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# THE GREATEST SQUARE FREE FACTOR OF A BINARY RECURSIVE SEQUENCE

By T. N. SHOREY

§ 1. For any sequence of integers  $u_0, u_1, \dots, u_m, \dots$  satisfying

$$u_m = r u_{m-1} + s u_{m-2}, \quad m = 2, 3, \dots$$

where  $r$  and  $s$  are rational integers with  $r^2 + 4s \neq 0$ , we have

$$(1) \quad u_m = a \alpha^m + b \beta^m, \quad m = 0, 1, 2, \dots$$

where  $\alpha$  and  $\beta$  are roots of the polynomial  $X^2 - rX - s$  and

$$(2) \quad a = \frac{u_0 \beta - u_1}{\beta - \alpha}, \quad b = \frac{u_1 - u_0 \alpha}{\beta - \alpha}.$$

The polynomial  $X^2 - rX - s$  is called the polynomial associated to the sequence  $\{u_m\}$ . The sequence  $\{u_m\}$  is said to be a non-degenerate binary recursive sequence if  $a, b, \alpha, \beta$  are non-zero and  $\alpha/\beta$  is not a root of unity. For a rational integer  $\kappa$  with  $|\kappa| > 1$ , denote by  $P(\kappa)$  the greatest prime factor of  $\kappa$  and by  $Q(\kappa)$  the greatest square free factor of  $\kappa$ . If  $p_1, \dots, p_r$  are all the distinct primes dividing  $\kappa$ , then  $Q(\kappa) = p_1 \dots p_r$ . For non-zero rational integers  $x$  and  $y$ , denote by  $[x, y]$  and  $(x, y)$ , respectively, the least common multiple and the greatest common divisor of  $x$  and  $y$ . Further we define  $P(1) = P(-1) = 1$  and

$$P\left(\frac{x}{y}\right) = P\left(\frac{\kappa}{(x, y)} \frac{y}{(\kappa, y)}\right) = P\left(\frac{[x, y]}{(x, y)}\right)$$

and

$$Q\left(\frac{x}{y}\right) = Q\left(\frac{[x, y]}{(x, y)}\right)$$

Let  $\{u_m\}$  be a non-degenerate binary recursive sequence given by (1). Stewart [4] proved that

$$Q(u_m) > C \left( \frac{m}{((\log m)^2)} \right)^{1/d}, m > C',$$

where  $d = [Q(a) : Q]$  and  $C > 0, C' > 0$  are effectively computable numbers depending only on  $a$  and  $b$ . Observe that  $d = 1$  or  $2$ . Further, if  $|a| > |\beta|$ , Stewart [4] proved that for any  $\theta$  with  $0 < \theta < 1$ ,

$$Q(u_m) > m^\theta, m > C'',$$

where  $C'' > 0$  is an effectively computable number depending only on  $\theta$  and the sequence  $\{u_m\}$ . We shall generalise and strengthen this result as follows :

### Theorem 1

Let  $\{u_m\}$  be a non-degenerate binary recursive sequence. There exist effectively computable numbers  $C_1 > 0$  and  $C_2 > 0$  depending only on the sequence  $\{u_m\}$  such that for every  $m > C_1$ , we have

$$\log Q(u_m) > C_2 (\log m)^2 (\log \log m)^{-1}.$$

The improvement depends on utilising the fact that the contribution from small primes in  $u_m$  is small. Stewart [5] proved theorem 1 for the greatest square free factor of the members of Lucas and Lehmer sequences. Further, for Lucas and Lehmer sequences, Stewart [5] proved that for almost all  $m$

$$\log Q(u_m) > (\log m)^{2+\log 2-\epsilon}, \epsilon > 0.$$

Theorem 1 is contained in the following result.

**Theorem 2**

Let  $\{u_m\}$  be a non-degenerate binary recursive sequence. There exist effectively computable numbers  $C_3 > 0$  and  $C_4 > 0$  depending only on the sequence  $\{u_m\}$  such that for every pair  $m, n$  with  $m > n$ ,  $m \geq C_3$  and  $u_n \neq 0$ , we have

$$\log Q(\Delta'_{m, n}) > C_4 (\log m)^2 (\log \log m)^{-1}$$

where

$$\Delta'_{m, n} = [u_m, u_n] / (u_m, u_n).$$

For a non-degenerate binary recursive sequence  $\{u_m\}$ , observe that the equation  $u_m = 0$  implies that  $m$  is bounded by an effectively computable number depending only on the sequence  $\{u_m\}$ . We apply theorem 2 with the least integer  $n$  ( $n$  is either 0 or 1) such that  $u_n \neq 0$  to obtain theorem 1. For estimates on  $P(u_m)$  and  $P(\Delta'_{m, n})$ , we refer to Stewart [4] and the author [3]. See also the next theorem.

Let  $\{u_m\}$  and  $\{v_m\}$  be non-degenerate binary recursive sequences whose associated polynomials are identically equal. Denote by  $\alpha$  and  $\beta$  the roots of their associated polynomial. Then the sequence  $\{u_m\}$  is given by (1) and (2). Further for  $m = 0, 1, 2, \dots$ , we have

$$v_m = a_1 \alpha^m + b_1 \beta^m$$

where

$$a_1 = \frac{v_0 \beta - v_1}{\beta - \alpha}, \quad b_1 = \frac{v_1 - v_0 \alpha}{\beta - \alpha}.$$

For  $m$  and  $n$  with  $u_m v_n \neq 0$ , put

$$\Delta_{m,n} = [u_m, v_n] / (u_m v_n)$$

Then theorem 2 is a particular case of the following result.

### Theorem 3

Let  $A > 0$  and  $0 < K < (d+1)^{-1}$  where  $d = [Q(\alpha) : Q]$ . There exist effectively computable numbers  $C_5 > 0$  and  $C_6 > 0$  depending only on  $A, K$ , the sequences  $\{u_m\}$  and  $\{v_m\}$  such that for every pair  $m, n$  with  $m > n, m \geq C_5, v_n \neq 0$  and

$$(3) \quad \frac{a \alpha^m}{a_1 \alpha^n} \neq \frac{b \beta^m}{b_1 \beta^n},$$

either

$$\log P(\Delta_{m,n}) > (\log m)^A$$

or

$$\sum_{\substack{p \mid \Delta_{m,n} \\ p > m^K}} 1 > C_6 \frac{\log m}{\log \log m}$$

where  $p$  runs through primes

For the proof of theorem 2, we may assume  $\log P(\Delta'_{m,n}) < (\log m)^2$ . Then we apply theorem 3 with  $\{u_m\} = \{v_m\}$ ,  $A = 2$  and  $K = \frac{1}{4}$ . Observe that (3) is satisfied, since  $\alpha/\beta$  is not a root of unity. Now the assertion of theorem 2 follows immediately.

The proof of theorem 3 depends on the theory of linear forms in logarithms. Let  $\alpha_1, \dots, \alpha_n$  be non-zero algebraic numbers. Let  $K$  be their splitting field over  $Q$ . Put  $D = [K:Q]$

We denote by  $A_1, \dots, A_n$  upper bounds for the heights of  $\alpha_1, \dots, \alpha_n$  respectively, where we assume that  $A_j > 3$  for  $1 < j < n$ . Write

$$\Omega' = \prod_{j=1}^{n-1} \log A_j, \quad \Omega = \Omega' \log A_n.$$

The proof of theorem 3 depends on the following theorem of Baker [1] on linear forms in logarithms.

**Theorem A.**

*There exist effectively computable absolute constants  $C_7 > 0$  and  $C_8 > 0$  such that the inequalities*

$$0 < \left| \alpha_1^{b_1} \dots \alpha_n^{b_n} - 1 \right| < \exp \left( - (C_7 n D)^{C_8 n} \Omega \log \Omega' \log B \right)$$

*have no solution in rational integers  $b_1, \dots, b_n$  with absolute values at most  $B (> 2)$ .*

We shall also need a  $p$ -adic analogue, due to van der Poorten [2], of theorem A.

**Theorem B.**

*Let  $\mathfrak{p}$  be a prime ideal of  $K$  lying above a rational prime  $p$ . There exist effectively computable absolute constants  $C_9 > 0$  and  $C_{10} > 0$  such that the inequalities*

$$\infty > \text{ord}_{\mathfrak{p}} \left( \alpha_1^{b_1} \dots \alpha_n^{b_n} - 1 \right) > (C_9 n D)^{C_{10} n} \frac{p^D}{\log p} \Omega (\log B)^2$$

*have no solution in rational integers  $b_1, \dots, b_n$  with absolute values at most  $B (> 2)$ .*

### § 2. Proof of theorem 3.

Let  $A > 0$  and  $0 < K < (d+1)^{-1}$ . Put

$$(4) \quad \tau = K(d+1).$$

Observe that  $0 < \tau < 1$ . Let  $\{u_m\}$  and  $\{v_m\}$  be as in theorem 3. There is no loss of generality in assuming that  $|\alpha| \geq |\beta|$ . Then, since  $\alpha/\beta$  is not a root of unity, we find that  $|\alpha| > 1$ . For algebraic integer  $\kappa \in Q(\alpha)$ , denote by  $[\kappa]$  the ideal generated by  $\kappa$  in the ring of integers of  $Q(\alpha)$ . There exists a positive rational integer  $k$  such that

$$([\alpha^2], [\beta^2]) = [k].$$

Put  $\alpha_1 = \alpha^2/k$  and  $\beta_1 = \beta^2/k$ . Then the ideals  $[\alpha_1]$  and  $[\beta_1]$  are relatively coprime. For  $m = 0, 1, 2, \dots$ , notice that

$$U_m = k^{-m} u_{2m} = a\alpha_1^m + b\beta_1^m,$$

$$U'_m = k^{-m} u_{2m+1} = a\alpha\alpha_1^m + b\beta\beta_1^m,$$

$$V_m = k^{-m} v_{2m} = a_1\alpha_1^m + b_1\beta_1^m,$$

$$V'_m = k^{-m} v_{2m+1} = a_1\alpha\alpha_1^m + b_1\beta\beta_1^m.$$

Observe that the sequences  $\{U_m\}$ ,  $\{U'_m\}$ ,  $\{V_m\}$  and  $\{V'_m\}$  are non-degenerate binary recursive sequences. By proving the theorem separately for sequences  $\{U_m\}$  and  $\{V_m\}$ ,  $\{U_m\}$  and  $\{V'_m\}$ ,  $\{U'_m\}$  and  $\{V_m\}$ ,  $\{U'_m\}$  and  $\{V'_m\}$ , there is no loss of generality in assuming that  $([\alpha], [\beta]) = [1]$ .

Denote by  $c_1, c_2, \dots$  effectively computable positive numbers depending only on  $A, K$ , the sequences  $\{u_m\}$  and

$\{v_m\}$ . We may assume that  $m > c_1$  with  $c_1$  sufficiently large. Then, since  $\{u_m\}$  is non-degenerate, we see that  $u_m \neq 0$ . Let  $0 < n < m$  satisfy  $v_n \neq 0$  and suppose that (3) is valid. We suppose

$$(5) \quad \text{lcm } P(\Delta_{m,n}) < (\log m)^A.$$

Let  $\pi_1, \dots, \pi_s$  be all the rational primes satisfying  $\pi_i | \Delta_{m,n}$  and  $\pi_i > m^K$  for  $1 < i < s$ . Let  $0 < \epsilon < 1$ . We suppose that

$$(6) \quad s < + \epsilon (\log m) (\log \log m)^{-1}.$$

We shall arrive at a contradiction for a suitable choice of  $\epsilon$  depending only on  $A, K$  the sequences  $\{u_m\}$  and  $\{v_m\}$ .

We write

$$(7) \quad B_1 = \frac{u_m}{(u_m, v_n)}, \quad B_2 = \frac{v_n}{(u_m, v_n)}, \\ \Delta = (u_m, v_n).$$

When

$$(8) \quad \frac{u_m}{v_n} = \frac{B_1}{B_2} \quad \text{and} \quad (B_1, B_2) = 1.$$

Further

$$(9) \quad \Delta_{m,n} = \pm B_1 B_2.$$

For a prime  $p$  dividing  $B_1$ , we see from (7) that

$$\text{ord}_p(B_1) < \text{ord}_p(u_m).$$

Let  $\wp$  be a prime ideal in the ring of integers of  $Q(\alpha)$  dividing  $p$ . Then, since the ideals  $[\alpha]$  and  $[\beta]$  are relatively prime, either  $\wp$  does not divide  $[\alpha]$  or  $\wp$  does not divide  $[\beta]$ . For simplicity assume that  $\wp$  does not divide  $[\alpha]$ . Then, by (1), we have



$$\begin{aligned} \text{ord}_p(u_m) &< \text{ord}_{\wp}(u_m) \\ &< c_2 + \text{ord}_{\wp} \left( -\frac{b}{a} \left( \frac{\beta}{\alpha} \right)^m - 1 \right). \end{aligned}$$

Now we apply theorem B with  $n = 2$ ,  $D = d$ ,  $\alpha_1 = -b/a$ ,  $\alpha_2 = \beta/\alpha$ ,  $b_1 = 1$  and  $b_2 = m$  to conclude that

$$\begin{aligned} \text{ord}_{\wp} \left( -\frac{b}{a} \left( \frac{\beta}{\alpha} \right)^m - 1 \right) \\ < c_3 p^d (\log p)^{-1} (\log m)^2. \end{aligned}$$

Therefore

$$\text{ord}_p(B_1) < c_4 p^d (\log p)^{-1} (\log m)^2.$$

This inequality follows similarly when  $\wp$  does not divide  $[\beta]$ . Consequently, by (4),

$$\sum_{\substack{p|B_1 \\ p < m^K}} \text{ord}_p(B_1) \log p < c_4 m^\tau (\log m)^2.$$

Similarly

$$\sum_{\substack{p|B_2 \\ p < m^K}} \text{ord}_p(B_2) \log p < c_5 m^\tau (\log m)^2.$$

Consequently, by (9), we may write

$$(10) \quad B_1 = B_3 \pi_1^{x_1} \dots \pi_s^{x_s}, \quad B_2 = B_4 \pi_1^{y_1} \dots \pi_s^{y_s}$$

where  $x_1, \dots, x_s, y_1, \dots, y_s$  are non-negative integers and

$B_3, B_4 \in \mathbb{Z}$  with

$$(11) \quad \log \max(|B_3|, |B_4|) < c_6 m^\tau (\log m)^2.$$

Further we see from (7) that

$$\log \max ( | B_1 | , | B_2 | ) < c_7 m$$

which, together with (10), implies that

$$(12) \quad \max ( x_1, \dots, x_s, y_1, \dots, y_s ) < c_8 m$$

with  $c_8 > 1$ .

We have

$$(13) \quad u_m a_1^{-1} a \alpha^{m-n} v_n \\ = -b_1 \beta^n (a_1^{-1} a \alpha^{m-n} - b_1^{-1} b \beta^{m-n})$$

and, by (7) and (1),

$$(14) \quad \Lambda B_1 - a \alpha^m = b \beta^m$$

In view of (3), we see that

$$(15) \quad u_m - a_1^{-1} a \alpha^{m-n} v_n \neq 0.$$

Put

$$T = a_1^{-1} a \alpha^{m-n} v_n u_m^{-1} - 1,$$

$$T_1 = a^{-1} \alpha^{-m} \Lambda B_1 - 1.$$

By (15) and (14), notice that

$$TT_1 \neq 0.$$

Further it follows from (8) and (10) that

$$T_1 = a_1^{-1} \alpha^{-m} \pi_1^{x_1} \dots \pi_s^{x_s} (B_3 \Lambda) - 1$$

$$\text{and } T = \frac{a}{a_1} \alpha^{m-n} \pi_1^{z_1} \dots \pi_s^{z_s} \frac{B_4}{B_3} - 1$$

where  $z_i = y_i - x_i$  for  $1 < i < s$ . Now we split the proof of theorem 3 in two cases.

*Case I.*  $|\alpha| > |\beta|$ . Dividing both the sides of (13) by  $u_m$ , we have

$$(16) \quad 0 < |T| < c_9^{-n}, \quad c_9 > 1.$$

We apply theorem  $\mu$  with

$$n = s + 3 < \varepsilon (\log m) (\log \log m)^{-1} + 3 \text{ by (6),}$$

$$D = d < 2, \quad \log A_1 = \log A_2 = c_{10}, \quad \log A_3 = \dots =$$

$$\log A_{n-1} = (\log m)^A \text{ by (5), } \log A_n = c_6 m^\tau (\log m)^2 \text{ by (11)}$$

and  $B = c_8 m$  by (12) to conclude that

$$(17) \quad |T| > \exp(-m^{\tau+c_{11}} \varepsilon (\log m)^5).$$

We shall choose  $\varepsilon$  to satisfy

$$(18) \quad \varepsilon < (1-\tau)/2 c_{11}.$$

Put

$$\tau_1 = (1 + \tau)/2.$$

Then, since  $0 < \tau < 1$ , we find that  $\tau < \tau_1 < 1$ .

Combining (16), (17) and (18), we have

$$n < c_{12} m^{\tau_1} (\log m)^5.$$

Then

$$(19) \quad \log |\wedge| < \log |v_n| < c_{13} m^{\tau_1} (\log m)^5.$$

Dividing both the sides of (14) by a  $d^m$ , we have

$$(20) \quad 0 < |T_1| < c_{14}^{-m}, \quad c_{14} > 1.$$

We apply theorem A with  $n = s + 3 < \varepsilon (\log m) (\log \log m)^{-1} + 3$  by (6),  $D = d < 2$ ,  $\log A_1 = \log A_2 = c_{15}$ ,

$$\log A_3 = \dots = \log A_{n-1} = (\log m)^A \text{ by (5), } \log A_n = 2c_{13} m^{\tau_1}$$

$(\log m)^5$  by (19), (11) and  $B = c_8 m$  by (12) to conclude that

$$(21) \quad |T_1| > \exp(-m^{\tau_1 + c_{16}\epsilon} (\log m)^8).$$

Let

$$\epsilon = \min \left( \frac{1 - \tau_1}{2c_{11}}, \frac{1 - \tau_1}{2c_{16}}, \frac{1}{2} \right)$$

Then (18) is satisfied. Put

$$\tau_2 = (1 + \tau_1) / 2.$$

Observe that  $\tau_1 < \tau_2 < 1$ . Now we combine (20) and (21) to conclude that

$$m \leq c_{17} m^{\tau_2} (\log m)^8$$

which, since  $\tau_2 < 1$ , implies that  $m < c_{18}$ . But this is not possible if  $c_1 > c_{18}$ .

### Case II

$|\alpha| = |\beta|$ . Let  $\tau_1$  and  $\tau_2$  be defined as in case I.

Observe that  $\beta$  is not a unit, since  $\alpha/\beta$  is not a root of unity. Therefore there exists a prime ideal  $\mathfrak{p}$  in the ring of integers of  $\mathbb{Q}(\alpha)$  such that  $\mathfrak{p} \mid [\beta]$ . Further, since the ideals  $[\alpha]$  and  $[\beta]$  are relatively coprime, observe that  $\mathfrak{p}$  does not divide  $[\alpha]$ . Consequently  $\text{ord}_{\mathfrak{p}}(u_m) \leq c_{19}$ . Now, by counting the power

of prime ideal  $\mathfrak{p}$  on both the sides in (13), we have

$$n \leq c_{20} + \text{ord}_{\mathfrak{p}}(u_m) + \text{ord}_{\mathfrak{p}}(T) \leq c_{21} + \text{ord}_{\mathfrak{p}}(T).$$

We apply theorem B with  $p < c_{22}$  and the same parameters as in case I for obtaining a lower bound for  $|\Gamma|$  by theorem A. We obtain

$$\text{ord}_{\wp}(\Gamma) < m^{\tau + c_{23}\epsilon} (\log m)^5.$$

We shall choose  $\epsilon$  to satisfy

$$(22) \quad \epsilon < \frac{1 - \tau}{2c_{23}}.$$

Then

$$n < c_{24} m^{\tau_1} (\log m)^5$$

which implies that

$$\log |\Lambda| < c_{25} m^{\tau_1} (\log m)^5.$$

Counting the power of prime ideal  $\wp$  on both the sides in (14), we obtain

$$m < c_{26} + \text{ord}_{\wp}(\Gamma_1).$$

We apply theorem B with  $p < c_{22}$ ,  $\log A_n = c_{25} m^{\tau_1} (\log m)^5$  and the same parameters as in case I for obtaining a lower bound for  $|\Gamma_1|$  by theorem A. We obtain

$$\text{ord}_{\wp}(\Gamma_1) < m^{\tau_1 + c_{27}\epsilon} (\log m)^8.$$

Let

$$\epsilon = \min \left( \frac{1 - \tau}{2c_{23}}, \frac{1 - \tau_1}{2c_{27}}, \frac{1}{2} \right)$$

Then (22) is satisfied. We obtain

$$m < c_{28} m^{\tau_2} (\log m)^8.$$

Consequently  $m < c_{29}$  which is not possible if  $c_1 > c_{29}$ .

This completes the proof of theorem 3.

**Remarks**

(i) Let  $\{u_m\}$  be a non-degenerate binary recursive sequence. For every pair  $m, n$  with  $m > n$ ,  $u_m u_n \neq 0$  and  $Q(u_m) = Q(u_n)$ , we have

$$m - n > c_{30} (\log m)^2 (\log \log m)^{-1}$$

where  $c_{30} > 0$  is an effectively computable number depending only on the sequence  $\{u_m\}$ . This follows immediately from theorem 1 and the relation (13) with  $a_1 = a$ ,  $b_1 = b$ .

(ii) Let  $P \geq 2$  and denote by  $S$  the set of all non-zero integers composed of primes not exceeding  $P$ . We can apply the argument of proof of theorem 1 to prove that for every  $x \in S, y \in S$  with  $(x, y) = 1, |x| > |y|$  and  $\log |x| > e^e$ ,

$$\log Q(x+y) \geq c_{31} (\log \log |x|)^2 (\log \log \log |x|)^{-1}$$

where  $c_{31} > 0$  is an effectively computable number depending only on  $P$ .

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