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## ON THE ZEROS OF THE RIEMANN ZETA-FUNCTION AND L-SERIES-II

By

R. BALASUBRAMANIAN AND K. RAMACHANDRA

To

Professors E. C. TITCHMARSH, Yu. V. LINNIK and  
J. E. LITTLEWOOD

In Memoriam

### § 1. Introduction

This paper is a continuation of Ramachandra's paper [2] with the same title. It is divided into two parts A and B for convenience. Both the parts deal with the clusters of zeros of the Riemann zeta-function (part A near  $\sigma = 1$  and part B in  $\frac{1}{2} < \sigma < 1$ ). The main result of part A is

#### Theorem A

Let  $\zeta(\beta_0 + i\gamma_0) = 0$  where  $\beta_0 > \frac{1}{2}$  and  $\gamma_0 > 100$ . Let  $0 < \lambda \leq \frac{1}{200}$ . Let  $u$  and  $v$  be positive and let  $N_v(u)$  denote the number of zeros  $\rho$  (counted with multiplicities) of the zeta function which satisfy  $|\rho - 1 - iv| < u$ . Define the function  $g(\gamma_0)$  by

$$g(\gamma_0) = N_{\gamma_0}(4\lambda) e^{-\lambda\gamma_0} +$$

$$\gamma_0 - 2\lambda \leq v \leq \gamma_0 + 2\lambda \left\{ \int_0^{\lambda Y + 1} N_v \left( \frac{2u}{Y} \right) e^{-u} du \right\}$$

Then there exist effective positive constants  $A_1, A_2, A_3$  independent of  $\lambda, \beta_0$ , and  $\gamma_0$  such that for all  $Y$  satisfying  $Y > A_3 \log \log \gamma_0 > (A_3 + 1000)^2$  we have

$$g(\gamma_0) > A_1 \lambda^2 \{ \lambda^4 Y^{-1} (1 - \beta_0)^{-1} - A_2 \}.$$

**Remark 1**

We can replace  $g(\gamma_0)$  by  $N_{\gamma_0}(4\lambda) e^{-\lambda Y} +$

$$\int_{\gamma_0 - 2\lambda}^{\gamma_0 + 2\lambda} \left[ \int_0^{\lambda Y + 1} N_v \left( \frac{2u}{Y} \right) e^{-u} du \right] dv.$$

The final inequality of the theorem still holds if  $\lambda Y > 1$  in addition to the other conditions of the theorem. The proof of this result can be achieved by considering a lower bound for

$$X < p \leq 2X \quad j = - \frac{[Y]}{\sum} \left| 1 + p^{\frac{\alpha_j}{j}} \right|^2$$

where  $\alpha_j = \gamma_0 + jY^{-1}$ . Lemma 2 gives for this sum the lower bound

$$\gg \sum_j |\gamma_0 - \alpha_j|^2 \gg Y.$$

**Remark 2**

This theorem improves Theorem 1 of [2]. The earlier history of this result is connected with the names N. LEVINSON, H. L. MONTGOMERY and K. RAMACHANDRA. For detailed historical comments see [2]. It should be mentioned that LEVINSON was the first to prove that  $N_{\gamma_0}(\lambda) > 2$  for all  $\gamma_0 \geq \gamma_0(\lambda)$ . In the lower bound for  $g(\gamma_0)$  the dependence on  $\lambda$  can be sharpened.

**Remark 3**

For convenience we state this theorem in part A again and point out its application to VINOGRADOV's zero-free region. Zero-free regions are not new. (For an exposition of the usual methods of getting zero-free regions see for instance [4]). But theorem A says something new in the following direction,

It may be looked upon as a new information about the zeros of  $\zeta(s)$  in  $\sigma > 1 - (\log \log t)^{-1}$ ,  $t \geq 100$ . A simple corollary to theorem A can be stated very easily. *Let us call a zero  $\rho$  isolated if in a disc with centre  $\rho$  and radius  $10^{-800}$  there are no other zeros. Then there are at most only finitely many isolated zeros  $\rho = \beta + i\gamma$  in the region  $\sigma \geq 1 - (\log \log t)^{-2}$ ,  $t > 100$ . Moreover  $N_\gamma(\lambda) \geq 10000$  for all  $\gamma$  exceeding an effective positive constant depending only on  $\lambda$ .*

**Remark 4**

The method adopted in part A is the same as that of [2] with an important difference. We replace the inequality

$$10^8 \left| 1 + p^{i\alpha} \right|^2 + \left| 1 + p^{2i\alpha} \right|^2 > 3,$$

valid for all primes  $p$  (This was used in [2]) by an averaged inequality

$$X < \sum_{p < 2X} \left( \left| 1 + p^{i\alpha_1} \right|^2 + \left| 1 + p^{i\alpha_2} \right|^2 \right) > \delta \sum_{p < 2X} 1,$$

where  $\alpha_1$  and  $\alpha_2$  are any two real numbers satisfying  $0 < \alpha_1 - \alpha_2 < 1$  and  $\delta = 10^{-5} (\alpha_1 - \alpha_2)^2$ . This inequality holds for all  $X \geq 10^{90}$ . The proof of this inequality

depends crucially on an important idea of BALASUBRAMANIAN which we utilized in [1]. BALASUBRAMANIAN'S idea makes it possible to prove this inequality by using the HARDY-LITTLEWOOD upper bound for  $\pi(x+b) - \pi(x)$  (the number of primes in small intervals) which was obtained by them by using the fundamental method of BRUN. For an exposition of BRUN'S method see [3]. We sketch the proof of this averaged inequality in part A. The rest of the work in part A consists in repeating without any essential changes the work done in [2] and the appendix therein.

**Remark 5**

The result in part A (and also those of part B) depend primarily on the EULER product and the functional equation. Hence they extend easily to more general situations where these two are available. In particular the results have obvious analogues to zeta-functions and L functions of algebraic number fields.

Part B deals with "good and bad zeros" of  $\zeta(s)$  in  $\sigma > \frac{1}{2}$ .

Our results of part B were inspired by a letter of Professor D. R. HEATH-BROWN, where he announced without proof the following theorem which he proved in September 1975. We quote from his letter in his notation.

"Theorem (D. R. HEATH-BROWN)

Let  $N_{\epsilon}(\sigma, T)$  denote the number of zeros  $\rho = \beta + i\gamma$  of  $\zeta(s)$  in the region  $|\gamma| \leq T$ ,  $\beta > \sigma$  for which  $\zeta(s)$  does not vanish on

$$\{s : |\operatorname{Im} s - \gamma| \leq (\log \gamma)^2, \operatorname{Re} s > \beta - \epsilon\}$$

except at  $s = \rho$ . Then  $N_{\epsilon}(\sigma, T) \ll T^{2-2\sigma + \epsilon}$ .

Here the constant implied by the Vinogradov symbol depends only on  $\epsilon$ .

We tried to reconstruct the proof of HEATH-BROWN'S theorem and we did succeed! Around August 1977, RAMACHANDRA wrote up an improved version of this theorem in his unpublished article "REMARKS ON GOOD AND BAD ZEROS OF  $\zeta(s)$ ". This improvement (which was possible by applying some techniques employed in [2]) can be stated as follows. (It is the main result of part B).

**Theorem B :**

Let  $M$  be a positive function of  $T$  such that  $M \log T$  lies between  $\text{Exp}(\psi(T) \log \log \log T)$  and  $(\log T)^c$  where  $c$  is any positive constant less than  $\frac{1}{2}$ , and  $\psi(T) < \frac{c}{2} \log \log T$  is any positive function which tends to infinity as  $T$  tends to infinity. Let us agree to call a zero  $\rho_0 = \beta_0 + i\gamma_0$  ( $T < \gamma_0 < 2T$ ,  $T > T_0$ ) "M-good" if

$$\left[ \sum_{\substack{p \\ |p - \rho_0| \leq M}} \frac{1}{p - \rho_0} \right] \left[ \sum_{\substack{p \\ |p - \rho_0| \leq M}} \frac{1}{p} \right]^{-1} = O(M(\log \log T)^{-1})$$

(in the sum over  $p$ ,  $p$  is counted according to its multiplicity), and

$$\zeta(s) \neq 0 \text{ in } \{ |\text{Im } s - \gamma_0| \leq (\log T)^2, \text{Re } s > \beta_0 + d\alpha \}$$

where  $d = 10^{-8}$ ,  $\alpha = 10^{-8000} m^{-1}$   $M$  and  $m = [\log \log T]$ . Then the number of "M-good" zeros (counted with multiplicity) in  $\{ \text{Re } s > \sigma_0, T < \text{Im } s \leq 2T \}$  ( $T > 10$ )

is  $O(T^{(2+\epsilon)(1-\sigma_0)})$ . Here the  $O$ -constant depends only on  $\epsilon$  and not on  $T$  and  $\sigma_0$  ( $> \frac{1}{2}$ ).

We then sketch a proof of the following

**Corollary 1 :**

Let  $\delta_0$  be any constant satisfying  $0 < \delta_0 < \frac{1}{2}$ . We shall fix it once for all. Let us agree to call a zero  $\rho$  "good" if  $\text{Re } \rho > \frac{1}{2}$

and further in a disc with centre  $\rho$  and radius  $(\log | \rho |)^{-1 + \delta_0}$  the number of zeros (of  $\zeta(s)$ ) counted with multiplicity, is

$$o \left[ \frac{\log \log | \rho |}{\log \log \log | \rho |} \right].$$

Let us call  $\rho$  to be "bad" if it is not good. If the number of bad zeros  $\rho$  with  $\text{Re } \rho > \sigma$  and  $| \text{Im } \rho | < T$ , is  $O(T^{B(1-\sigma)} + \epsilon)$  (where  $T > 10$ ,  $\sigma > \frac{1}{2}$ ,  $\epsilon > 0$ , and  $B$  is a positive constant independent of all parameters and the  $O$ -constant depends only on  $\epsilon$ , which is arbitrary) then  $N(\sigma, T) = O(T^{B(1-\sigma)} + 2\epsilon)$  is true. Here, as usual  $N(\sigma, T)$  is the number of zeros  $\rho$  of  $\zeta(s)$  with  $\text{Re } \rho > \sigma$ , and  $| \text{Im } \rho | < T$ .

In particular taking  $B = 2$  in the above corollary we can state qualitatively (but vaguely) the following

**Corollary 2 :**

If the thick clusters of zeros are rather rare in  $\sigma > \frac{1}{2}$  then density hypothesis is true.

We are very much thankful to Professor D. R. HEATH-BROWN for his letter. Since his result is not intended to be published we would like to regard the results of part B as joint work with him.

## PART - A

§ 2. We begin with

**Lemma 1.**

The number of primes  $p$  satisfying  $X \leq p \leq 2X$ , for which

$$\left| p^{i(\alpha_1 - \alpha_2)} - 1 \right| < \theta, \quad (0 < \theta < \frac{\pi}{10}),$$

does not exceed  $20J (\log J)^{-1}$ , where  $J = X \theta (\alpha_1 - \alpha_2)^{-1}$ .

$0 < \alpha_1 - \alpha_2 < 1$ , and  $J > 10^{80}$ .

**Proof.**

Treating  $p$  as a continuous variable we see that (as  $p$  varies over  $X < p < 2X$ ),  $p^{i(\alpha_1 - \alpha_2)}$  makes at most  $[(\alpha_1 - \alpha_2) (2\pi)^{-1} \log 2] + 1$  revolutions on the unit circle. In each revolution the inequality of the lemma defines a  $p$ -interval of length not exceeding  $5X\theta (\alpha_1 - \alpha_2)^{-1}$ . The Hardy-Littlewood upper bound (obtained by the fundamental method of Brun) for the number of primes in small intervals completes the proof of lemma 1. (However we have used the Selberg upper bound at the last step). For an exposition of Brun's method see [3].

**Lemma 2.**

Let  $0 < \alpha_1 - \alpha_2 < 1$ ,  $\delta = 10^{-5} (\alpha_1 - \alpha_2)^2$ , and  $X$  any real number  $> 10^{90}$ . Then there holds,

$$X < p < 2X \quad \left( \left| 1 + p^{i\alpha_1} \right|^2 + \left| 1 + p^{i\alpha_2} \right|^2 \right) \geq \delta \sum_{X < p < 2X} 1$$



**Proof.**

Suppose the lemma is false: Then the number of primes  $p$  for which  $|1 + p^{i\alpha_1}| > \eta$ , plus the number of primes  $p$  for which  $|1 + p^{i\alpha_2}| > \eta$ , does not exceed  $\delta\eta^{-2} X < \frac{\Sigma 1}{p} < 2X$ .

Hence at least  $(1 - \delta\eta^{-2}) \frac{\Sigma 1}{X} < \frac{\Sigma 1}{p} < 2X$  of the primes satisfy both

$|1 + p^{i\alpha_1}| \leq \eta$  and  $|1 + p^{i\alpha_2}| \leq \eta$  and hence  $|p^{i\alpha_1} - p^{i\alpha_2}| = |p^{i(\alpha_1 - \alpha_2)} - 1| \leq 2\eta$ . Thus we obtain from this and lemma 1, the inequality

$$(1 - \delta\eta^{-2}) \frac{\Sigma 1}{X} < \frac{\Sigma 1}{p} < 2X \leq \frac{40X\eta}{a_1 - a_2} \left[ \log \left[ \frac{2X\eta}{a_1 - a_2} \right] \right]^{-1}$$

provided  $2X\eta (a_1 - a_2)^{-1} \geq 10^{80}$ . We now set  $\eta = (2\delta)^{\frac{1}{2}}$ ,

where  $\delta = 10^{-5} (a_1 - a_2)^2$  and get

$$\frac{1}{3} \frac{X}{\log X} \leq \frac{X}{5} \left( \log \left( \frac{X}{200} \right) \right)^{-1},$$

which is a contradiction for all  $X \geq 10^{90}$ . This completes the proof of lemma 2.

We now almost copy down the section §2 of [2], with the changes sketched in the appendix of that paper. We denote by  $s$  a real number lying in  $\left(1, \frac{101}{100}\right)$ , and by  $\alpha$  a real number exceeding 100, and  $w$  will denote a complex variable. We denote by  $\beta_0 + i\alpha$  a fixed zero of  $\zeta(w)$  with  $\beta_0 > \frac{1}{2}$ ,  $\alpha > 100$ , and by  $C_1, C_2, C_3, \dots$  effective positive constants,

The 0-constants will be effective absolute constants. These constants are absolute numerical constants with the exception of  $C_4$  (and  $C_2$ ) which may depend on  $\lambda$  ( $0 < \lambda < \frac{1}{100}$ ), and we will specify how it depends on  $\lambda$ . The number  $\lambda$  occurs in the following way. We set  $\alpha = \alpha_1$ , and subject  $\alpha_2$  to

$$0 < \alpha_1 - \alpha_2 < 1. \quad D_\lambda(1 + i\alpha_1) \text{ and}$$

$D_\lambda(1 + i\alpha_2)$  denote discs with centres  $1 + i\alpha_1$  and  $1 + i\alpha_2$  respectively with radius  $\lambda$  each. We now start with lemma 1 of [2] and continue as in that paper.

**Lemma 3.** *Let*

$$\phi(s) = \phi_1(s) \phi_2(s), \text{ where } \phi_1(s) = \zeta^2(s) \zeta(s - i\alpha_1) \zeta(s + i\alpha_1), \text{ and } \phi_2(s) = \zeta^2(s) \zeta(s - i\alpha_2) \zeta(s + i\alpha_2).$$

*Then*

$$-\frac{\phi'(s)}{\phi(s)} = \sum_{n=1}^{\infty} \frac{b_n}{n^s},$$

where  $b_n$  are non-negative real numbers. Further there holds for

all  $X \geq 10^{90}$ , with  $\delta = 10^{-5} (\alpha_1 - \alpha_2)^2$ , the inequality,

$$\sum_{X < n < 2X} b_n \geq \frac{1}{2} \delta \sum_{X < n < 2X} \Lambda(n)$$

**Proof**

The proof is trivial except the last inequality which follows from lemma 2.

**Lemma 4.** *For all  $X \geq 1$ , we have by writing*

$$F(s) = \frac{\phi'(s)}{\phi(s)},$$

$$\frac{1}{2\pi i} \int_{\text{Re } w = 2} F(s+w) X^w \Gamma\left(\frac{w}{2}\right) dw =$$

$$-2 \sum_{n=1}^{\infty} \frac{b_n}{n^s} \text{Exp}\left(-\left(\frac{n}{X}\right)^2\right).$$

**Proof.**

Follows from  $\frac{1}{2\pi i} \int_{\text{Re } w = 2} \left(\frac{X}{n}\right)^w \Gamma\left(\frac{w}{2}\right) dw$

$$= 2 \text{Exp}\left(-\left(\frac{n}{X}\right)^2\right).$$

Moving the line of integration to  $\text{Re } w = -2 + \frac{1}{100}$ , and assuming that  $1 < s < 1 + \frac{1}{100}$ , we have

**Lemma 5.** If  $1 < s \leq 1 + \frac{1}{100}$ , then for  $X > 1$ ,  $\alpha_1 > 100$ ,

$$2 \sum_{n=1}^{\infty} \frac{b_n}{n^s} \left(1 - \text{Exp}\left(-\left(\frac{n}{X}\right)^2\right)\right) = 2\mathfrak{z}_1 + 2\mathfrak{z}_2 +$$

$$O\left(X^{1-s} + \frac{\log \alpha_1}{X}\right)$$

where

$$\mathfrak{z}_1 = \text{Re}\left(-X^{1-s} \Gamma\left(\frac{1-s}{2}\right) + \sum_{\rho} X^{\rho-s-i\alpha_1} \Gamma\left[\frac{\rho-s-i\alpha_1}{2}\right]\right)$$

and  $\mathfrak{z}_2$  is the same as  $\mathfrak{z}_1$  with  $\alpha_1$  replaced by  $\alpha_2$ . Here  $\rho = \beta + i\gamma$

runs over all the zeros of  $\zeta(w)$  with positive real part. (In future the same convention will be in force. Also the restriction of the sums to zeros which satisfy some conditions will be indicated below the summation symbol or explained at appropriate places)

We now write  $\rho_0 = \beta_0 + i\alpha_1$  and

$$\sum_3 = 2 \sum_{n=1}^{\infty} \frac{b_n}{n^s} \left(1 - \text{Exp} \left(-\left(\frac{n}{X}\right)^2\right)\right)$$

$$\sum_4 = \text{Re} \sum_{\rho \neq \rho_0} X^{\rho-s-i\alpha_1} \Gamma \left[ \frac{\rho-s-i\alpha_1}{2} \right],$$

$$\sum_5 = \text{Re} \sum_{\rho} X^{\rho-s-i\alpha_2} \Gamma \left[ \frac{\rho-s-i\alpha_2}{2} \right],$$

and we will see that lemmas 3 and 5 give

**Lemma 6 :**

Let  $1 < s < 1 + \frac{1}{100}$ ,  $X \geq (\log a_1)^{1000}$ . Then

$$2 \cdot 10^{-6} (a_1 - a_2)^2 \frac{X^{1-s}}{s-1} < 2(-X^{1-s} \Gamma\left(\frac{1-s}{2}\right) + X^{\beta_0-s} \Gamma\left(\frac{\beta_0-s}{2}\right)) + 2 \sum_4 + 2 \sum_5 + o(X^{1-s}).$$

**Proof :**

Follows from  $\sum_3 > 2 \cdot 10^{-6} (a_1 - a_2)^2 X^{1-s} (s-1)^{-1}$ ,

We now impose conditions on  $s$  and  $X$  so as to satisfy

$$\begin{aligned}
 & -10^{-6} (a_1 - a_2)^2 \frac{X^{1-s}}{s-1} - X^{1-s} \Gamma\left(\frac{1-s}{2}\right) \\
 & + X^{\beta_0 - s} \Gamma\left(\frac{\beta_0 - s}{2}\right) \\
 & < -\frac{1}{2} 10^{-6} (a_1 - a_2)^2 \frac{X^{1-s}}{s-1}.
 \end{aligned}$$

This requires

$$\begin{aligned}
 & -\frac{1}{2} 10^{-6} (a_1 - a_2)^2 + (s-1) \left( X^{\beta_0 - 1} \Gamma\left(\frac{\beta_0 - s}{2}\right) \right. \\
 & \left. - \Gamma\left(\frac{1-s}{2}\right) \right) < 0
 \end{aligned}$$

i. e.

$$\begin{aligned}
 & -\frac{1}{2} 10^{-6} (a_1 - a_2)^2 - 2X^{\beta_0 - 1} \Gamma\left(\frac{\beta_0 - s}{2} + 1\right) \\
 & - (1 - \beta_0) X^{\beta_0 - 1} \Gamma\left(\frac{\beta_0 - s}{2}\right) + 2\Gamma\left(\frac{1-s}{2} + 1\right) \leq 0
 \end{aligned}$$

i. e.

$$\begin{aligned}
 & -\frac{1}{2} 10^{-6} (a_1 - a_2)^2 + C_1 \left( X^{\beta_0 - 1} \frac{1 - \beta_0}{s - \beta_0} + \right. \\
 & \left. (1 - \beta_0) \log X \right) < 0.
 \end{aligned}$$

By putting  $s \geq 1 + C_2 (1 - \beta_0)$ , we are led to the requirement

$$-\frac{1}{2} 10^{-6} (a_1 - a_2)^2 + C_1 \left( \frac{X^{\beta_0 - 1}}{C_2} + (1 - \beta_0) \log X \right) \leq 0$$

We now record

**Lemma 7.** *We have*

$$0 \leq -\frac{1}{2} 10^{-6} (\alpha_1 - \alpha_2)^2 \frac{X^{1-s}}{s-1} + \Sigma_4 + \Sigma_5 + O(X^{1-s})$$

subject to the conditions

$$\alpha_1 \geq 100, X \geq (\log \alpha_1)^{1000},$$

$$1 + C_2 (1 - \beta_0) \leq s \leq 1 + \frac{1}{100} \text{ and}$$

$$-\frac{1}{2} 10^{-6} (\alpha_1 - \alpha_2)^2 + C_1 \left( \frac{X^{\beta_0 - 1}}{C_2} + (1 - \beta_0) \log X \right) \leq 0.$$

Here  $C_1$  and the 0-constants are absolute positive constants and  $C_2$  is any positive constant (in fact any positive variable) not exceeding  $(100(1 - \beta_0))^{-1}$ .

The next lemma is essentially due to Montgomery

**Lemma 8.**

*We have, for  $X > 1$ ,*

$$\operatorname{Re} \left( X^{\rho - s - i\alpha_1} \Gamma \left( \frac{\rho - s - i\alpha_1}{2} \right) \right) \leq C_3 X^{\beta - s} \operatorname{lg} X,$$

where  $C_3$  is a certain positive constant.

**Proof**

Let  $a$  denote the complex number  $\rho - s - i\alpha_1$ . Observe that the real part of  $a$  lies between  $-3/2$  and zero and write

$$\begin{aligned} X^a \Gamma \left( \frac{a}{2} \right) &= X^a \left( \Gamma \left( \frac{a}{2} \right) - \frac{2}{a} \right) + \frac{2}{a} (X^a - X^{\beta - s}) \\ &\quad + \frac{2}{a} X^{\beta - s} \end{aligned}$$

The first two terms on the right are  $O(X^{\beta-s} \log X)$  and the real part of the last term is negative. These remarks complete the proof of lemma 8

We next write

$$\begin{aligned} \Sigma_4 &= \operatorname{Re} \sum_{\rho \neq \rho_0} X^{\rho-s-ia_1} \Gamma\left(\frac{\rho-s-ia_1}{2}\right) \\ &= \Sigma_6 + \Sigma_7 \end{aligned}$$

$$\text{and } \Sigma_5 = \operatorname{Re} \sum_{\rho} X^{\rho-s-ia_2} \Gamma\left(\frac{\rho-s-ia_2}{2}\right) = \Sigma_8 + \Sigma_9$$

$$\text{where } \Sigma_6 = \operatorname{Re} \sum_{\rho \neq \rho_0} \rho \text{ in } D_{\lambda} (1 + ia_1)$$

$$\text{and } \Sigma_8 = \operatorname{Re} \sum_{\rho \text{ in } D_{\lambda}} (1 + ia_2).$$

We now write  $X = \operatorname{Exp}(Y + u_1 + u_2 + \dots + u_m + u_{m+1})$  where  $Y \geq 1000 \log a_1$ ,  $0 \leq u_k \leq C_4$  ( $k = 1, 2, \dots, m$ ),  $m = [Y]$ , and  $0 \leq u_{m+1} \leq Y$ . As  $u_1, u_2, \dots, u_{m+1}$  vary over this  $(m+1)$  dimensional box the maximum and the minimum values of  $X$  are  $X_1 = \operatorname{Exp}(C_4[Y] + 2Y)$  and  $X_0 = \operatorname{Exp}(Y)$ .

We now wish to average the inequality of lemma 7 over the  $(m+1)$  dimensional box defined already. To do this we have to impose the condition

$$-\frac{1}{2} 10^{-6} (\alpha_1 - \alpha_2)^2 + C_1 \left[ \frac{X_0^{\beta_0-1}}{C_2} + (1 - \beta_0) \log X_1 \right] < 0$$

We satisfy this by imposing the conditions

$$1000 \log \alpha_1 \leq Y \leq C_1^{-1} (C_4 + 2)^{-1} \left\{ \frac{1}{4} 10^{-6} (\alpha_1 - \alpha_2)^2 \right\} (1 - \beta_0)^{-1} \quad (1)$$

$$\text{and } \text{Exp}(-Y(1 - \beta_0)) \leq C_2 C_1^{-1} \left\{ \frac{1}{4} 10^{-6} (\alpha_1 - \alpha_2)^2 \right\} \quad (2)$$

$$\begin{aligned} \text{Trivially } -\frac{1}{2} 10^{-6} (\alpha_1 - \alpha_2)^2 \frac{X^{1-s}}{s-1} &\leq \\ &-\frac{1}{2} 10^{-6} (\alpha_1 - \alpha_2)^2 \frac{X_1^{1-s}}{s-1} \end{aligned} \quad (3)$$

$$\text{and } X^{1-s} \leq X_0^{1-s} \quad (4)$$

Our next task is to obtain an upper bound for the average

$$C_4^{-m} Y^{-1} \int \int \dots \int (\zeta_4 + \zeta_5) du_1 du_2 \dots du_{m+1} \quad (5)$$

We do this in lemmas 9 and 10 to follow. Before proceeding further we record the following relations.

$$m = [Y], X_0 = \text{Exp}(Y) \text{ and } X_1 \leq \text{Exp}((C_4 + 2)Y). \quad (6)$$

**Lemma 9**

*We have*

$$\begin{aligned} &C_4^{-m} Y^{-1} \int \int \dots \int (\zeta_7 + \zeta_9) du_1 \dots du_{m+1} \\ &\leq 4 C_5 C_4^2 Y^{-1} \left[ \frac{2}{C_4 \lambda} \right]^{m+2} (\log \alpha_1) X_0^{1-s}. \end{aligned}$$

Here  $C_5$  is an absolute positive constant and  $C_4$  is a positive constant still at our choice.



**Proof**

Trivially,

$$\begin{aligned}
 \text{LHS} &= 0 \left[ C_4^{-m} Y^{-1} \left\{ \sum_{|p-1-i\alpha_1| > \lambda} \frac{2^{m+1} X_0^{\beta-s}}{|p-s-i\alpha_1|^{m+2}} + \sum_{|p-1-i\alpha_2| > \lambda} \frac{2^{m+1} X_0^{\beta-s}}{|p-s-i\alpha_2/2|^{m+2}} \right\} \right] \\
 &\leq \frac{4C_5}{C_4^m Y \lambda} \left(\frac{2}{\lambda}\right)^{m+1} (\log \alpha_1) X_0^{1-s} \zeta(m+2) \\
 &\leq 4C_5 C_4^2 Y^{-1} \left(\frac{2}{C_4 \lambda}\right)^{m+2} (\log \alpha_1) X_0^{1-s}.
 \end{aligned}$$

This proves the lemma completely.

**Lemma 10.**

For any real number  $V$  let  $N_V(r)$  denote the number of zeros of the zeta-function in the disc  $D_r(1+iv)$ , with  $\lambda$  ( $0 < \lambda < \frac{1}{100}$ ),  $Y \geq 2$  and  $V \geq 10$  define

$$\begin{aligned}
 f(V) = f(\lambda, Y, V) &= \max_{V-\lambda \leq V \leq V+\lambda} \left\{ N_V(2\lambda) e^{-\lambda Y} \right. \\
 &\quad \left. + \int_0^{\lambda Y+1} N_V\left(\frac{2u}{Y}\right) e^{-u} du \right\}.
 \end{aligned}$$

Then

$$C_4^{-m} Y^{-1} \int \int \dots \int (\xi_6 + \xi_8) du_1 \dots du_{m+1}$$

$$< C_6 (C_4 + C_5 + 2) (f(a_1) + f(a_2)) Y X_0^{1-s},$$

where  $C_6$  and  $C_5$  are certain absolute positive constants and  $C_4 > 0$  is still at our choice.

**Proof.**

We shall get the upper bound

$$C_4^{-m} Y^{-1} \int \int \dots \int (\xi_6) du_1 \dots du_{m+1} \\ < C_6 (C_4 + C_5 + 2) f(a_1) Y$$

and leave the remaining part of the work containing  $\xi_8$  as a trivial imitation.

We start by writing

$$\xi_6 = \sum_n \operatorname{Re} \xi \dots = \sum_n S_n \text{ say,} \\ \frac{n}{Y} < |\gamma - a_1| < \frac{n+1}{Y}$$

where  $n$  runs over  $0, 1, 2, \dots, [\lambda Y] + 1$  and for these values of  $n$  the inner sum is over those zeros which satisfy the summation condition for  $\gamma$  indicated, plus of course the condition that  $\beta + i\gamma$  shall lie in  $D_\lambda(1 + ia_1)$ . The average of  $S_n$  is easily seen to be

$$< \begin{cases} C_7 (C_4 + C_5 + 2) Y \left\{ \sum_{|\gamma - a_1| < \frac{1}{Y}} X_0^{\beta-1} \right\} X_0^{1-s}, & \text{if } n = 0, \\ C_7 (C_4 + C_5 + 2) Y \left\{ \sum_{\frac{n}{Y} < |\gamma - a_1| < \frac{n+1}{Y}} X_0^{\beta-1} Y^{-2} |\gamma - a_1|^{-2} \right\} X_0^{1-s}, & \text{if } n > 1, \end{cases}$$

Here we have used lemma 8 in the case  $n = 0$ . Hence the average of  $\Sigma_6$  does not exceed

$$2C_7 (C_4 + C_5 + 2) Y (1 + \zeta(2)) X_0^{1-s} \left[ \max_I \sum_{\substack{\gamma - \alpha_1 \text{ in } I, \\ \rho \text{ in } D_\lambda(1 + i\alpha_1)}} X_0^{\beta-1} \right]$$

where  $I$  runs over all intervals of length  $\frac{1}{Y}$  contained in  $[\alpha_1 - \lambda, \alpha_1 + \lambda]$ . It remains to appraise the maximum.

We denote by  $N(\sigma, I, \frac{1}{Y})$  the number of zeros counted (with multiplicity) in the last sum (in  $\max_I$ ) with real parts  $> \sigma$ , and we have

$$\begin{aligned} \Sigma X_0^{\beta-1} &= \int_1^{1-\lambda} \text{Exp}(-Y(1-\sigma)) d N(\sigma, I, \frac{1}{Y}) \\ &= N(1-\lambda, I, \frac{1}{Y}) \text{Exp}(-\lambda Y) + \\ &\quad Y \int_{1-\lambda}^1 N(\sigma, I, \frac{1}{Y}) \text{Exp}(-Y(1-\sigma)) d\sigma \\ &< N_{\alpha_3} (2\lambda) \text{Exp}(-\lambda Y) + Y \int_0^\lambda N(1-u, I, \frac{1}{Y}) \\ &\quad \text{Exp}(-uY) du \\ &< N_{\alpha_3} (2\lambda) \text{Exp}(-\lambda Y) + Y \int_0^\lambda N_{\alpha_3} (2u + \frac{2}{Y}) \\ &\quad \text{Exp}(-uY) du \\ &< N_{\alpha_3} (2\lambda) \text{Exp}(-\lambda Y) + \int_0^{\lambda Y+1} N_{\alpha_3} (\frac{2u}{Y}) e^{-u} du \end{aligned}$$

where  $\alpha_3$  is any real number which is the end point of any interval  $I$  under consideration. A similar argument applies to the average of  $\mathfrak{z}_g$ . This completes the proof of the lemma.

Collecting the results of averaging the inequality of lemma 7 we state

**Lemma 11**

Let  $\zeta(\beta_0 + i\alpha_1) = 0$ , where  $\beta_0 > \frac{1}{2}$ ,  $\alpha_1 \geq 100$ ,

$1 + C_2(1 - \beta_0) \leq s \leq 1 + \frac{1}{100}$ , where  $C_2$  satisfies

$0 < C_2 \leq (100(1 - \beta_0))^{-1}$ ,  $0 < \alpha_1 - \alpha_2 < 1$ ,

$1000 \log \log \alpha_1 \leq Y \leq C_1^{-1} (C_4 + 2)^{-1} \left\{ \frac{1}{4} 10^{-6} (\alpha_1 - \alpha_2)^2 \right\} (1 - \beta_0)^{-1}$ ,

$\text{Exp}(-Y(1 - \beta_0)) \leq C_2 C_1^{-1} \left\{ \frac{1}{4} 10^{-6} (\alpha_1 - \alpha_2)^2 \right\}$

Then, we have with  $m = [Y]$ .

$0 \leq -\frac{1}{2} 10^{-6} (\alpha_1 - \alpha_2)^2 (s-1)^{-1} \text{Exp}(-(s-1)(C_4+1)Y)$

$+ 4C_5 C_4^2 Y^{-1} \left( \frac{2}{C_4 \lambda} \right)^{m+2} (\log \alpha_1) + C_8$

$+ C_6 (C_4 + C_5 + 2) (f(\alpha_1) + f(\alpha_2)) Y.$

In the above equations  $C_1, C_5, C_6$  and  $C_8$  are certain effective absolute positive constants and  $C_4 > 0$  is still arbitrary.

The number  $C_2$  is also arbitrary but is subject to the restriction stated above.

**Proof :**

Follows by (3), (4), (6) and lemmas 9 and 10. We have divided throughout by  $X_0^{1-s}$  after averaging, and used the inequality

$$- \left[ \frac{X_1}{X_0} \right]^{1-s} < - \text{Exp} (-(s-1) (C_4 + 1)Y).$$

This completes the proof of lemma 11.

Lemma 11 is the fundamental result that we have arrived at. We will now record a special case of lemma 11 as

**Theorem 1 :**

Let  $\zeta(\beta_0 + i\gamma_0) = 0$ , where  $\beta_0 > \frac{1}{2}$  and  $\gamma_0 \geq 100$ . Let  $0 < \lambda \leq \frac{1}{100}$ . Then there exist positive constants  $C_9'$ ,  $C_{10}$  and  $C_{11}$ , independent of  $\lambda$ ,  $\beta_0$  and  $\gamma_0$  such that if  $Y > C_9 \log \log \gamma_0 > C_9^2$ , there holds, for all such  $Y$ , the inequality

$$g(\gamma_0) > \frac{C_{10} \lambda^6}{Y(1-\beta_0)} - C_{11} \lambda^2,$$

where  $g(\gamma_0) = N_{\gamma_0}(4\lambda) e^{-\lambda Y} + \max_{(\gamma_0 - 2\lambda \leq v \leq \gamma_0 + 2\lambda)}$

$$- \int_0^{\lambda Y + 1} N_V \left( \frac{2u}{Y} \right) e^{-u} du,$$

and  $N_V(u)$  is the number of zeros  $\rho$  of the zeta-function which satisfy  $|\rho - 1 - iV| \leq u$ .

**Proof:**

Given any  $\lambda$  satisfying  $0 < \lambda < \frac{1}{200}$  we select  $\alpha_1 = \gamma_0$ ,  
 $\alpha_2 = \gamma_0 - \frac{\lambda}{100}$ . Put  $C_2 = 4 \cdot 10^6 C_1 (\alpha_1 - \alpha_2)^{-2}$ ,  
 $s = 1 + C_2(1 - \beta_0)$  and  $C_4 = 400 \lambda^{-2}$ . We note that since  
 $Y > C_9 \log \log \gamma_0$  the second term in the first inequality of  
lemma 11, and the term  $C_8$  can be replaced by another positive  
constant  $D_1$ . The resulting inequality is easily seen to be

$$g(\gamma_0) > \frac{D_2 \lambda^6}{Y(1 - \beta_0)} - D_3.$$

The upperbound  $Y = O((1 - \beta_0)^{-1} \lambda^4)$  for  $Y$  is unnecessary  
since we can increase  $D_3$  to make the RHS negative when  $Y$   
exceeds this bound. Also by the same reason <sup>the</sup> condition  
 $C_2 = O((1 - \beta_0)^{-1})$  is unnecessary. This completes the  
proof of Theorem 1.

**Corollary**

*If for all  $t \geq 30$  and all  $\sigma$  in  $\frac{3}{4} \leq \sigma \leq 1$  we have*

*$|\log |\zeta(\sigma + it)| \leq D_4 (1 - \sigma)^{3/2} \log t + D_5 \log \log t$ ,*  
*where  $D_4$  and  $D_5$  are positive numerical constants independent*  
*of  $\sigma$  and  $t$ , then*

$$\frac{1}{1 - \beta_0} = O((\log \gamma_0)^{2/3} (\log \log \gamma_0)^{1/3}).$$

**Remark 1.**

The statement involving  $D_4$  and  $D_5$  is actually true and it is a very deep result and is almost completely due to the ideas of I. M. VINOGRADOV. Hence we have the Vinogradov zero-free region

$$\sigma > 1 - D_6 (\log t)^{-2/3} (\log \log t)^{-1/3}$$

**Remark 2.**

For the history of Theorem 1 see [2]. Here it suffices to say that theorem 1 contains some extra information apart from giving the well known zero-free region of Vinogradov.

**Proof of the Corollary**

We choose  $\lambda$  to be a positive constant and suppose  $Y > \frac{100}{\lambda} \log \log Y_0$ . Then we can neglect the first term in  $g(Y_0)$ . The integral which occurs in  $\max \dots$  is

$$Y \int_0^{\lambda + \frac{1}{Y}} N(r) e^{-rY} dr$$

where  $N(r) = N_V(r)$ . Integrating by parts we get, introducing

$H(u) = \int_0^u \frac{N(r)}{r} dr$ , the expression

$$Y H(u) e^{-uY} \Big|_{u=0}^{u = \lambda + \frac{1}{Y}} - Y \int_0^{\lambda + \frac{1}{Y}} H(u) (ue^{-uY})' du$$

$$\leq Y H\left(\lambda + \frac{1}{Y}\right) e^{-\lambda Y} - Y \int_{\frac{1}{Y}}^{\lambda + \frac{1}{Y}} H(u) (ue^{-uY})' du$$

Now  $-(ue^{-uY})' > 0$  for  $u > \frac{1}{Y}$ . We now see by using

$$H(u) =$$

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left\{ \left| \zeta(1+iv + ue^{i\theta}) \right| \left| \zeta(1+iv) \right|^{-1} \right\} d\theta$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} (D_4 u^{3/2} \log \gamma_0 + (D_5 + 1000) \log \log t) d\theta$$

that 
$$\frac{1}{Y(1-\beta_0)} = O \left[ \frac{\log \gamma_0}{Y^{3/2}} + \log \log \gamma_0 \right].$$

i. e. 
$$\frac{1}{1-\beta_0} = O \left( \frac{\log \gamma_0}{Y^{1/2}} + Y \log \log \gamma_0 \right)$$

Choosing  $Y$  ( $Y \geq \frac{100}{\lambda} \log \log \gamma_0$ ), so as to minimise the

0-term we get the result stated in the corollary. For the result

on  $H(u)$  expressing it as  $\frac{1}{2\pi} \int_0^{2\pi} \dots$  see page 126 of

Titchmarsh's book 'The Theory of Functions', Oxford (1939).

This result is called Jensen's formula. An alternative method

is to apply maximum modulus principle to the function



$$\frac{\zeta(s) (\zeta(s_0))^{-1}}{\prod_{\rho} \left[ 1 - \frac{s - s_0}{\rho - s_0} \right]}, \quad \text{where}$$

$\rho$  runs over all the zéros counted in  $N(r)$  and  $s_0 = 1 + iV$ .

According to the maximum modulus principle the absolute value at  $s = s_0$  is not more than its maximum modulus on a circle of radius  $10r$  and centre  $S_0$ . This gives an upper bound for  $N(r)$ . This is enough to give Vinogradov zero-free region.

## PART-B

The notation in this part is independent of that of part A.

### Some preparations

3. We write  $F(s) = -\frac{\zeta'(s)}{\zeta(s)}$  and proceed to count the zeros  $\rho$  of  $\zeta(s)$  in  $\text{Re } s > \sigma$ ,

$T \leq \text{Im } s \leq 2T$  ( $\frac{1}{2} \leq \sigma \leq 1$ ). We write  $\rho = \beta + i\gamma$  for a typical zero of  $\zeta(s)$ . Let  $M$  be a positive function of  $T$  such that  $M \log T$  does not exceed  $(\log T)^c$  (where  $c$  is a positive constant  $< \frac{1}{2}$ ) but exceeds every fixed positive power of  $\log \log T$ .

$g$  and  $d$  will be positive constants which are not more than  $10^{-8}$ . Let  $\alpha, H, m$  and  $Y$  be positive functions of  $T$  to be chosen later. We write  $Z = e^Y, Y = e^{Y + u_1 + \dots + u_m}$  where  $u_1, \dots, u_m$  are independent real variables varying between 0 to

$H$  both inclusive.  $\alpha$  will be less than  $10^{-8}$  and  $H, m, Y$  tend to infinity with  $T$ . Also  $mH$  shall not exceed  $gY$ .

By well known techniques we have, for any fixed zero,  
 $\rho_0 = \beta_0 + i\gamma_0$  ( $\equiv$  denotes definition),

$$\begin{aligned} \phi(X, \rho_0) &\equiv \sum_{n=1}^{\infty} \Lambda(n) n^{-\rho_0} \left( e^{-\frac{n}{2X}} - e^{-\frac{n}{X}} \right) \\ &= \frac{1}{2\pi i} \int_{\text{Re } W = 2} F(\rho_0 + W) \Gamma(W) X^W (2^W - 1) dW \\ &= n_0 \log 2 + \sum_{\rho \neq \rho_0} \Gamma(\rho - \rho_0) X^{\rho - \rho_0} (2^{\rho - \rho_0} - 1) \\ &\quad + O\left(X^{-\frac{1}{2}}\right) \end{aligned}$$

(where  $n_0$  is the order of  $\rho_0$  and  $\rho$  runs over all the complex zeros of  $\zeta(s)$ ). Breaking the series  $\phi(X, \rho_0)$  suitably we get

$$\begin{aligned} \phi_1(X, \rho_0, Z) &\equiv \sum_{\substack{n \\ Z^{80} < n < Z^{80}}} \Lambda(n) n^{-\rho_0} \left( e^{-\frac{n}{2X}} - e^{-\frac{n}{X}} \right) \\ &= (n_0 + n_1) \log 2 + O\left(X^{-\frac{1}{10}}\right) \end{aligned}$$

$$\begin{aligned} &+ \sum_{|\rho - \rho_0| < M} (X^{\beta - \beta_0 + 1})^{|\rho - \rho_0|} Y \\ &+ \sum_{|\rho - \rho_0| > M} \Gamma(\rho - \rho_0) X^{\rho - \rho_0} (2^{\rho - \rho_0} - 1) \end{aligned}$$

where  $n_0 + n_1$  is the number of zeros  $P$  with  $|P - P_0| < M$ .

We multiply both sides by  $X^{-\alpha}$  and integrate with respect to  $u_1, \dots, u_m$ . We assume that there is no zero  $P$  with

$$|\gamma - \gamma_0| < (\log T)^2, \beta > \beta_0 + \alpha d, \text{ and also } \alpha < \frac{M}{2}.$$

Writing

$$G(s) = H^{-m} \int_0^H \dots \int_0^H X^{-\alpha} \phi_1(X, s, Z) du_1 \dots du_m$$

and  $R \equiv \sum_{|P - P_0| < M} |P - P_0|$ , we get,

$$\begin{aligned} |G(P_0)| &> (n_0 + n_1) \log 2 e^{-\alpha(1+g)Y} \\ &\quad + O(RY e^{-\alpha(1-d)Y} + Z^{\frac{1}{20}}) \\ &\quad + O\left(\frac{(\log T)^{50} 4^m e^{-\alpha(1-d)Y}}{(MH)^m}\right) \\ &> e^{-\alpha(1+g)Y} (n_0 + n_1) \log 2 \end{aligned}$$

$$\left[ 1 + O\left(\frac{RY e^{-\alpha(d+g)Y}}{n_0 + n_1} + Z^{-\frac{1}{40}}\right) + O\left(\frac{e^{\alpha(d+g)Y}}{(\log T)^{20}}\right) \right]$$

provided  $m = [\log \log T]$ , and  $H = 10^{8000} M^{-1}$ . Having

fixed  $m$  and  $H$  we satisfy  $mH < gY$  by writing  $Y = \frac{mH}{g}$ . We

can now fix  $d$  and  $g$  to be any two constants which may be

chosen to be equal. Next we put  $\alpha = \frac{1}{gY} = \frac{1}{MH}$

(note that  $\alpha < \frac{M}{2}$  is satisfied).

We next make the assumption

$$R = \sum_{\substack{|p-p_0| = 0 \\ |p-p_0| < M}} (M \log \log T)^{-1} \sum_{\substack{1 \\ |p-p_0| < M}}$$

(Note that we have already made an assumption about  $p_0$ ).

Under these two assumptions we see that  $G(s)$  serves as a zero detecting function which detects zeros of the type  $p_0$ .

Note  $\alpha d = \frac{1}{Y} = \frac{g}{MH}$  and that the conditions imposed on

$M$  enable one to deduce the estimate  $O(T^{(2+\epsilon)(1-\alpha)})$  for the number of such zeros  $p_0$ . Hence we state

**Theorem 2.** *The number of "M-good zeros" (counted with multiplicity) to be defined below, whose real part is  $> \sigma$  is  $O(T^{(2+\epsilon)(1-\sigma)})$ , where the 0-constant depends only on  $\epsilon$ , which is an arbitrary positive constant.*

**4. Definitions.** Accordingly we define three notions

**Definition: 1** A zero  $p_0 = \beta_0 + i\gamma_0$  ( $T < \gamma_0 < 2T$ ,  $1 > T_0$ ) of  $\zeta(s)$  is said to be M-good if

$$\begin{aligned} 1) \quad R_1 = R_1(p_0, M) &= \frac{\sum_{\substack{|p-p_0| \\ |p-p_0| < M}} 1}{\sum_{\substack{1 \\ |p-p_0| < M}} 1} \\ &= O(M (\log \log T)^{-1}) \end{aligned}$$

and

$$(ii) \zeta(s) \neq 0 \text{ in } \begin{cases} |\operatorname{Im} s - \gamma_0| < (\log T)^2 \\ \operatorname{Re} s > \beta_0 + d\alpha \end{cases}$$

where  $\alpha = 10^{-8000} m^{-1} M$  where  $m = [\log \log T]$  and  $d = 10^{-8}$ .

**Definition 2.** A zero  $\rho_0$  is said to be  $M_1 - M_2$  - good (where  $M_1 < M_2$  are two possible values of  $M$  such that

$$\frac{\log(M_2 M_1^{-1})}{\log \log \log T} \text{ tends to infinity) if}$$

$$n(\rho_0, M_2) \equiv \sum_{|\rho - \rho_0| < M_2} 1 = e \left( \frac{\log(M_1 M_2^{-1})}{\log \log \log T} \right)$$

**Definition 3.** A zero  $\rho_0$  is said to be good ( $\delta$  a positive constant, which is once for all fixed subject to  $0 < \delta < \frac{1}{2}$ ) if

$$\begin{aligned} n(\rho_0, (\log |\rho_0|)^{-1+\delta}) &= \sum_{|\rho - \rho_0| < (\log |\rho_0|)^{-1+\delta}} 1 \\ &= e \left( \frac{\log \log |\rho_0|}{\log \log \log |\rho_0|} \right). \end{aligned}$$

Otherwise  $\rho_0$  is said to be a bad zero.

The motivation behind the second definition is this. Put  $P_0 = M_1$ ,  $P_1 = P_0 U$ ,  $P_2 = P_0 U^2, \dots$  where  $U = (\log \log T)^3$  and stop at the highest  $j$  say  $J$  with  $P_j < M_2$ . If  $\rho_0$  is not  $P_j$ -good for any  $j$  then there are at least  $\frac{1}{8} J$  zeros  $\rho$  with

$|\rho - \rho_0| < M_2$ . For, suppose there is no zero  $\rho$  satisfying  $P_j < |\rho - \rho_0| < P_{j+1}$ . Then it follows

$$R_1(\rho_0, P_{j+1}) \gg P_{j+1} (\log \log T)^{-1}.$$

$$\begin{aligned} \text{But } R_1(\rho_0, P_{j+1}) &= \frac{\sum_{|\rho - \rho_0| < P_j} |\rho - \rho_0|}{\sum_{|\rho - \rho_0| < P_j} 1} \\ &< P_j = \frac{P_{j+1}}{(\log \log T)^3} \end{aligned}$$

Hence the  $M_1 - M_2$ -good zeros can be classified into  $P_j$ -good zero for at least one  $j$  and counted separately. Hence this number is again  $O(T^{(2+\delta)(1-\sigma)})$ . Of course we have the condition (ii) in the first definition. To meet this, we take the least  $\alpha_j = \alpha$  and count only the zeros  $\rho_0 = \beta_0 + i\gamma_0$  with  $\sigma < \beta_0 < \sigma + da$ . Making a suitable inductive hypothesis (such inductive arguments were first employed by M. JUTILA in connection with some density results) on  $N(\sigma + da, T)$  we can secure (ii) for most zeros  $\rho_0$ . In definition 3 we can take

$$M_1, M_2 \text{ suitably (e.g. } M_1 = (\log T)^{-1+\delta_1}, M_2 = (\log T)^{-1+\delta},$$

$0 < \delta_1 < \delta < \frac{1}{2}$ ) so that definition 3 and the results about good and bad zeros follow from definition 2.

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