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► **To cite this version:**

Marielle Simon. Hydrodynamic limit for the velocity-flip model. Cédric Bernardin and Patricia Gonçalves. From Particle Systems to Partial Differential Equations I, France. 75, Springer Berlin Heidelberg, pp.269-284, 2013, From Particle Systems to Partial Differential Equations, 978-3-642-54270-1. 10.1007/978-3-642-54271-8 . hal-01104050

HAL Id: hal-01104050

<https://hal.science/hal-01104050>

Submitted on 15 Jan 2015

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Hydrodynamic Limit for the Velocity-flip Model

Marielle Simon

Abstract We review the proof of the hydrodynamic limit for the velocity-flip model which is given in [12]. We study the diffusive scaling limit for a chain of N coupled harmonic oscillators. In order to provide the system with good ergodic properties, we perturb the Hamiltonian dynamics with random flips of velocities, so that the energy is still conserved. We derive the hydrodynamic equations by estimating the relative entropy with respect to the local equilibrium state modified by a correction term.

Key words: Hydrodynamic limit, Hamiltonian system, Velocity-flip model.

1 Introduction

We consider a Hamiltonian system of N coupled oscillators with the same mass that we set equal to 1. Since the ergodic properties of Hamiltonian dynamics are poorly understood, especially when the size of the system goes to infinity, we perturb it by an additional conservative mixing noise, as it has been proposed for the first time by Olla, Varadhan and Yau ([11]) in the context of gas dynamics, and then in [6] in the context of Hamiltonian lattice dynamics.

We are interested in the macroscopic behavior of this system as N goes to infinity, after rescaling space and time. The system is considered under periodic boundary conditions, more precisely we work on the one-dimensional discrete torus $\mathbb{T}_N := \{0, \dots, N-1\}$. A typical configuration is given by $\omega = (p_x, r_x)_{x \in \mathbb{T}_N}$ where p_x stands for the velocity of the oscillator at site x , and r_x represents the distance between oscillator x and oscillator $x+1$. The deterministic dynamics is described by the harmonic Hamiltonian

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$$\mathcal{H}_N = \sum_{x=0}^{N-1} \left[\frac{p_x^2 + r_x^2}{2} \right]. \quad (1)$$

The stochastic perturbation is added only to the velocities, in such a way that the energy of particles is still conserved. Nevertheless, the momentum conservation is no longer valid, so that we can hope for a normal diffusion of energy¹. The added noise can be easily described: each particle independently waits an exponentially distributed time interval and then flips the sign of velocity. The strength of the noise is regulated by the parameter $\gamma > 0$. The total deformation $\sum r_x$ and the total energy $\sum (p_x^2 + r_x^2) / 2$ are the only two conserved quantities. Thus, the Gibbs states are parametrized by two potentials, temperature and tension: for $\beta > 0$ and $\lambda \in \mathbb{R}$, the equilibrium Gibbs measures $\mu_{\beta, \lambda}^N$ on the configuration space $\Omega^N := (\mathbb{R} \times \mathbb{R})^{\mathbb{T}_N}$ are products of Gaussians (see (10)).

The goal is to prove that the two empirical profiles associated to the conserved quantities converge in the thermodynamic limit $N \rightarrow \infty$ to the macroscopic profiles $\mathbf{r}(t, \cdot)$ and $\mathbf{e}(t, \cdot)$, which satisfy an autonomous system of coupled parabolic equations. More precisely, let $\mathbf{r}_0 : \mathbb{T} \rightarrow \mathbb{R}$ and $\mathbf{e}_0 : \mathbb{T} \rightarrow \mathbb{R}$ be respectively the initial macroscopic deformation profile and the initial macroscopic energy profile defined on the one-dimensional torus $\mathbb{T} = [0, 1]$ and denote by μ_0^N the Gibbs local equilibrium associated to \mathbf{r}_0 and \mathbf{e}_0 (see (14) for the explicit formula). If the initial law of the process is μ_0^N , then the law of the process in the diffusive scale, namely at time tN^2 , is close in the large N limit, to the Gibbs local equilibrium associated to the functions $\mathbf{r}(t, q)$ and $\mathbf{e}(t, q)$ (defined on $\mathbb{R}_+ \times \mathbb{T}$), which are solutions of

$$\begin{cases} \partial_t \mathbf{r} = \frac{1}{\gamma} \partial_q^2 \mathbf{r}, \\ \partial_t \mathbf{e} = \frac{1}{2\gamma} \partial_q^2 \left(\mathbf{e} + \frac{\mathbf{r}^2}{2} \right), \end{cases} \quad q \in \mathbb{T}, t \in \mathbb{R}_+, \quad (2)$$

with the initial conditions $\mathbf{r}(0, \cdot) = \mathbf{r}_0(\cdot)$ and $\mathbf{e}(0, \cdot) = \mathbf{e}_0(\cdot)$.

We approach this problem by using the relative entropy² method introduced for the first time by H. T. Yau [15] for a gradient³ diffusive Ginzburg-Landau dynamics. Roughly speaking, we measure the distance between the Gibbs local equilibrium⁴ $\mu_{\mathbf{e}(t, \cdot), \mathbf{r}(t, \cdot)}^N$ and the state μ_t^N by their relative entropy $H_N(t)$ (see (28)). The strategy consists in proving that $\lim_{N \rightarrow \infty} H_N(t)/N = 0$ and deducing that the hydrodynamic limit holds. In the context of diffusive systems, the relative entropy method works if the following conditions are satisfied.

¹ If the momentum is conserved, anomalous behaviors can emerge, see for example [1], or [3].

² The relative entropy of the probability measure μ with respect to the probability measure ν is denoted by $H(\mu|\nu)$ and is defined in (15).

³ A conservative system is called gradient if the currents corresponding to the conserved quantities are gradients.

⁴ For the sake of readability, in the following sections we will denote it by $\mu_{\beta_t(\cdot), \lambda_t(\cdot)}^N$ (see (14)).

- First, the dynamics has to be *ergodic*: the only time and space invariant measures for the infinite system, with finite local entropy, are given by mixtures of the Gibbs measures in infinite volume $\mu_{\beta,\lambda}$ (see (16)). From [6], we know that the velocity-flip model is ergodic in the sense above (see Theorem 3).
- Next, we need to establish the so-called *fluctuation-dissipation equations* in the mathematics literature (for example, in [10]). Such equations express the microscopic current of energy (which here is not a discrete gradient) as the sum of a discrete gradient and a fluctuating term. More precisely, the microscopic current of energy, denoted by $j_{x,x+1}$, is defined by the local energy conservation law: $\mathcal{L}e_x = \nabla j_{x-1,x}$, where \mathcal{L} is the generator of the infinite dynamics. The standard approach consists in proving that there exist local functions f_x and h_x such that the following decomposition holds:

$$j_{x,x+1} = \nabla f_x + \mathcal{L}h_x . \quad (3)$$

Equation (3) is called a microscopic fluctuation-dissipation equation. The term $\mathcal{L}h_x$, when integrated in time, is a martingale. Roughly speaking, $\mathcal{L}h_x$ represents rapid fluctuation, whereas ∇f_x represents dissipation. Gradient models are systems for which $h_x = 0$ with the previous notations.

In general, these equations are not explicit but we are able to compute them in our model (see [12], App. A).

- Finally, since we observe the system on a diffusive scale and the system is non-gradient, we need second order approximations. If we want to obtain the entropy estimate of order $o(N)$, we can not work with the measure $\mu_{\mathbf{e}(t,\cdot),\mathbf{r}(t,\cdot)}^N$: we have to correct the Gibbs local equilibrium state with a small term. This idea was first introduced in [7] and then used in [13] for interacting Ornstein-Uhlenbeck processes, and in [9] for the asymmetric exclusion process. However, as far as we know, it is the first time that this is applied for a system with several conservation laws.

Up to present, the derivation of hydrodynamic equations for the harmonic oscillators perturbed by the velocity-flip noise is not rigorously achieved (see e.g. [5]), because the control of large energies has not been considered so far. Along the proof, we need to control all the following moments,

$$\int \left[\frac{1}{N} \sum_{x \in \mathbb{T}_N} |p_x|^k \right] d\mu_t^N , \quad (4)$$

uniformly in time and with respect to N . In fact, the only first moments are necessary to cut-off large energies and we need all the others to obtain the Taylor expansion that appears in the relative entropy method (Proposition 1). Usually, the following entropy inequality (true for any $\alpha > 0$ and any positive measurable function f)

$$\int f d\mu \leq \frac{1}{\alpha} \left\{ \log \left(\int e^{\alpha f} dv \right) + H(\mu|v) \right\} \quad (5)$$

reduces the control of (4) to the estimate of the following equilibrium exponential moments

$$\int \exp(\delta |p_x|^k) d\mu_{1,0}^N, \text{ with } \delta > 0 \text{ small.} \quad (6)$$

Unfortunately, in our model, these integrals are infinite for all $k \geq 3$ and all $\delta > 0$.

Bernardin [2] deals with a harmonic chain perturbed by a stochastic noise which is different from ours but has the same motivation: energy is conserved, momentum is not. He derives the hydrodynamic limit for a particular value of the intensity of the noise. In this case the hydrodynamic equations are simply given by two decoupled heat equations. The author highlights that good energy bounds are necessary to extend his work to other values of the noise intensity. In fact, in [2], only the following weak form is proved:

$$\lim_{N \rightarrow +\infty} \int \left[\frac{1}{N^2} \sum_{x \in \mathbb{T}_N} p_x^4 \right] d\mu_t^N = 0. \quad (7)$$

In [12], we get uniform control of (4) for our model (Theorem 2). Let us notice that the harmonicity of the chain is crucial to get this result: roughly speaking, it ensures that the set of mixtures of Gaussian probability measures is left invariant during the time evolution. The article is divided into two parts: after the main results being stated, we give the ideas of proof. All the results discussed here are in [12] to which we refer for the details.

2 The Velocity-flip Model

We consider the unpinned harmonic chain perturbed by the momentum-flip noise. Each particle has the same mass that we set equal to 1. A typical configuration is $\omega = (\mathbf{r}, \mathbf{p}) \in \Omega^N := (\mathbb{R} \times \mathbb{R})^{\mathbb{T}_N}$, where $\mathbf{r} = (r_x)_{x \in \mathbb{T}_N}$ and $\mathbf{p} = (p_x)_{x \in \mathbb{T}_N}$.

The generator of the dynamics is given by $\mathcal{L}_N := \mathcal{A}_N + \gamma \mathcal{S}_N$, where for any continuously differentiable function $f : \Omega^N \rightarrow \mathbb{R}$,

$$\mathcal{A}_N(f)(\mathbf{r}, \mathbf{p}) := \sum_{x \in \mathbb{T}_N} [(p_{x+1} - p_x) \partial_{r_x} f(\mathbf{r}, \mathbf{p}) + (r_x - r_{x-1}) \partial_{p_x} f(\mathbf{r}, \mathbf{p})], \quad (8)$$

$$\mathcal{S}_N(f)(\mathbf{r}, \mathbf{p}) := \frac{1}{2} \sum_{x \in \mathbb{T}_N} [f(\mathbf{r}, \mathbf{p}^x) - f(\mathbf{r}, \mathbf{p})]. \quad (9)$$

Here \mathbf{p}^x is the configuration obtained from \mathbf{p} by the flip of p_x into $-p_x$. The parameter $\gamma > 0$ regulates the strength of the random flip of momenta.

The operator \mathcal{A}_N is the Liouville operator of a chain of harmonic oscillators, and \mathcal{S}_N is the generator of the stochastic part of the dynamics that flips at random time the velocity of one particle. The dynamics conserves two quantities: the total deformation of the lattice $\mathcal{R} := \sum_{x \in \mathbb{T}_N} r_x$ and the total energy $\mathcal{E} := \sum_{x \in \mathbb{T}_N} e_x$, where $e_x = (p_x^2 + r_x^2)/2$. Observe that the total momentum is no longer conserved. The

deformation and the energy define a family of invariant measures depending on two parameters. For $\beta > 0$ and $\lambda \in \mathbb{R}$, we denote by $\mu_{\beta, \lambda}^N$ the Gaussian product measure on Ω^N given by

$$\mu_{\beta, \lambda}^N(\mathbf{dr}, \mathbf{dp}) := \prod_{x \in \mathbb{T}_N} \frac{e^{-\beta e_x - \lambda r_x}}{Z(\beta, \lambda)} dr_x dp_x, \quad (10)$$

where $Z(\beta, \lambda)$ is the partition function.

In the following, we shall denote by $\mu[\cdot]$ the expectation with respect to the measure μ . The thermodynamic relations between the averaged conserved quantities $\bar{\mathbf{r}} \in \mathbb{R}$ and $\bar{\mathbf{e}} \in (0, +\infty)$, and the potentials $\beta \in (0, +\infty)$ and $\lambda \in \mathbb{R}$ are given by

$$\begin{cases} \bar{\mathbf{e}}(\beta, \lambda) := \mu_{\beta, \lambda}^N[e_x] = \frac{1}{\beta} + \frac{\lambda^2}{2\beta^2}, \\ \bar{\mathbf{r}}(\beta, \lambda) := \mu_{\beta, \lambda}^N[r_x] = -\frac{\lambda}{\beta}. \end{cases} \quad (11)$$

Notice that

$$\forall \beta \in (0, +\infty), \forall \lambda \in \mathbb{R}, \bar{\mathbf{e}}(\beta, \lambda) > \frac{\bar{\mathbf{r}}^2(\beta, \lambda)}{2}. \quad (12)$$

Remark 1. There exists a bijection between the two sets $\{(\beta, \lambda) \in \mathbb{R}^2; \beta > 0\}$ and $\{(\mathbf{e}, \mathbf{r}) \in \mathbb{R}^2; \mathbf{e} > \mathbf{r}^2/2\}$. The equations above can be inverted according to the functional

$$\begin{aligned} \Psi : \{(\mathbf{e}, \mathbf{r}) \in \mathbb{R}^2; \mathbf{e} > \mathbf{r}^2/2\} &\rightarrow \{(\beta, \lambda) \in \mathbb{R}^2; \beta > 0\} \\ (\mathbf{e}, \mathbf{r}) &\mapsto \left(\frac{1}{\mathbf{e} - \mathbf{r}^2/2}, -\frac{\mathbf{r}}{\mathbf{e} - \mathbf{r}^2/2} \right). \end{aligned}$$

We assume that the system is initially close to a *local equilibrium*.

Definition 1. A sequence $(\mu^N)_N$ of probability measures on Ω^N is a *local equilibrium* associated to a deformation profile $\mathbf{r}_0 : \mathbb{T} \rightarrow \mathbb{R}$ and an energy profile $\mathbf{e}_0 : \mathbb{T} \rightarrow (0, +\infty)$ if for every continuous function $G : \mathbb{T} \rightarrow \mathbb{R}$ and for every $\delta > 0$, we have

$$\begin{cases} \lim_{N \rightarrow \infty} \mu^N \left[\left| \frac{1}{N} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) r_x - \int_{\mathbb{T}} G(q) \mathbf{r}_0(q) dq \right| > \delta \right] = 0, \\ \lim_{N \rightarrow \infty} \mu^N \left[\left| \frac{1}{N} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) e_x - \int_{\mathbb{T}} G(q) \mathbf{e}_0(q) dq \right| > \delta \right] = 0. \end{cases} \quad (13)$$

Example 1. For any integer N we define the probability measures

$$\mu_{\beta_0(\cdot), \lambda_0(\cdot)}^N(\mathbf{dr}, \mathbf{dp}) := \prod_{x \in \mathbb{T}_N} \frac{\exp(-\beta_0(x/N)e_x - \lambda_0(x/N)r_x)}{Z(\beta_0(x/N), \lambda_0(x/N))} dr_x dp_x, \quad (14)$$

where the two profiles β_0 and λ_0 are related to \mathbf{e}_0 and \mathbf{r}_0 by (11). Then, this sequence of probability measures is a local equilibrium, and it is called the *Gibbs local equilibrium state* associated to the macroscopic profiles β_0, λ_0 . Both profiles are assumed to be continuous.

To establish the hydrodynamic limits, we look at the process with generator $N^2 \mathcal{L}_N$, namely in the diffusive scale. The configuration at time tN^2 is denoted by ω_t^N , and the law of the process $(\omega_t^N)_{t \geq 0}$ is denoted by μ_t^N .

2.1 Hydrodynamic Equations

Let μ and ν be two probability measures on the same measurable space (X, \mathcal{F}) . We define the relative entropy $H(\mu|\nu)$ of the probability measure μ with respect to the probability measure ν by

$$H(\mu|\nu) := \sup_f \left\{ \int_X f \, d\mu - \log \left(\int_X e^f \, d\nu \right) \right\}, \quad (15)$$

where the supremum is carried over all bounded measurable functions f on X . The Gibbs states in infinite volume are the probability measures $\mu_{\beta, \lambda}$ on $\Omega = (\mathbb{R} \times \mathbb{R})^{\mathbb{Z}}$ given by

$$\mu_{\beta, \lambda}(\mathbf{dr}, \mathbf{dp}) := \prod_{x \in \mathbb{Z}} \frac{e^{-\beta e_x - \lambda r_x}}{Z(\beta, \lambda)} \mathbf{dr}_x \mathbf{dp}_x. \quad (16)$$

We denote by $\tau_x \varphi$ the shift of φ : $(\tau_x \varphi)(\omega) = \varphi(\tau_x \omega) = \varphi(\omega(x + \cdot))$. Hereafter, all statements involving time t assume that t belongs to a compact set $[0, T]$. In [12] the following theorem is proved.

Theorem 1. *Let $(\mu_0^N)_N$ be a sequence of probability measures on Ω^N which is a local equilibrium associated to a deformation profile \mathbf{r}_0 and an energy profile \mathbf{e}_0 such that $\mathbf{e}_0 > \mathbf{r}_0^2/2$. We denote by β_0 and λ_0 the potential profiles associated to \mathbf{r}_0 and \mathbf{e}_0 : $(\beta_0, \lambda_0) := \Psi(\mathbf{e}_0, \mathbf{r}_0)$.*

We assume that the initial profiles are continuous, and that

$$H\left(\mu_0^N | \mu_{\beta_0(\cdot), \lambda_0(\cdot)}^N\right) = o(N). \quad (17)$$

We also assume that the energy moments are bounded: let us suppose that there exists a positive constant C which does not depend on N and t , such that

$$\forall k \geq 1, \mu_t^N \left[\sum_{x \in \mathbb{T}_N} e_x^k \right] \leq (Ck)^k \times N. \quad (18)$$

Let G be a continuous function on the torus \mathbb{T} and φ be a local function which satisfies the following property: there exists a finite subset $\Lambda \subset \mathbb{Z}$ and a constant $C > 0$ such that, for all $\omega \in \Omega^N$, $\varphi(\omega) \leq C(1 + \sum_{i \in \Lambda} e_i(\omega))$. Then,

$$\mu_t^N \left[\left[\frac{1}{N} \sum_x G\left(\frac{x}{N}\right) \tau_x \varphi - \int_{\mathbb{T}} G(q) \tilde{\varphi}(\mathbf{e}(t, q), \mathbf{r}(t, q)) dq \right] \right] \xrightarrow{N \rightarrow \infty} 0 \quad (19)$$

where $\tilde{\varphi}$ is the grand-canonical expectation of φ : in other words, for any $(\mathbf{e}, \mathbf{r}) \in \mathbb{R}^2$, if $(\beta, \lambda) = \Psi(\mathbf{e}, \mathbf{r})$ then

$$\tilde{\varphi}(\mathbf{e}, \mathbf{r}) = \mu_{\beta, \lambda}[\varphi] = \int_{(\mathbb{R} \times \mathbb{R})^{\mathbb{Z}}} \varphi(\omega) d\mu_{\beta, \lambda}(\omega). \quad (20)$$

Besides, \mathbf{e} and \mathbf{r} are defined on $\mathbb{R}_+ \times \mathbb{T}$ and are solutions of

$$\begin{cases} \partial_t \mathbf{r} = \frac{1}{\gamma} \partial_q^2 \mathbf{r}, \\ \partial_t \mathbf{e} = \frac{1}{2\gamma} \partial_q^2 \left(\mathbf{e} + \frac{\mathbf{r}^2}{2} \right), \end{cases} \quad q \in \mathbb{T}, t \in \mathbb{R}_+, \quad (21)$$

with the initial conditions $\mathbf{r}(\cdot, 0) = \mathbf{r}_0(\cdot)$ and $\mathbf{e}(\cdot, 0) = \mathbf{e}_0(\cdot)$.

Remark 2. Let us notice that the functions $\mathbf{e}, \mathbf{r}, \beta$ and λ are smooth when $t > 0$, since the system of partial differential equations is parabolic.

In Sec. 4, we will see that the hypothesis on moments bounds (18) holds for a large class of initial local equilibrium states. Before stating the theorem, we give some definitions. We denote by $\mathfrak{S}_N(\mathbb{R})$ the set of real symmetric matrices of size N . The correlation matrix $C \in \mathfrak{S}_{2N}(\mathbb{R})$ of a probability measure ν on Ω^N is the symmetric matrix $C = (C_{i,j})_{1 \leq i, j \leq 2N}$ defined by

$$C_{i,j} := \begin{cases} \nu[r_i r_j] & i, j \in \{1, \dots, N\}, \\ \nu[r_i p_j] & i \in \{1, \dots, N\}, j \in \{N+1, \dots, 2N\}, \\ \nu[p_i r_j] & i \in \{N+1, \dots, 2N\}, j \in \{1, \dots, N\}, \\ \nu[p_i p_j] & i, j \in \{N+1, \dots, 2N\}. \end{cases} \quad (22)$$

Let us denote by Σ_N the subset of $\mathbb{R}^{2N} \times \mathfrak{S}_{2N}(\mathbb{R})$ defined by the following condition:

$$(m, C) \in \Sigma_N \Leftrightarrow \begin{cases} m_k = 0 & \text{for all } k = N+1 \dots 2N, \\ C_{i,j} = 0 & \text{for all } i \neq j, \\ C_{i,i} > 0 & \text{for all } i = 1 \dots 2N, \\ C_{i,i} - m_i^2 = C_{i+N, i+N} & \text{for all } i = 1 \dots N. \end{cases} \quad (23)$$

Precisely, it means that m is written as $m = (m_1, \dots, m_N, 0, \dots, 0)$, and C is a diagonal matrix whose components are $(m_1^2 + \alpha_1, \dots, m_N^2 + \alpha_N, \alpha_1, \dots, \alpha_N)$, where $\alpha_i > 0$ for all $i = 1 \dots N$. For $(m, C) \in \Sigma_N$, we denote by $G_{m,C}(\cdot)$ the Gaussian measure with mean m and correlations given by the matrix C . The covariance matrix of $G_{m,C}(\cdot)$ is thus $C - m' m$. In [12] the following lemma is proved.

Lemma 1. *Let λ and β be two functions of class \mathcal{C}^1 defined on \mathbb{T} , and $\mu_{\beta(\cdot),\lambda(\cdot)}^N$ be the Gibbs local equilibrium defined by (14). If we denote by $m_{\beta(\cdot),\lambda(\cdot)}$ and $C_{\beta(\cdot),\lambda(\cdot)}$ respectively the mean vector and the correlation matrix of the probability measure $\mu_{\beta(\cdot),\lambda(\cdot)}^N$, then we have*

$$(m_{\beta(\cdot),\lambda(\cdot)}, C_{\beta(\cdot),\lambda(\cdot)}) \in \Sigma_N \quad \text{and} \quad \mu_{\beta(\cdot),\lambda(\cdot)}^N = G_{m_{\beta(\cdot),\lambda(\cdot)}, C_{\beta(\cdot),\lambda(\cdot)}}. \quad (24)$$

Now we state our second main theorem.

Theorem 2. *We assume that the initial probability measure μ_0^N is a Gibbs local equilibrium state, defined by (14).*

Then, (18) holds, and the conclusions of Theorem 1 are valid.

In the following, we will denote by $\mathbf{e}_t(\cdot)$, $\mathbf{r}_t(\cdot)$, $\lambda_t(\cdot)$ and $\beta_t(\cdot)$ respectively the functions $q \rightarrow \mathbf{e}(t, q)$, $q \rightarrow \mathbf{r}(t, q)$, $q \rightarrow \lambda(t, q)$, and $q \rightarrow \beta(t, q)$ defined on \mathbb{T} .

2.2 Ergodicity of the Infinite Volume Velocity-flip Model

We conclude this part by giving the ergodicity theorem, which is proved in [3], Secs 2.2 and 2.4.2, by following the ideas of [6]. We define, for all finite subsets $\Lambda \subset \mathbb{Z}$, and for two probability measures ν and μ on $\Omega = (\mathbb{R} \times \mathbb{R})^{\mathbb{Z}}$, the restricted relative entropy $H_\Lambda(\nu|\mu) := H(\nu_\Lambda|\mu_\Lambda)$ where ν_Λ and μ_Λ are the marginal distributions of ν and μ on Ω . The Gibbs states in infinite volume are the probability measures $\mu_{\beta,\lambda}$ on Ω given by (16). The formal generator of the infinite dynamics is denoted by \mathcal{L} (respectively \mathcal{A} and \mathcal{S} for the antisymmetric and the symmetric part).

Theorem 3. *Let ν be a probability measure on the configuration space Ω such that*

1. *ν has finite density entropy: there exists $C > 0$ such that for all finite subsets Λ of \mathbb{Z} , $H_\Lambda(\nu|\mu_*) \leq C|\Lambda|$, with $\mu_* := \mu_{1,0}$ a reference Gibbs measure on $(\mathbb{R} \times \mathbb{R})^{\mathbb{Z}}$;*
2. *ν is translation invariant ;*
3. *ν is stationary: for any compactly supported and differentiable function $F(\mathbf{r}, \mathbf{p})$,*

$$\int \mathcal{A}(F) d\nu = 0; \quad (25)$$

4. *the conditional probability distribution of \mathbf{p} given the probability distribution of \mathbf{r} , denoted by $\nu(\mathbf{p}|\mathbf{r})$, is invariant by any flip $\mathbf{p} \rightarrow \mathbf{p}^x$, with $x \in \mathbb{Z}$.*

Then, ν is a mixture of infinite Gibbs states.

Corollary 1. *If ν is a probability measure on Ω satisfying 1, 2 and if ν is stationary in the sense that: for any compactly supported and differentiable function $F(\mathbf{r}, \mathbf{p})$,*

$$\int \mathcal{L}(F) d\nu = 0, \quad (26)$$

then ν is a mixture of infinite Gibbs states.

3 The Relative Entropy Method

For the sake of simplicity, we denote all couples of the form $(\beta(\cdot), \lambda(\cdot))$ by $\chi(\cdot)$. First, we introduce the corrected local Gibbs state $\nu_{\chi_t(\cdot)}^N$ defined by

$$\frac{d\nu_{\chi_t(\cdot)}^N}{d\mathbf{r}d\mathbf{p}} := \frac{1}{Z(\chi_t(\cdot))} \prod_{x \in \mathbb{T}_N} \exp \left(-\beta_t \left(\frac{x}{N} \right) e_x - \lambda_t \left(\frac{x}{N} \right) r_x + \frac{1}{N} F \left(t, \frac{x}{N} \right) \cdot \tau_x h(\mathbf{r}, \mathbf{p}) \right) \quad (27)$$

where $Z(\chi_t(\cdot))$ is the partition function. Functions F and h should be judiciously chosen, and are explicitly defined in [12].

We are going to use the relative entropy method, with the corrected local Gibbs state $\nu_{\chi_t(\cdot)}^N$ instead of the usual one $\mu_{\chi_t(\cdot)}^N$. We define

$$H_N(t) := H \left(\mu_t^N | \nu_{\chi_t(\cdot)}^N \right) = \int_{\Omega^N} f_t^N(\omega) \log \frac{f_t^N(\omega)}{\phi_t^N(\omega)} d\mu_{1,0}^N(\omega), \quad (28)$$

where f_t^N is the density of μ_t^N with respect to the reference measure $\mu_{1,0}^N$. In the same way, ϕ_t^N is the density of $\nu_{\chi_t(\cdot)}^N$ with respect to $\mu_{1,0}^N$ (which here is easily computable). The objective is to prove a Gronwall estimate of the entropy production of the form

$$\partial_t H_N(t) \leq C H_N(t) + o(N), \quad (29)$$

where $C > 0$ does not depend on N . In order to prove Theorem 1, we show in [12] that $H_N(t) = o(N)$ and this implies the existence of the hydrodynamic limit in the sense given in the theorem, by using the relative entropy inequality (5). For a proof of this last step, we refer the reader to [3], Proposition 3.3.2. and [8]. Thus, our purpose now is to prove (29).

We begin with the following lemma, proved in [8], Chap. 6, Lemma 1.4 and [4], Sec. 3.2. The operator $\mathcal{L}_N^* = -\mathcal{A}_N + \gamma \mathcal{S}_N$ is the adjoint of \mathcal{L}_N in $\mathbb{L}^2(\mu_{1,0}^N)$.

Lemma 2.

$$\partial_t H_N(t) \leq \int \frac{1}{\phi_t^N} (N^2 \mathcal{L}_N^* \phi_t^N - \partial_t \phi_t^N) f_t^N d\mu_{1,0} = \mu_t^N \left[\frac{1}{\phi_t^N} (N^2 \mathcal{L}_N^* \phi_t^N - \partial_t \phi_t^N) \right].$$

We define $\xi_x := (e_x, r_x)$ and $\eta(t, q) := (\mathbf{e}(t, q), \mathbf{r}(t, q))$. If f is a vectorial function, we denote its differential by Df . In [12], we prove that we can choose the correction term to obtain the following technical result.

Proposition 1. *The term $(\phi_t^N)^{-1} (N^2 \mathcal{L}_N^* \phi_t^N - \partial_t \phi_t^N)$ is given by the sum of five terms in which a microscopic expansion up to the first order appears.*

In other words,

$$\begin{aligned}
& (\phi_t^N)^{-1} (N^2 \mathcal{L}_N^* \phi_t^N - \partial_t \phi_t^N) \\
&= \sum_{k=1}^5 \sum_{x \in \mathbb{T}_N} v_k \left(t, \frac{x}{N} \right) \left[J_x^k - H_k \left(\eta \left(t, \frac{x}{N} \right) \right) - (DH_k) \left(\eta \left(t, \frac{x}{N} \right) \right) \cdot \left(\xi_x - \eta \left(t, \frac{x}{N} \right) \right) \right] \\
& \qquad \qquad \qquad + o(N) \quad (30)
\end{aligned}$$

where

k	J_x^k	$H_k(\mathbf{e}, \mathbf{r})$	$v_k(t, q)$
1	$p_x^2 + r_x r_{x-1} + 2\gamma p_x r_{x-1}$	$\mathbf{e} + \mathbf{r}^2/2$	$-(2\gamma)^{-1} \partial_q^2 \beta(t, q)$
2	$r_x + \gamma p_x$	\mathbf{r}	$-\gamma^{-1} \partial_q^2 \lambda(t, q)$
3	$p_x^2 (r_x + r_{x-1})^2$	$(2\mathbf{e} - \mathbf{r}^2) (\mathbf{e} + 3\mathbf{r}^2/2)$	$(4\gamma)^{-1} [\partial_q \beta(t, q)]^2$
4	$p_x^2 (r_x + r_{x-1})$	$\mathbf{r} (2\mathbf{e} - \mathbf{r}^2)$	$\gamma^{-1} \partial_q \beta(t, q) \partial_q \lambda(t, q)$
5	p_x^2	$\mathbf{e} - \mathbf{r}^2/2$	$\gamma^{-1} [\partial_q \lambda(t, q)]^2$

(31)

Remark 3. Along the proof, the so-called fluctuation-dissipation equations will play a crucial role, in particular for the choice of functions F, h .

A priori the first term on the right-hand side of (30) is of order N , but we want to take advantage of these microscopic Taylor expansions. First, we need to cut-off large energies in order to work with bounded variables only. Second, the strategy consists in performing a one-block estimate: we replace the empirical truncated current which is averaged over a microscopic box centered at x by its mean with respect to a Gibbs measure with the parameters corresponding to the microscopic averaged profiles. This is achieved thanks to the ergodicity of the dynamics (see Theorem 3). A one-block estimate is performed for each term of the form

$$\sum_{x \in \mathbb{T}_N} v_k \left(t, \frac{x}{N} \right) \left[J_x^k - H_k \left(\eta \left(t, \frac{x}{N} \right) \right) - (DH_k) \left(\eta \left(t, \frac{x}{N} \right) \right) \cdot \left(\xi_x - \eta \left(t, \frac{x}{N} \right) \right) \right]. \quad (32)$$

We deal with error terms by taking advantage of the following equality

$$H_k \left(\eta \left(t, \frac{x}{N} \right) \right) = \mathbf{v}_{\mathcal{X}_t(x/N)}^N (J_0^k) \quad (33)$$

and by using the large deviation properties of the probability measure $\mathbf{v}_{\mathcal{X}_t(\cdot)}^N$, that locally is almost homogeneous. Along the proof, we will need to control, uniformly in N , the quantity

$$\int \sum_{x \in \mathbb{T}_N} \exp \left(\frac{e_x}{N} \right) d\mu_t^N. \quad (34)$$

In fact, to get the convenient estimate, it is not difficult to see that it is sufficient to prove (18). For all the details, we refer the reader to [12], where the proof is written following the lines of [3], Sec. 3.3 and inspired from [11].

4 Proof of Theorem 2: Moments Bounds

Now we review how to prove the two conditions on the moments bounds for a class of local equilibrium states. Hereafter we assume that the initial law μ_0^N is the Gibbs local equilibrium state $\mu_{\beta_0(\cdot), \lambda_0(\cdot)}^N$.

We need to control the moments $\mu_t^N [\sum_x e_x^k]$ for all $k \geq 1$. The first two bounds would be sufficient to justify the cut-off of the currents, but here we need more bounds because of the Taylor expansion (Proposition 1). Precisely, the moments bounds are necessary to get the term of order $o(N)$ in the right hand-side of (32). Since the chain is harmonic, Gibbs states are gaussian. Remarkably, all Gaussian moments can be expressed in terms of variances and covariances. We start with an other representation of the dynamics of the process, and then we prove the bounds and describe their dependence on k . Let us highlight that, from now on, we consider the process with generator \mathcal{L}_N : it is not accelerated any more. The law of this new process $(\tilde{\omega}_t)_{t \geq 0}$ is denoted by $\tilde{\mu}_t^N$. Theorem 2 will be easily deduced since all estimates will not depend on t , and the following equality still holds: $\mu_t^N = \tilde{\mu}_{t/N^2}^N$.

Remark 4. 1. In the following, we always respect the decomposition of the space $\Omega^N = \mathbb{R}^N \times \mathbb{R}^N$. Let us recall that the first N components stand for \mathbf{r} and the last N components stand for \mathbf{p} . All vectors and matrices are written according to this decomposition. Let ν be a measure on Ω^N . We denote by $m \in \mathbb{R}^{2N}$ its mean vector and by $C \in \mathfrak{M}_{2N}(\mathbb{R})$ its correlation matrix (see (22)). There exist $\rho := \nu[\mathbf{r}] \in \mathbb{R}^N$, $\pi := \nu[\mathbf{p}] \in \mathbb{R}^N$ and $U, V, Z \in \mathfrak{M}_N(\mathbb{R})$ such that

$$m = (\rho, \pi) \in \mathbb{R}^{2N} \quad \text{and} \quad C = \begin{pmatrix} U & Z^* \\ Z & V \end{pmatrix} \in \mathfrak{S}_{2N}(\mathbb{R}). \quad (35)$$

Hereafter, we denote by Z^* the transpose of the matrix Z .

2. Thanks to the convexity inequality $(a+b)^k \leq 2^{k-1} (a^k + b^k)$ ($a, b > 0$, k a positive integer), we have

$$e_x^k \leq \frac{1}{2} (p_x^{2k} + r_x^{2k}). \quad (36)$$

Thus, instead of proving (18) we can show

$$\mu_t^N \left[\sum_{x \in \mathbb{T}_N} p_x^{2k} \right] \leq (Ck)^k \times N \quad \text{and} \quad \mu_t^N \left[\sum_{x \in \mathbb{T}_N} r_x^{2k} \right] \leq (Ck)^k \times N. \quad (37)$$

4.1 Poisson Process and Gaussian Measures

We start by giving a graphical representation of the process $(\tilde{\omega}_t)_{t \geq 0}$. Let us define

$$A := \begin{pmatrix} 0 & \cdots & \cdots & 0 & -1 & 1 & & (0) \\ \vdots & & & \vdots & 0 & \ddots & \ddots & \\ \vdots & & & \vdots & 0 & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & \ddots & & 0 & \vdots & & & \vdots \\ & \ddots & \ddots & 0 & \vdots & & & \vdots \\ (0) & & -1 & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \in \mathfrak{M}_{2N}(\mathbb{R}). \quad (38)$$

We consider the Markov process $(m_t, C_t)_{t \geq 0}$ on $\mathbb{R}^{2N} \times \mathfrak{S}_{2N}(\mathbb{R})$ defined by its generator \mathcal{G} , which can be written as follows.

Take $m = (\rho, \pi) \in \mathbb{R}^{2N}$ and $C = \begin{pmatrix} U & Z^* \\ Z & V \end{pmatrix} \in \mathfrak{S}_{2N}(\mathbb{R})$, with two vectors $\rho, \pi \in \mathbb{R}^N$ and three matrices $U, V \in \mathfrak{S}_N(\mathbb{R})$, $Z \in \mathfrak{M}_N(\mathbb{R})$. The generator \mathcal{G}_N is given by

$$(\mathcal{G}_N v)(m, C) := (\mathcal{K}_N v)(m, C) + \gamma (\mathcal{H}_N v)(m, C), \quad (39)$$

where

$$\mathcal{K}_N := \sum_{i,j \in \mathbb{T}_N} (-AC + CA)_{i,j} \partial_{C_{i,j}} + \sum_{i \in \mathbb{T}_N} \{(\pi_{i+1} - \pi_i) \partial_{\rho_i} + (\rho_i - \rho_{i-1}) \partial_{\pi_i}\}, \quad (40)$$

$$(\mathcal{H}_N v)(m, C) := \frac{1}{2} \sum_{k \in \mathbb{T}_N} [v(m^k, C^k) - v(m, C)]. \quad (41)$$

In these formulas, we define $m^k := (\rho, \pi^k)$ and $C^k := \Sigma_k^* \cdot C \cdot \Sigma_k = \begin{pmatrix} U & Z^{k*} \\ Z^k & V^k \end{pmatrix}$, where π^k is the vector obtained from π by the flip of π_k into $-\pi_k$, and Σ_k is

$$\Sigma_k := \begin{pmatrix} I_n & 0_n \\ 0_n & I_n - 2E_{k,k} \end{pmatrix}. \quad (42)$$

Here, $E_{i,j}$ denotes the (n, n) -matrix which has only one non-zero entry, the component (i, j) , equal to 1.

We denote by \mathbb{P}_{m_0, C_0} the law of the process $(m_t, C_t)_{t \geq 0}$ starting from (m_0, C_0) , and by $\mathbb{E}_{m_0, C_0}[\cdot]$ the expectation with respect to \mathbb{P}_{m_0, C_0} . For $t \geq 0$ fixed, let $\theta_{m_0, C_0}^t(\cdot, \cdot)$ be the law of the random variable $(m_t, C_t) \in \mathbb{R}^{2N} \times \mathfrak{S}_{2N}(\mathbb{R})$, knowing that the process starts from (m_0, C_0) .

The following lemma, which is proved in [12], gives the link between the two Markov processes defined in this paper. The proof is based on the Harris description.

Lemma 3. *Let $\mu_0^N := \mu_{\beta_0(\cdot), \lambda_0(\cdot)}^N$ be the Gibbs equilibrium state defined by (14), where $\lambda_0(\cdot)$ and $\beta_0(\cdot)$ are two macroscopic potential profiles.*

Then,

$$\tilde{\mu}_t^N = \int G_{m,C}(\cdot) d\theta_{m_0,C_0}^t(m,C), \quad (43)$$

where the components of $(m_0, C_0) \in \Sigma_N$ can be explicitly expressed (and depend on λ_0 and β_0).

Remark 5. Observe that we have, from (43),

$$\tilde{\mu}_t^N[p_x] = \int G_{m,C}(p_x) d\theta_{m_0,C_0}^t(m,C) = \int \pi_x d\theta_{m_0,C_0}^t(m,C) = \mathbb{E}_{m_0,C_0}[\pi_x(t)], \quad (44)$$

$$\tilde{\mu}_t^N[r_x] = \int G_{m,C}(r_x) d\theta_{m_0,C_0}^t(m,C) = \int \rho_x d\theta_{m_0,C_0}^t(m,C) = \mathbb{E}_{m_0,C_0}[\rho_x(t)]. \quad (45)$$

Finally, thanks to the Harris description and Lemma 3, it is proved in [12] that we can control the quantities $\pi_y(t)$ and $\rho_y(t)$ for all $t > 0$. More precisely,

Lemma 4. *Let $(m_t, C_t)_{t \geq 0}$ be the Markov process defined above. As previously done, we introduce $\rho(t), \pi(t) \in \mathbb{R}^N$ and $U(t), V(t), Z(t) \in \mathfrak{M}_N(\mathbb{R})$ such that*

$$m_t = (\rho(t), \pi(t)) \text{ and } C_t = \begin{pmatrix} U(t) & Z^*(t) \\ Z(t) & V(t) \end{pmatrix}. \quad (46)$$

Then,

$$\mathbb{P}_{m_0,C_0} \text{ - a. s. , } \forall t \geq 0, \begin{cases} \pi_y^2(t) \leq V_{y,y}(t), \\ \rho_y^2(t) \leq U_{y,y}(t). \end{cases} \quad (47)$$

4.2 Evolution of the Process $(m_t, C_t)_{t \geq 0}$

According to Lemma 4, the problem is reduced to estimate $U_{y,y}(t)$ and $V_{y,y}(t)$ for $t > 0$. Thanks to the regularity of β_0 and λ_0 , we know that there exists a constant K which does not depend on N such that

$$\frac{1}{N} \sum_i [(U_{i,i})^k(0) + (V_{i,i})^k(0)] \leq K^k, \text{ for all } k \geq 1. \quad (48)$$

It is easy to see that the above inequality is uniform in $t > 0$, in the sense that

$$\frac{1}{N} \mathbb{E}_{m_0,C_0} \left[\sum_i [(U_{i,i})^k(t) + (V_{i,i})^k(t)] \right] \leq K^k, \text{ for all } k \geq 1. \quad (49)$$

We are going to see how this last inequality can be used to show (18). We denote by $u_k(t)$ and $v_k(t)$ the two quantities

$$u_k(t) := \mathbb{E}_{m_0,C_0} \left[\sum_{i \in \mathbb{T}_N} U_{i,i}^k(t) \right], \quad v_k(t) := \mathbb{E}_{m_0,C_0} \left[\sum_{i \in \mathbb{T}_N} V_{i,i}^k(t) \right]. \quad (50)$$

Let us make the link with (18). We are going to focus on $u_k(t)$. The same ideas work for $v_k(t)$. In view of (43), we can write

$$\tilde{\mu}_t^N \left[p_y^{2k} \right] = \int G_{m,C} \left[p_y^{2k} \right] d\theta_{m_0, C_0}^t(m, C). \quad (51)$$

We use the convexity inequality to get

$$\begin{aligned} \tilde{\mu}_t^N \left[p_y^{2k} \right] &= \int G_{m,C} \left[(p_y - \pi_y + \pi_y)^{2k} \right] d\theta_{m_0, C_0}^t(m, C) \\ &\leq 2^{2k-1} \left[\int G_{m,C} \left[(p_y - \pi_y)^{2k} \right] d\theta_{m_0, C_0}^t(m, C) + \int \pi_y^{2k} d\theta_{m_0, C_0}^t(m, C) \right]. \end{aligned}$$

We deal with the two terms of the sum separately. First, observe that Gaussian centered moments are easily computable:

$$G_{m,C} \left[(p_y - \pi_y)^{2k} \right] = (V_{y,y} - \pi_y^2)^k \frac{(2k)!}{k! 2^k}. \quad (52)$$

Hence,

$$\sum_{y \in \mathbb{T}_N} \int (V_{y,y} - \pi_y^2)^k \frac{(2k)!}{k! 2^k} d\theta_{m_0, C_0}^t(m, C) \leq \frac{(2k)!}{k! 2^k} \left(v_k(t) + \mathbb{E}_{m_0, C_0} \left[\sum_{y \in \mathbb{T}_N} \pi_y^{2k}(t) \right] \right). \quad (53)$$

Lemma 4 shows that

$$\mathbb{E}_{m_0, C_0} \left[\sum_{y \in \mathbb{T}_N} \pi_y^{2k}(t) \right] \leq \mathbb{E}_{m_0, C_0} \left[\sum_{y \in \mathbb{T}_N} V_{y,y}^k(t) \right] = v_k(t). \quad (54)$$

As a result,

$$\sum_y \tilde{\mu}_t^N \left[p_y^{2k} \right] \leq \frac{(2k)!}{k!} v_k(t) \sim 2 \left(\frac{4}{e} \right)^k k^k v_k(t). \quad (55)$$

As a result, in order to get (18) we need to estimate the two quantities $u_k(t)$ and $v_k(t)$, which are related to C_t . In [12], the following final technical lemma is proved.

Lemma 5. *For any integer k not equal to 0, there exists a positive constant K which does not depend on N and t such that*

$$\begin{cases} v_k(t) \leq K^k N, \\ u_k(t) \leq K^k N. \end{cases} \quad (56)$$

Acknowledgements I thank Cédric Bernardin and Stefano Olla for giving me this problem, and for the useful discussions and suggestions on this work.

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