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Mangala J Narlikar

► **To cite this version:**

Mangala J Narlikar. On a theorem of Erdos and Szemerédi. Hardy-Ramanujan Journal, 1980, Volume 3 - 1980, pp.41 - 47. 10.46298/hrj.1980.90 . hal-01103868

**HAL Id: hal-01103868**

**<https://hal.science/hal-01103868>**

Submitted on 15 Jan 2015

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# ON A THEOREM OF ERDÖS AND SZEMEREDI

By M. J. NARLIKAR

## § 1. Introduction

In 1951, K. F. Roth [7] proved that if  $1 = q_1 < q_2 < \dots$  is the sequence of all square-free integers, then

$$q_{n+1} - q_n = O\left(n^{\frac{3}{13}} (\log n)^{\frac{4}{13}}\right)$$

and this was improved to  $O\left(n^{\frac{2}{9}}\right)$  by H. E. Richert [6]. In

an attempt to put these in more general setting P. Erdős [1] introduced with any sequence  $B: 2 < b_1 < b_2 < \dots$ , the sequence  $Q: 1 = q_1 < q_2 < \dots$  of all integers  $q_i$  not divisible by any  $b_j$  and proved (subject only to  $(b_i, b_j) = 1$  unless  $i = j$

and  $\sum \frac{1}{b_i} < \infty$ ) that

$$q_{n+1} - q_n = O\left(q_n^\theta\right),$$

with some  $\theta < 1$ , where  $\theta$  is independent of  $B$ . His  $\theta$  was close to 1. E. Szemerédi made an important progress and showed that this is true for every fixed  $\theta > \frac{1}{2}$ . As in all previous results of this kind, he showed that if  $Q(x) = \sum_{q_i < x} 1$ ,

then  $Q(x+h) - Q(x) \gg h$ , where  $h > x^\theta$ .

Using the ideas of Szemerédi with some refinements, we prove

### Theorem 1

Let  $p$  be any prime and  $r_p$  denote the number of  $b_i$  divisible by  $p$  and suppose that as  $p$  varies,  $r_p$  does not exceed  $p^A$  where  $A$  is any positive constant. Let  $\sum \frac{1}{b_i} < \infty$ . Then

$Q(x+h) - Q(x) \gg h$ , where  $x > h > x^\theta$   
and  $\theta > \frac{1}{2}$  is any constant.

Further if for some  $\alpha < 1$ , we have  $\sum b_i^{-\alpha} < \infty$ , then

$$Q(x+h) - Q(x) \gg h.$$

where  $h > x^\theta$ , and  $\theta > \frac{\alpha}{1+\alpha} = \beta$  say.

Next using the ideas of Jutila [4], the results of Brun [1] and the zero-free region for  $\zeta(s)$  due to I. M. Vinogradov [9] we prove

### Theorem 2

Let  $r_p \leq p^A$  as before and in place of

$\sum b_i^{-1} < \infty$  let  $\lim_{y \rightarrow \infty} \sum_{y \leq b_i \leq y^2} b_i^{-1} = 0$ . Then

$$Q(x+h) - Q(x) \gg h / \omega_3 x,$$

where  $h > x^\theta$ , with  $\theta > \frac{1}{2}$ .

The improvement Theorem 2 of Theorem 1 was suggested by Professor K. Ramachandra and I am thankful to him for explaining the same. Also I express my gratitude to him for encouragement and useful suggestions. He and I, in a joint paper to appear have improved Theorem 2 in several ways. These researches will appear in Acta Arithmetica in due course.

## § 2. Proofs of Theorems 1 &amp; 2

We begin with some notations.

(1) We can assume without loss of generality that  $b_1, b_2, \dots, b_k$  are primes (because we can replace them by their greatest prime factor and select distinct ones amongst them). Next we assume

$$\sum_{i > k} b_i^{-1} < \frac{1}{2} \text{ and define } j_0 \text{ by}$$

$$\sum_{i > j_0} b_i^{-1} < \eta, \text{ where } \eta \text{ is sufficiently small.}$$

(2) Let  $n > 10$  be any large integer constant and for  $i = 1, 2, \dots, n$  put

$$C_i = \{ p/x(2n)^{-1} + (i-1)(8n^4)^{-1} < p \\ < x(2n)^{-1} + i(8n^4)^{-1} \},$$

$$C'_i = \{ p/x\beta(2n)^{-1} + (i-1)(8n^4)^{-1} < p \\ < x\beta(2n)^{-1} + i(8n^4)^{-1} \}.$$

Let  $g$  run over integers of the form  $\pi p$ , where  $p$  are

chosen one from each  $C_i$ . Let  $g'_i$  be defined in the same way with respect to  $C'_i$ . For any fixed integer  $g$  the number of integers in  $[x, x+h]$  which are divisible by  $g$  (respectively  $g'_i$ ),  $k < i \leq j_0$  but coprime to  $b_1, b_2, \dots, b_k$  is

$$\frac{h}{g} \pi_{i \leq k} \left( 1 - \frac{1}{b_i} \right) + O(2^{-k}).$$

$$\left( \text{respectively } \frac{h}{g b_j} \pi_{i \leq k} \left( 1 - \frac{1}{b_i} \right) + O(2^{-k}) \right)$$

Also the number of integers in  $[x, x+h]$  divisible by  $b_i$  for any fixed  $i > j_0$  such that  $b_i < h$  is  $< \frac{2h}{b_i}$ . Hence the number of integers (counted with certain multiplicities) in  $[x, x+h]$  which are coprime to  $b_1, b_2, \dots, b_k$ , but divisible by some  $g$  or other, but not divisible by any  $b_i (i > j_0, b_i < h)$

$$\begin{aligned}
 &> \sum_g \left\{ \frac{h}{g} \sum_{i \leq k} \left( 1 - \frac{1}{b_i} \right) - \right. \\
 &\quad \left. \sum_{k < i < j_0} \left( \frac{h}{g b_i} \sum_{i \leq k} \left( 1 - \frac{1}{b_i} \right) + O(2^k) \right) \right\} \\
 &\quad - \sum_{\substack{b_{j_0} < b_j < h \\ j > j_0}} \frac{2h}{b_j} \\
 &> \sum_g \left( \frac{h}{2g} \sum_{i \leq k} \left( 1 - \frac{1}{b_i} \right) \right) - \sum_{b_{j_0} < b_i} \frac{2h}{b_i} \\
 &\quad + O\left( 2^k \sum_{j_0} \frac{1}{g} \right) \\
 &= h \left\{ \frac{1}{2} \left( \sum_{i \leq k} \left( \sum_{p \in C_i} \frac{1}{p} \right) \right) \sum_{i \leq k} \left( 1 - \frac{1}{b_i} \right) - 2\eta \right\} \\
 &\quad + O\left( 2^k \sum_{j_0} x^{\frac{1}{2}} + \frac{1}{8n^2} \right) \\
 &> h \left\{ \left( \frac{1}{1000n} \right)^{10n} \sum_{i \leq k} \left( 1 - \frac{1}{b_i} \right) - 2\eta \right\} \\
 &\quad + O\left( 2^k \sum_{j_0} x^{\frac{1}{2}} + \frac{1}{8n^2} \right)
 \end{aligned}$$

The multiplicities are  $\leq (4n)!$  since the number of prime factors  $> x^{\frac{1}{2n}}$  is  $\leq 4n$  for the integers counted. Now let us look at integers in  $[x, x + h]$  which are of the form  $mg$  but divisible by some  $b_i$  or the other with  $b_i > h$ . Now a given  $b_i$  can divide atmost one integer in the interval and so it suffices to count the number of  $b_i$  to get an upper bound for the number of integers in question. If  $(b_i, g) = 1$ , then  $b_i g < 2x$  and hence  $hg \leq 2x$ . This is impossible since  $h > x^\theta$ . Hence  $(b_i, g) > 1$  and the number of possible  $b_i$ 's is therefore

$$\begin{aligned} &< \sum_{p \leq x^{\frac{1}{n}}} p^A < x^{\frac{A+1}{n}} \end{aligned}$$

A large choice of  $n$  now completes the proof of the first part of Theorem 1. The second part can be proved similarly using  $C_i'$ .

The proof of Theorem 2 to put it briefly starts with

$$\begin{aligned} &\frac{1}{2\pi i} \int \left( - \sum_p (p^{-s} \log p) \right) \left( \sum_{X < p < 2X} p^{-s} \right)^N \\ &\quad \times \frac{(x+h)^s - x^s}{s} ds \end{aligned}$$

where the line of integration is  $\sigma = 1 + (\log x)^{-1}$ ,

$|t| \leq T$ . We then move the line of integration to  $\sigma = 1 - (\log T)^{-\frac{99}{100}}$ . Rough estimations are enough to

show that the number of numbers of the form  $p p_1 p_2 \dots p_N$

( $X < p_i < 2X$ ) lying in  $[x, x+h]$  is  $\gg \frac{h}{\log x}$  (provided

$X = \tau \frac{1}{n}$  and  $N = n - 1$  and  $h = x^{\frac{1}{2} + \eta}$  where  $\eta > 0$  is small provided  $n$  is large). If  $b_i$  divides a number of the

counted type then  $x^{\frac{1}{2n}} < b_i < x^2$  and by Brun's sieve the number of counted numbers divisible by such  $b_i$  is

$$\ll \sum_{x^{\frac{1}{2n}} < b_i < \frac{h}{(\log x)^2}} \left( \frac{h}{b_i \log x} + 1 \right) + \sum'_{\frac{h}{(\log x)^2} < b_i < x^2} 1,$$

where the second sum is over those  $b_i$  which divide a number of the counted type. This proves Theorem 2.

*Remark:* In the joint paper [5] to appear, the present paper has been referred to under the title '**An Analytic Approach to Szemerédi's Theorem**'. The interested readers will please note this change of title. The paper was written in 1977 and could not be published earlier because of some reasons.

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701, Colaba Housing Colony,  
Homi Bhabha Road,  
Bombay 400 005  
India.