) = 1} x^m$$

and * denotes the omission of some integers m.

Theorem 2

$$\sum_{a_i=1}^{p_1-1} \cdots \sum_{a_n=1}^{p_n-1} \sum_{0<b<K} x^{J^2} = 2x + 2\phi(x)$$

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where $\phi(x) = \sum^{*} x^{m^2}$

$$p_{n+1}^2 \text{ (illegible)} < m \leq KQ, (m, Q) = 1$$

$$+ \sum x^{m^2}$$

$$p_n^{\text{(illegible)}} + 1 \leq m \leq p_{n+1}^2 - 1, (m, Q) = 1$$

and $*$ *denotes the omission of some integers* m.

Theorem 3

Let $(a'_1, \ldots, a'_n)$ *be the unique solution of*

$$-2 \equiv \sum_{i=1}^{n} a'_i Q_i \bmod Q, \text{ with}$$

$$0 \leq a'_i \leq p_i - 1, (i = 1, 2, \ldots, n).$$

Then

$$\sum_{a_1=1}^{p_1-1} \sum_{a_2=1}^{p_2-1} \cdots \sum_{a_n=1}^{p_n-1} \sum_{0 \leq b < K} x^{J}$$

$$a_1 \neq a'_1 \quad a_2 \neq a'_2 \quad a_n \neq a'_n$$

$$= \sum_{(m(m+2), Q) = 1} x^{m}$$

where the sum on the right sums over the relevant range for m.

Theorem 4

We have,

$$\sum_{a_1} \cdots \sum_{a_n} \sum_{b} x^{J^2} = \sum_{(m(m+2), Q) = 1} x^{m^2}$$

The sum over m *on the right bring over the same set of integers as in Theorem 3.*

Remark 1

Note that $a'_1 = 0$ and that $1 \leq a'_i \leq p_i - 1$

$(i = 2, 3, \ldots, n)$

Remark 2

In the first two theorems the m's that satisfy

$p_n + 1 \leq m \leq p_{n+1}^2 - 1$ are precisely *all the primes in this interval.* In the next two theorems they are all the twin primes in this interval.

§ 2 Proof

The proofs of theorems 1 and 2 follow from the following two remarks.

First given any integer c there is a unique solution of

$$\sum_{i=1}^{n} a_i Q_i \equiv c \pmod{Q} \text{ subject to } 0 \leq a_i \leq p_i - 1$$

$(i = 1, 2, \ldots, n)$. Out of these solutions $(c, Q) = 1$ is satisfied if and only if $1 \leq a_i \leq p_i - 1$ $(i = 1, 2, 3, \ldots, n)$.

The proof of theorems 3 and 4 follow from the following remark.

Subject to $1 \leq a_i \leq p_i - 1$ for all i we have already secured $(m, Q) = 1$. If in addition $m + 2$ is to be coprime to Q we should have

$$(\Sigma a_i Q_i - \Sigma a'_i Q_i, Q) = 1, \text{ i. e. } a_i - a'_i \neq 0 \text{ for each i.}$$

§ 3. Further Remarks

We can find by the method above conditions to ensure $(m\,(m+2)\,(m+6), Q) = 1$ and so on. Next one can easily get a formula for the n^{th} prime from Theorem 2. It is :

$$p_{n+1}^2 - 1 = \left[-\log \left(\frac{1}{2} \sum_{a_1=1}^{p_1-1} \sum_{a_2=1}^{p_2-2} \cdots \right. \right.$$

$$\left. \left. \cdots \sum_{a_n=1}^{p_n-1} \sum_{b=1}^{n} e^{-(\sum a_i Q_i - bQ)^2} - \frac{1}{e} \right) \right]$$

What we have done corresponds nearly to the Eratosthanese sieve. It will be interesting to modify our investigations in a way which correspond to Brun's sieve.

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Reference

1) K. Ramachandra, Viggo Brun (13—10—1885 to 15—8—1978), The Mathematics Student, Vol. 49, No. 1 (1981) p 87—95

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