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ON A THEOREM OF ERDÖS AND SZEMEREDI

By M. J. NARLIKAR

§ 1. Introduction

In 1951, K. F. Roth [7] proved that if $1 = q_1 < q_2 < \dots$ is the sequence of all square-free integers, then

$$q_{n+1} - q_n = O\left(n^{\frac{3}{13}} (\log n)^{\frac{4}{13}}\right)$$

and this was improved to $O\left(n^{\frac{2}{9}}\right)$ by H. E. Richert [6]. In

an attempt to put these in more general setting P. Erdős [1] introduced with any sequence $B: 2 < b_1 < b_2 < \dots$, the sequence $Q: 1 = q_1 < q_2$ of all integers q_i not divisible by any b_j and proved (subject only to $(b_i, b_j) = 1$ unless $i = j$

and $\sum \frac{1}{b_i} < \infty$) that

$$q_{n+1} - q_n = O\left(q_n^\theta\right),$$

with some $\theta < 1$, where θ is independent of B . His θ was close to 1. E. Szemerédi made an important progress and showed that this is true for every fixed $\theta > \frac{1}{2}$. As in all previous results of this kind, he showed that if $Q(x) = \sum_{q_i < x} 1$,

then $Q(x+h) - Q(x) \gg h$, where $h > x^\theta$.

Using the ideas of Szemerédi with some refinements, we prove

Theorem 1

Let p be any prime and r_p denote the number of b_i divisible by p and suppose that as p varies, r_p does not exceed p^A where A is any positive constant. Let $\sum \frac{1}{b_i} < \infty$. Then

$Q(x+h) - Q(x) \gg h$, where $x > h > x^\theta$ and $\theta > \frac{1}{2}$ is any constant.

Further if for some $\alpha < 1$, we have $\sum b_i^{-\alpha} < \infty$, then

$$Q(x+h) - Q(x) \gg h.$$

where $h > x^\theta$, and $\theta > \frac{\alpha}{1+\alpha} = \beta$ say.

Next using the ideas of Jutila [4], the results of Brun [1] and the zero-free region for $\zeta(s)$ due to I. M. Vinogradov [9] we prove

Theorem 2

Let $r_p \leq p^A$ as before and in place of

$\sum b_i^{-1} < \infty$ let $\lim_{y \rightarrow \infty} \sum_{y \leq b_i \leq y^2} b_i^{-1} = 0$. Then

$$Q(x+h) - Q(x) \gg h / \omega_3 x,$$

where $h > x^\theta$, with $\theta > \frac{1}{2}$.

The improvement Theorem 2 of Theorem 1 was suggested by Professor K. Ramachandra and I am thankful to him for explaining the same. Also I express my gratitude to him for encouragement and useful suggestions. He and I, in a joint paper to appear have improved Theorem 2 in several ways. These researches will appear in Acta Arithmetica in due course.

§ 2. Proofs of Theorems 1 & 2

We begin with some notations.

(1) We can assume without loss of generality that b_1, b_2, \dots, b_k are primes (because we can replace them by their greatest prime factor and select distinct ones amongst them). Next we assume

$$\sum_{i > k} b_i^{-1} < \frac{1}{2} \text{ and define } j_0 \text{ by}$$

$$\sum_{i > j_0} b_i^{-1} < \eta, \text{ where } \eta \text{ is sufficiently small.}$$

(2) Let $n > 10$ be any large integer constant and for $i = 1, 2, \dots, n$ put

$$C_i = \{ p/x(2n)^{-1} + (i-1)(8n^4)^{-1} < p \\ < x(2n)^{-1} + i(8n^4)^{-1} \},$$

$$C'_i = \{ p/x\beta(2n)^{-1} + (i-1)(8n^4)^{-1} < p \\ < x\beta(2n)^{-1} + i(8n^4)^{-1} \}.$$

Let g run over integers of the form πp , where p are

chosen one from each C_i . Let g'_i be defined in the same way with respect to C'_i . For any fixed integer g the number of integers in $[x, x+h]$ which are divisible by g (respectively $g b_i$), $k < i \leq j_0$ but coprime to b_1, b_2, \dots, b_k is

$$\frac{h}{g} \pi_{i \leq k} \left(1 - \frac{1}{b_i} \right) + O(2^{-k}).$$

$$\left(\text{respectively } \frac{h}{g b_j} \pi_{i \leq k} \left(1 - \frac{1}{b_i} \right) + O(2^{-k}) \right)$$

Also the number of integers in $[x, x+h]$ divisible by b_i for any fixed $i > j_0$ such that $b_i < h$ is $< \frac{2h}{b_i}$. Hence the number of integers (counted with certain multiplicities) in $[x, x+h]$ which are coprime to b_1, b_2, \dots, b_k , but divisible by some g or other, but not divisible by any $b_i (i > j_0, b_i < h)$

$$\begin{aligned}
 &> \sum_g \left\{ \frac{h}{g} \sum_{i \leq k} \left(1 - \frac{1}{b_i} \right) - \right. \\
 &\quad \left. \sum_{k < i < j_0} \left(\frac{h}{g b_i} \sum_{i \leq k} \left(1 - \frac{1}{b_i} \right) + O(2^k) \right) \right\} \\
 &\quad - \sum_{\substack{b_{j_0} < b_j < h \\ j > j_0}} \frac{2h}{b_j} \\
 &> \sum_g \left(\frac{h}{2g} \sum_{i \leq k} \left(1 - \frac{1}{b_i} \right) \right) - \sum_{b_{j_0} < b_i} \frac{2h}{b_i} \\
 &\quad + O\left(2^k \sum_{j_0} \frac{1}{g} \right) \\
 &= h \left\{ \frac{1}{2} \left(\sum_{i \leq k} \frac{1}{p_i} \right) \sum_{p \in C_i} \left(1 - \frac{1}{p} \right) - 2\eta \right\} \\
 &\quad + O\left(2^k \sum_{j_0} \frac{1}{x^2} + \frac{1}{8n^2} \right) \\
 &> h \left\{ \left(\frac{1}{1000n} \right)^{10n} \sum_{i \leq k} \left(1 - \frac{1}{b_i} \right) - 2\eta \right\} \\
 &\quad + O\left(2^k \sum_{j_0} \frac{1}{x^2} + \frac{1}{8n^2} \right)
 \end{aligned}$$

The multiplicities are $\leq (4n)!$ since the number of prime factors $> x^{\frac{1}{2n}}$ is $\leq 4n$ for the integers counted. Now let us look at integers in $[x, x + h]$ which are of the form mg but divisible by some b_i or the other with $b_i > h$. Now a given b_i can divide atmost one integer in the interval and so it suffices to count the number of b_i to get an upper bound for the number of integers in question. If $(b_i, g) = 1$, then $b_i g < 2x$ and hence $hg \leq 2x$. This is impossible since $h > x^\theta$. Hence $(b_i, g) > 1$ and the number of possible b_i 's is therefore

$$\begin{aligned} &< \sum_{p \leq x^{\frac{1}{n}}} p^A < x^{\frac{A+1}{n}} \end{aligned}$$

A large choice of n now completes the proof of the first part of Theorem 1. The second part can be proved similarly using C_i' .

The proof of Theorem 2 to put it briefly starts with

$$\begin{aligned} &\frac{1}{2\pi i} \int \left(- \sum_p (p^{-s} \log p) \right) \left(\sum_{X < p < 2X} p^{-s} \right)^N \\ &\quad \times \frac{(x+h)^s - x^s}{s} ds \end{aligned}$$

where the line of integration is $\sigma = 1 + (\log x)^{-1}$, $|t| \leq T$. We then move the line of integration to $\sigma = 1 - (\log T)^{-\frac{99}{100}}$. Rough estimations are enough to

show that the number of numbers of the form $p p_1 p_2 \dots p_N$

($X < p_i < 2X$) lying in $[x, x+h]$ is $\gg \frac{h}{\log x}$ (provided

$X = \tau^{\frac{1}{n}}$ and $N = n - 1$ and $h = x^{\frac{1}{2} + \eta}$ where $\eta > 0$ is small provided n is large). If b_i divides a number of the

counted type then $x^{\frac{1}{2n}} < b_i < x^2$ and by Brun's sieve the number of counted numbers divisible by such b_i is

$$\ll \sum_{x^{\frac{1}{2n}} < b_i < \frac{h}{(\log x)^2}} \left(\frac{h}{b_i \log x} + 1 \right) + \sum'_{\frac{h}{(\log x)^2} < b_i < X^2} 1,$$

where the second sum is over those b_i which divide a number of the counted type. This proves Theorem 2.

Remark: In the joint paper [5] to appear, the present paper has been referred to under the title '**An Analytic Approach to Szemerédi's Theorem**'. The interested readers will please note this change of title. The paper was written in 1977 and could not be published earlier because of some reasons.

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