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## ON A QUESTION OF RAMACHANDRA

By Hugh L MONTGOMERY \*

Let  $a_k(n)$  be the Dirichlet series coefficients defined by the relation

$$(1) \quad \left( \sum_p \frac{1}{p^s} \right)^k = \sum_{n=1}^{\infty} \frac{a_k(n)}{n^s} \quad (\text{Re } s > 1).$$

Here  $k$  is a positive integer, and we see more explicitly that

if  $n = p_1^{a_1} \dots p_t^{a_t}$  then

$$(2) \quad a_k(n) = \begin{cases} \frac{k!}{a_1! \dots a_t!} & \text{if } \Omega(n) = k, \\ 0 & \text{otherwise,} \end{cases}$$

With possible applications to the Riemann zeta function in mind, Ramachandra has wanted to know the asymptotic size of the quantity

$$\max_k \left( \sum_{n \leq N} \frac{a_k(n)^2}{n^{2\sigma}} \right)^{\frac{1}{2k}}$$

as a function of  $N$  and  $\sigma$ , where  $\sigma$  is fixed,  $\frac{1}{2} < \sigma < 1$ . We settle this question by demonstrating the following

**Theorem:** Let  $\sigma$  be fixed,  $\frac{1}{2} < \sigma < 1$ . With the  $a_k(n)$  defined by (1), we have

$$(3) \quad \max_k \left( \sum_{n \leq N} \frac{a_k(n)^2}{n^{2\sigma}} \right)^{\frac{1}{2k}} \approx \frac{(\log N)^{1-\sigma}}{\log \log N}$$

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In addition, for any integer  $k > 1$

$$(4) \quad \left( \sum_n \frac{a_k(n)^2}{n^{2\sigma}} \right)^{\frac{1}{2k}} \approx \frac{k^{1-\sigma}}{(\log k)^\sigma}$$

Here the implicit constants may depend only on  $\sigma$ .

With a little more care one could show that the values of  $k$  for which the maximum in (3) is attained satisfy

$$k \approx (\log N) / \log \log N.$$

To establish that the right hand side majorizes the left above we shall require the following two lemmas.

**Lemma 1:** Let  $Y > 2$ , and put

$$R = R(Y) = \{r: p \mid r \Rightarrow p < Y\}.$$

Then for any  $k > 1$ , and fixed  $\sigma$ ,  $\frac{1}{2} < \sigma < 1$ ,

$$\left( \sum_{r \in R} \frac{a_k(r)^2}{r^{2\sigma}} \right)^{\frac{1}{2k}} \ll \frac{Y^{1-\sigma}}{\log Y}.$$

**Proof:** Let  $X_2, X_3, \dots, X_p$  be independent random variables, each uniformly distributed on the circle  $|z| = 1$ .

Then

$$\begin{aligned} \left( \sum_{r \in R} \frac{a_k(r)^2}{r^{2\sigma}} \right)^{\frac{1}{2k}} &= \left\| \sum_{p < y} \frac{X_p}{p^\sigma} \right\|_{2k}^{\frac{1}{2k}} \leq \left\| \sum_{p < y} \frac{X_p}{p^\sigma} \right\|_{\infty}^{\frac{1}{2k}} \\ &= \sum_{p < y} \frac{1}{p^\sigma} \ll \frac{Y^{1-\sigma}}{\log Y}. \end{aligned}$$

**Lemma 2:** Let  $Y > 2$ , and put

$$S = S(y) = \{s : p | s \Rightarrow p > y\}.$$

Then for any  $k > 1$ , and fixed  $\sigma$ ,  $\frac{1}{2} < \sigma < 1$ .

$$\left( \sum_{s \in S} \frac{a_k(s)^2}{s^{2\sigma}} \right)^{\frac{1}{2k}} \ll \frac{k^{1/2} Y^{\frac{1}{2}-\sigma}}{(\log Y)^{1/2}}$$

**Proof:** From (2) we see that  $a_k(s) < k!$  for all  $s$ .

Thus

$$\begin{aligned} \sum_{s \in S} \frac{a_k(s)^2}{s^{2\sigma}} &< k! \sum_{s \in S} \frac{a_k(s)}{s^{2\sigma}} = k! \left( \sum_{p > Y} \frac{1}{p^{2\sigma}} \right)^k \\ &< k! \left( \frac{c Y^{1-2\sigma}}{\log Y} \right)^k, \end{aligned}$$

and the stated bound follows on taking the  $2k$ -th root.

We now prove the Theorem. We first show that the left hand side of (3) is at least as large as the right hand side. Let  $Y$  be the largest integer such that

$$P = \prod_{p < Y} p < N.$$

Then by the prime number theorem with remainder,

$$Y = \log N + O\left(\frac{\log N}{(\log \log N)^A}\right)$$

Take  $k = \Omega(P) = \pi(Y)$ . Then

$$\sum_{n < N} \frac{a_k(n)^2}{n^{2\sigma}} > \frac{a_k(P)^2}{P^{2\sigma}} = \frac{k!^2}{P^{2\sigma}} > \frac{k!^2}{N^{2\sigma}}$$

$$\text{so that } \left( \sum_{n < N} \frac{a_k(n)^2}{n^{2\sigma}} \right)^{\frac{1}{2k}} \gg \frac{k}{N^{\sigma/k}}$$

Clearly  $k \sim \frac{\log N}{\log \log N}$ , and by more careful use of the prime number theorem we see that  $k > \frac{\log N}{\log \log N}$  for all large  $N$ . Thus  $N^{1/k} < \log N$ , and we have the desired lower bound. In (4) the value of  $k$  is prescribed; we choose  $Y$  so that  $\pi(Y) = k$ , we take  $N = P$ , and proceed as above. We obtain the desired lower bound since

$$\frac{(\log N)^{1-\sigma}}{\log \log N} \sim \frac{k^{1-\sigma}}{(\log k)^\sigma}$$

We now complete the proof of (3). Let  $R$  and  $S$  be as in Lemmas 1 and 2, where  $Y$  is a parameter to be chosen later. Any  $n$  is uniquely of the form  $n = rs$  with  $r \in R$ ,  $s \in S$ . If  $\Omega(n) = k$  and  $\Omega(r) = m$  then by (2)

$$a_k(n) = \binom{k}{m} a_m(r) a_{k-m}(s).$$

Hence

$$\sum_{n \leq N} \frac{a_k(n)^2}{n^{2\sigma}} = \sum_{m=0}^k \binom{k}{m}^2 \sum_{r \in R} \frac{a_m(r)^2}{r^{2\sigma}} \sum_{\substack{s \in S \\ rs \leq N}} \frac{a_{k-m}(s)^2}{s^{2\sigma}}$$

Since  $\binom{k}{m} < 2^k$ , this is

$$< 2^k \sum_{m=0}^k \binom{k}{m} \left( \sum_{r \in R} \frac{a_m(r)^2}{r^{2\sigma}} \right) \left( \sum_{\substack{s \in S \\ s \leq N}} \frac{a_{k-m}(s)^2}{s^{2\sigma}} \right)$$

$$(5) \quad < 2^k (R^2 + s^2)^k$$

where

$$R = \max_m \left( \sum_{r \in R} \frac{a_m(r)^2}{r^{2\sigma}} \right)^{\frac{1}{2m}}$$

and

$$S = \max_m \left( \sum_{\substack{s \in S \\ s < N}} \frac{a_m(s)^2}{s^{2\sigma}} \right)^{\frac{1}{2m}}$$

By Lemma 1

$$(6) \quad R \ll \frac{Y^{1-\sigma}}{\log Y}$$

To treat  $S$  we note that  $\Omega(s) = m$ ,  $s \in S$ , then  $s > Y^m$ .

If  $m > \frac{\log N}{\log Y}$  then it follows that  $s > N$ , and the sum

defining  $S$  will be empty. Thus we may suppose that  $m < \frac{\log N}{\log Y}$ .

From this bound and Lemma 2 we see that

$$(7) \quad S \ll \frac{(\log N)^{\frac{1}{2}} Y^{\frac{1}{2} - \sigma}}{\log Y}$$

On combining (5), (6), and (7) it follows that

$$\left( \sum_{n < N} \frac{a_k(n)^2}{n^{2\sigma}} \right)^{\frac{1}{2k}} \ll \frac{Y^{1-\sigma}}{\log Y} + \frac{(\log N)^{\frac{1}{2}} Y^{\frac{1}{2} - \sigma}}{\log Y},$$

and it suffices to take  $Y = \log N$ .

To complete the proof of (4) we may proceed more simply. With the  $X_p$  as in the proof of Lemma 1 we see that

$$\left( \sum_n \frac{a_k(n)^2}{n^{2\sigma}} \right)^{\frac{1}{2k}} = \left\| \sum_p \frac{X_p}{p^\sigma} \right\|_{2k}.$$

By the triangle inequality this is

$$\begin{aligned} &< \left\| \sum_{p < Y} \frac{X_p}{p^\sigma} \right\|_{2k} + \left\| \sum_{p > Y} \frac{X_p}{p^\sigma} \right\|_{2k} \\ &= \left( \sum_{r \in R} \frac{a_k(r)^2}{r^{2\sigma}} \right)^{\frac{1}{2k}} + \left( \sum_{s \in S} \frac{a_k(s)^2}{s^{2\sigma}} \right)^{\frac{1}{2k}} \end{aligned}$$

By Lemmas 1 and 2 this is

$$\ll \frac{Y^{1-\sigma}}{\log Y} + \frac{k^{\frac{1}{2}} Y^{\frac{1}{2}-\sigma}}{(\log Y)^{\frac{1}{2}}}$$

and it suffices to take  $Y = k \log k$ .

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