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# PROGRESS TOWARDS A CONJECTURE ON THE MEAN-VALUE OF TITCHMARSH SERIES-II

By K. RAMACHANDRA

## § 1. Introduction

The result of this paper may be considered as complementary to that of my earlier paper [2], on Titchmarsh series. Although not as interesting as the earlier result, the result of the present paper finds a nice application, (See [1]). In [3] I defined a class of series called Titchmarsh series and I now start by recalling its definition.

**Titchmarsh Series.** (or briefly K D T series).

Let  $A > 10$  be a constant.

Let  $\frac{1}{A} < \mu_1 < \mu_2 < \dots$  where

$$\frac{1}{A} < \mu_{n+1} - \mu_n < A \text{ (for } n = 1, 2, 3, \dots \text{)}.$$

(In [2] the notation is slightly different and we have used there  $\lambda_n$  instead of  $\mu_n$  and for simplicity assumed  $\lambda_1 = 1$ . Also we have used there  $a_1, a_2, \dots$  in place of our present  $b_1, b_2, \dots$  and assumed for simplicity  $a_1 = 1$ . We have written there  $F(s)$  instead of  $F_0(s)$ ). Let  $b_1, b_2, \dots$  be a sequence of complex numbers possibly depending on a parameter  $H > 10$  such that

$$|b_n| < (\mu_n H)^A. \text{ Put } F_0(s) = \sum_{n=1}^{\infty} b_n \mu_n^{-s} \text{ where } s = \sigma + it.$$

$F_0(s)$  is called a KDT series if there exists a constant  $A > 10$  and a system of rectangles  $R(T, T+H)$  defined by  $\{\sigma > 0, T \leq t \leq T+H\}$  where  $10 < H \leq T$ , and  $T$  (which may be related to  $H$ ) tends to infinity, and  $F_0(s)$  admits an analytic continuation into these rectangles and the maximum of  $|F_0(s)|$  taken over  $R(T, T+H)$  does not exceed  $\text{Exp}(H^A)$ .

I then made the following conjecture

### Conjecture

For a KDT series  $F_0(s)$ , we have,

$$\frac{1}{H} \int_L |F_0(it)|^2 dt > C_A \mu_n^{\frac{1}{X}} |b_n|^2,$$

where  $X = 2 + D_A H$ ,  $L$  denotes the side ( $\sigma = 0, T \leq t \leq T+H$ ) of  $R(T, T+H)$ , and  $C_A$  and  $D_A$  are positive constants depending only on  $A$ , provided  $\mu_1 = b_1 = 1$ .

I proved the following theorem.

### Theorem 1

Under the restrictions  $\mu_1 = b_1 = 1$ , we have,

$$\frac{1}{H} \int_L |F_0(it)|^2 dt > C_A \mu_n^{\frac{1}{X}} |b_n|^2 \left( 1 - \frac{\log \mu_n}{\log H} + \frac{1}{\log \log H} \right),$$

where  $X = 2 + D_A H$  and,  $C_A$  and  $D_A$  are effective positive constants depending only on  $A$ .

I now prove the following theorem.

**Theorem 2**

For some convenience let us assume in the definition of Titchmarsh series  $F_0(s)$  the rectangles  $R(T, T + H)$  to be  $(\sigma > \beta, T < t < T + H)$  where  $\beta$  is a positive constant such that  $0 < \beta < \alpha_1 < \frac{1}{2}$ , where  $\alpha_1$  is another constant. Let

$k \geq 2$  be an integer and write  $F(s) = (F_0(s))^k =$

$$\sum_{n=1}^{\infty} (a_n \lambda_n^{-s}),$$

a series which is surely convergent where  $F_0(s)$

is absolutely convergent. Put  $Y = (M + H)^{\lambda}$  where

$$\lambda = kA^{100} (\alpha_1 - \beta)^{-10}$$

and  $M = \text{maximum of } |F_0(s)|$

taken over  $R(T, T + H)$ . Define the entire function  $\phi(s)$  by

$$\phi(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \Delta\left(\frac{Y}{\lambda_n}\right)$$

where for  $X > 0$ ,  $\Delta(X)$  is defined by

$$\Delta(X) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} X^W \text{Exp}(W^{4a+2}) \frac{dW}{W},$$

$a$  being a suitable positive integer constant at our choice. We now suppose that  $\alpha, \beta, \alpha_1, \alpha_2, \alpha_3$  are constants satisfying

$$\alpha < \beta < \alpha_1 < \alpha_2 < \alpha_3 < \alpha + \frac{1}{2}.$$

Put  $X = [2^{-A-100} H] + 2$  and

$$V(\sigma) = \frac{1}{H} \sum_{\mu_n < X} |b_n|^2 \left(\frac{H}{\mu_n}\right)^{2\sigma}.$$

Then we have,

$$\frac{1}{H} \int_L |\phi(\alpha + it)|^{2k-1} dt$$

$$\gg V(\alpha) H^{1-2\alpha} \left\{ \frac{V(\alpha_1)}{V(\alpha_3)} \right\}^{\frac{\alpha_2 - \alpha}{\alpha_3 - \alpha_2}},$$

provided only that  $V(\alpha) \neq 0$  and  $(\log V(\alpha)) (\log H)^{-1}$  is bounded below by a negative constant. Here the constant implied by the Vinogradov symbol  $\gg$  depends only on  $A, k, \alpha, \beta, \alpha_1, \alpha_2, \alpha_3$  and the negative constant referred to just now. Further it is effective.

**Remark:** It will be clear from the proof that if

$$\mu_n = n \quad (n = 1, 2, 3, \dots)$$

then we can choose  $X = \left[ \frac{H}{100} \right] + 2$ .

The object of this paper is to prove theorem 2. The proof of the theorem is fairly long. The proof depends upon a special case of a convexity theorem of R. M. Gabriel which we state below (in the notation of D. R. Heath-Brown's paper [2]; for the more general theorem of Gabriel see the reference in [2] or Titchmarsh's famous book [5] pages 203 and 337.)

### Theorem 3

Let  $f(z)$  be regular in the infinite strip  $\alpha < \operatorname{Re} z < \beta$  and continuous for  $\alpha < \operatorname{Re} z < \beta$ . Suppose  $f(z) \rightarrow 0$  as  $|\operatorname{Im} z| \rightarrow \infty$ , uniformly in  $\alpha < \operatorname{Re} z < \beta$ . Then for  $\alpha \leq \gamma \leq \beta$ , and any  $q > 0$ , we have

$$\int_{-\infty}^{\infty} |f(\gamma + it)|^q dt < \left( \int_{-\infty}^{\infty} |f(\alpha + it)|^q dt \right)^{\frac{\beta - \gamma}{\beta - \alpha}} \left( \int_{-\infty}^{\infty} |f(\beta + it)|^q dt \right)^{\frac{\gamma - \alpha}{\beta - \alpha}}$$

provided the right hand side is finite.

Apart from this we have to use a well-known theorem of Montgomery and Vaughan. For reference see for instance my paper [4], where I give a simple proof of a weaker result which is sufficient for the purposes of this paper.

We now split up the proof of theorem 2 into several steps and give a brief sketch of these steps.

§ 2. Proof of Theorem 2

Step 1. Let

$$I(\sigma) = \frac{1}{H} \int_T^{T+H} \left( \int_{-\infty}^{\infty} |\phi(s)|^{2/k} | \text{Exp}((s - it_0)^{4a+2}) | dt \right) dt_0,$$

and assume that  $I(\alpha) < V(\alpha) H^{1-2\alpha}$ . The constant  $a$  shall be a sufficiently large positive integer. As already stated we set

$$X = [2^{-A-100} H] + 2, \text{ and}$$

$$V(\sigma) = \frac{1}{H} \sum_{\mu_n < X} |b_n|^2 \left( \frac{H}{\mu_n} \right)^{2\sigma} \text{ and}$$

we impose  $\alpha < \beta < \alpha_1 < \alpha_2 < \alpha_3 < \alpha + \frac{1}{2}$ .

**Step 2.** Next we write

$$J(\sigma) = \frac{1}{H} \int_T^{T+H} \left( \int_{-\infty}^{\infty} \left| \phi(s) - P^k(s) \right| \left| \text{Exp}((s-it_0)^{4a+2}) \right| dt \right) dt_0$$

where  $P(s) = \sum_{\mu_n < X} b_n \mu_n^{-s}$ . It is easily seen that

$\Delta(X) = O(X^B)$  and also  $1 + O(X^{-B})$  where  $B > 0$  is an arbitrary constant and the  $O$ -constant depends only on  $B$  and  $a$ .

Again  $a_n = \mu_{n_1} \dots \mu_{n_k} = \lambda_n (b_{n_1} b_{n_2} \dots b_{n_k})$  and for

all  $N > 1$ , we have  $N < \mu_{n_1} \dots \mu_{n_k} \leq 2N = O(N^k)$ . From

these remarks it is clear that  $\phi(s) - P^k(s)$  decays fast enough to ensure  $J(\sigma) < 1$  when  $\sigma$  is large enough. Now from an easy application of a theorem of Gabriel (Theorem 3 above) it follows that in  $\sigma > \alpha$ ,  $J(\sigma)$  is  $\ll_{\epsilon} [J(\alpha)]^{1-\epsilon}$  for every positive constant  $\epsilon$  uniformly in  $\sigma$ , and so in  $(\sigma > \beta, T < t \leq T+H)$ ,  $|\phi(s)|$  is bounded above by a constant power of  $H$ . (Here for getting the last bound we have to use the fact that for any analytic function  $\phi(s)$ ,  $|\phi(s)|^{2/k}$  is bounded by its mean value over a disc of (positive but sufficiently small) constant radius with  $s$  as centre).

**Step 3.** An easy application of a well-known Montgomery-Vaughan theorem (refer [4] for instance) shows that

$$\begin{aligned} \frac{1}{H} \int_T^{T+H} \left( \int_{-\infty}^{\infty} \left| P(s) \right|^2 \left| \text{Exp}[(s-it_0)^{4a+2}] \right| dt \right) dt_0 \\ = O(V(\sigma) H^{1-2\sigma}). \end{aligned}$$

From this and the estimate  $J(\sigma) \ll_{\epsilon} (J(\alpha))^{1-\epsilon}$  it follows that

$$\int_{\alpha}^{A^3} \left( \frac{1}{H} \int_T^{T+H} \left( \int_{-\infty}^{\infty} (|\phi(s) - P^k(s)|^{2/k} + |P(s)|^2) |\text{Exp}((s-it_0)^{4a+2})| dt \right) dt_0 \right) d\sigma = O((J(\alpha))^{1-\epsilon} + V(\alpha) H^{1-2\alpha}).$$

Note that  $V(\sigma)$  and  $V^*(\sigma) H^{1-2\sigma}$  are respectively monotonic increasing and monotonic decreasing functions of  $\sigma$ , where  $V^*(\sigma)$  is the same as  $V(\sigma)$  with the terms  $\mu_n \leq 1$  omitted. From now on we assume that  $V(\alpha)$  is bounded below by a constant negative power of  $H$ . Under this assumption it follows that the integral just considered is

$$O_{\epsilon}(V(\alpha) H^{1-2\alpha+\epsilon}) \text{ for every positive constant } \epsilon.$$

Hence there exist intervals  $I_1$  and  $I_2$  contained in  $(T, T + \frac{H}{10})$  and  $(T + H - \frac{H}{10}, T + H)$  respectively, for which the lengths are  $4H^{\delta}$  ( $\delta$  being any fixed constant satisfying  $0 < \delta < \frac{1}{100}$ ) each, and further  $I(I_1, \alpha)$

$$= \int_{\alpha}^{A^3} \left( \frac{1}{H} \int_{I_1} \int_{-\infty}^{\infty} (|\phi(s) - P^k(s)|^{2/k} + |P(s)|^2) |\text{Exp}((s-it_0)^{4a+2})| dt dt_0 \right) d\sigma$$

and  $I(I_2, \alpha)$  defined similarly (by replacing  $I_1$  by  $I_2$ ) satisfy

$$I(I_1, \alpha) + I(I_2, \alpha) = O(V(\alpha) H^{1-2\alpha+\epsilon}).$$



Hence by the principle for the mean value over discs referred to in the second step, we see that in  $(\beta \leq \sigma \leq A^2, t$  in any of the intervals  $I_1, I_2)$  we have

$$|\phi(s)|^{2/k} = O(V(\alpha) H^{-2\alpha + 2\epsilon + \delta}).$$

**Step 4.** Let  $H_1$  and  $H_2$  be the mid points of  $I_1$  and  $I_2$  respectively. We now obtain a lower bound for at least one of the mean-values  $K(\alpha_1)$  or  $K(\alpha_2)$  where  $K(\sigma)$  is defined by

$$K(\sigma) = \frac{1}{H_2 - H_1} \int_{H_1}^{H_2} |F_0(s)|^2 dt, \quad (\beta < \sigma < A^2).$$

Note that when we replace  $H_1$  and  $H_2$  by other points in  $I_1$  and  $I_2$  the mean value  $K(\sigma)$  changes by an amount which is at most  $O(E)$  where  $E = V(\alpha) H^{-2\alpha + 2\epsilon + 2\delta}$ . Hence if  $\sigma$  denotes any of  $\alpha_2$  or  $\alpha_3$  and  $H_1 < t < H_2$ , we see that if  $j$  is a large positive integer constant,

$$(1) \quad \frac{j!}{2\pi i} \int_{L_0} \frac{F_0(s+w) (2X)^w}{w(w+1)\dots(w+j)} dw \\ = \sum_{\mu_n \leq 2X} b_n \mu_n^{-s} \left(1 - \frac{\mu_n}{2X}\right)^j,$$

where  $L_0$  is the line  $\text{Re } w = A^2$ . Deform the line  $L_0$  to the contour described by the lines  $L_1, L_2, L_3, L_4, L_5$  (in this order) defined as follows. Let  $H_3 = H_1 - H^\delta$ ,  $H_4 = H_2 + H^\delta$  where  $\delta$  is a small positive constant.  $L_1$  and

$L_5$  are the portions  $\text{Im } w \leq -H_3$  and  $\text{Im } w \geq H_4$  respectively of  $L_0$ .  $L_2$  is the line segment

$$(\text{Im } w = -H_3, \alpha_1 - \sigma \leq \text{Re } w \leq A^2)$$

and  $L_4$  is the line segment

$$(\text{Im } w = H_4, \alpha_1 - \sigma \leq \text{Re } w \leq A^2).$$

$L_3$  is the line segment ( $\text{Re } w = \alpha_1 - \sigma, -H_3 < \text{Im } w < H_4$ ).

Taking the mean square after deformation of  $L_0$  we find from the equation (1), (Note that the only pole to be taken care of is  $w = 0$ ),

$$(2) \begin{cases} K(\sigma) \ll V_1(\sigma) H^{1-2\sigma} + (K(\alpha_1) + E) H^{-2(\sigma-\alpha_1)} \\ \quad \text{(where } V_1(\sigma) \text{ is defined below) and also} \\ V(\sigma) H^{1-2\sigma} \ll K(\sigma) + (K(\alpha_1) + E) H^{-2(\sigma-\alpha_1)}. \end{cases}$$

The reason for this is that the mean square of the RHS of (1) is  $\gg$  and  $\ll V_1(\sigma) H^{1-2\sigma}$  where  $V_1(\sigma)$  is defined by

$$V_1(\sigma) = \frac{1}{H} \sum_{\mu_n \leq X} |b_n|^2 \left(1 - \frac{\mu_n}{2X}\right)^{2j} \left(\frac{H}{\mu_n}\right)^{2\sigma}.$$

Since  $V(\alpha) \ll V(\sigma)$  we may omit the term containing  $E$  in the second of the equations (2), provided  $\alpha_1 - \alpha < \frac{1}{2}$  (which is true because of our assumptions). This gives us  $V(\sigma) H^{1-2\sigma} \ll K(\sigma) + K(\alpha_1) H^{-2(\sigma-\alpha_1)}$ ; ( $\sigma = \alpha_2, \alpha_3$ )

If we put  $\sigma = \alpha_2$  we get a lower bound for one at least of the quantities  $K(\alpha_1)$  or  $K(\alpha_2)$ .

We now deduce from the last inequality

$$V(\sigma) H^{1-2\sigma} \ll I(\sigma) + I(\alpha_1) H^{-2(\sigma-\alpha_1)},$$

$$(\sigma = \alpha_2, \alpha_3).$$

This is possible since in the range  $(\sigma > \alpha_1, H_3 < t \leq H_4)$ ,

$|F_0(s)|^2$  is a very good approximation (in the mean) to

$|\phi(s)|^{2/k}$  and we leave the details as an exercise. From the

last inequality it follows that  $I(\sigma) \gg V(\sigma) H^{1-2\sigma}$ , for one at least of the values  $\sigma = \alpha_1$  or  $\alpha_2$ .

Next by the convexity theorem of Gabriel (Theorem 3), we find that with the value of  $\sigma$  ( $\alpha_1$  or  $\alpha_2$  determined),

$$(3) \quad (I(\sigma))^{\alpha_3 - \alpha} < (I(\alpha))^{\alpha_3 - \sigma} (I(\alpha_3))^{\sigma - \alpha}.$$

Moreover by the arguments used in the first of the inequalities in (2) we get (by taking  $X$  in place of  $2X$  in (1)).

$$I(\alpha_3) \ll V(\alpha_3) H^{1-2\alpha_3} + (I(\sigma) + E) H^{-2(\alpha_3 - \sigma)} + E.$$

$$\text{Now } E = V(\alpha) H^{-2\alpha + 2\varepsilon + 2\delta} < V(\alpha_3) H^{1-2\alpha_3}$$

(since by our assumptions,  $\alpha_3 - \alpha < \frac{1}{2}$ ) by a small choice of the positive constants  $\varepsilon, \delta$ . Thus we get

$$I(\alpha_3) \ll V(\alpha_3) H^{1-2\alpha_3} + I(\sigma) H^{-2(\alpha_3 - \sigma)},$$

and so by (3)

$$(4) \quad (I(\sigma))^{\alpha_3 - \alpha} \ll (I(\alpha))^{\alpha_3 - \sigma} \left( V(\alpha_3) H^{1-2\alpha_3} + I(\sigma) H^{-2(\alpha_3 - \sigma)} \right)^{\sigma - \alpha}.$$

This holds for either  $\sigma = \alpha_1$  or  $\sigma = \alpha_2$  and gives us

$$(5) \left\{ \begin{array}{l} \text{Either } I(\alpha) \gg V(\sigma) H^{1-2\alpha} > V(\alpha) H^{1-2\alpha}, \\ \text{or } (I(\alpha))^{\alpha_3 - \sigma} \gg (V(\sigma) H^{1-2\sigma} \alpha_3^{-\alpha}) \\ \qquad \qquad \qquad (V(\alpha_3) H^{1-2\alpha_3} \alpha^{-\sigma}). \end{array} \right.$$

The second of these inequalities gives

$$I(\alpha) \gg H^{1-2\alpha} V(\sigma) \left\{ \frac{V(\sigma)}{V(\alpha_3)} \right\}^{\frac{\sigma - \alpha}{\alpha_3 - \sigma}}$$

Since  $V(\alpha) < V(\alpha_1) < V(\sigma) < V(\alpha_3)$  and since  $\frac{u - \alpha}{\alpha_3 - u}$

is an increasing function of  $u$  in  $\beta < u < \alpha_3$ , we get finally

$$I(\alpha) \gg V(\alpha) H^{1-2\alpha} \left\{ \frac{V(\alpha_1)}{V(\alpha_3)} \right\}^{\frac{\alpha_2 - \alpha}{\alpha_3 - \alpha_2}}$$

**Step 5.** Step 4 nearly completes the proof. For we could have started with a slight modification of  $I(\sigma)$  by averaging over a slightly smaller interval contained in  $(T, T + H)$  instead of  $(T, T + H)$ . For instance by cutting off bits of length  $H^\delta$  on either side. The decaying factor  $\text{Exp}((s - it_0)^{4a+2})$  enables us to replace the modified

$$I(\sigma) \text{ by } \frac{1}{H} \int_T^{T+H} |\phi(s)|^{\frac{2}{k}} dt \quad \sigma = \alpha$$

in the last lower bound.

Steps 1, 2, 3, 4 and 5 complete the proof of theorem 2.

**Theorem 4**

With the notation of theorem 2, we have,

$$\frac{1}{H} \int_T^{T+H} |\phi(\alpha + it)|^2 dt \gg \left( v(\alpha) H^1 - 2\alpha \left\{ \frac{v(\alpha_1)}{v(\alpha_3)} \right\} \frac{\alpha_2 - \alpha}{\alpha_3 - \alpha_2} \right)^k$$

where the constant implied by the Vinogradov symbol is effective

*Remark.* This theorem will be used in [1].

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