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# ONE MORE PROOF OF SIEGEL'S THEOREM

By K. RAMACHANDRA

To the Academician Ivan Matveevich Vinogradov  
 A Humble Dedication on his Ninetieth Birthday

## § 1. Introduction

The object of this note is to give a trivial proof of the following theorem and apply it to obtain slightly a sharper version of Siegel's Theorem on  $L(1, \chi)$  for real characters  $\chi \pmod k$ . See Theorem 2 at the end of this section.

### Theorem 1

Let  $a_1, a_2, \dots$  be an infinite sequence of complex numbers satisfying  $\sum_{n \leq x} a_n \leq Cx^\phi$ , where  $C$  and  $\phi$  are positive constants satisfying  $C > 0, 0 < \phi < 1$ . Let  $\phi < s < 1$ .

Then for  $x > 1$ ,

$$\sum_{n \leq x} (n^{-s} - \sum_{m \leq \frac{x}{n}} a_m m^{-s}) = \frac{x^{1-s}}{1-s} f(1) + \zeta(s) f(s) + O(x^{1-\phi})^{-1},$$

where  $|\theta| \leq 1, f(s) = \sum_{n=1}^{\infty} (a_n n^{-s}),$

$$\zeta(s) = \sum_{n=1}^{\infty} \left( n^{-s} - \int_n^{n+1} \frac{du}{u^s} \right) + \frac{1}{s-1},$$

$$\text{and } E = x^{\lambda} 10^{-3} \left( 1160 + \frac{96(1-\phi)}{(s-\phi)x^s - \phi + \lambda} + \frac{192(\log(x+3))(1-\phi)}{(s-\phi)^2 x^s - \phi + \lambda} \right)$$

$$\lambda \text{ being } \frac{\phi+1}{2} - s.$$

This theorem will be proved in § 2 in a simple way. We now state a lemma (to be proved in § 3 in a trivial way).

**Lemma 1:** Let  $3 < k_1 < k_2$  where  $k_1$  and  $k_2$  are two integers. Let  $\chi_1$  and  $\chi_2$  be two non-principal real characters mod  $k_1$  and mod  $k_2$  respectively such that the character  $\chi_3 = \chi_1 \chi_2$  defined by  $\chi_3(n) = \chi_1(n) \chi_2(n)$  is non principal (we can verify that  $\chi_3$  is actually a character mod  $k_3 = k_1 k_2$ ).

Then for  $x \geq 1$  we have,

$$\begin{aligned} \sum_{n < x} \chi_1(n) &< k_1 \quad \text{and} \\ \sum_{lmn < x} \chi_1(l) \chi_2(m) \chi_3(n) &< 25 x^{\frac{3}{2}} (k_1 k_2)^2. \end{aligned}$$

*Remark.* The lemma is true for complex characters as well. Also the estimate can be improved by the use of the Polya-Vinogradov inequality.

**Lemma 2:** We have, under the assumptions of Lemma 1,

$$\begin{aligned} \sum_{mn < x} (\chi_1(n) (mn)^{-s}) &= \frac{x^{1-s}}{1-s} L_1 \\ &+ \zeta(s) L(s, \chi_1) + 1000 k_1 E_1 \theta, \quad (0 < s < 1) \end{aligned}$$

$$\begin{aligned} \text{and } \sum_{lmn\nu < x} \chi_1(l) \chi_2(m) \chi_3(n) (lmn\nu)^{-s} \\ = \frac{x^{1-s}}{1-s} L_1 L_2 L_3 + \zeta(s) L(s, \chi_1) \end{aligned}$$

$$L(s, \chi_2) L(s, \chi_3) + 200000 (k_1 k_2)^2 E_2 \theta, \left(\frac{3}{4} < s < 1\right),$$

where  $L(s, \chi_j) = \sum_{n=1}^{\infty} (\chi_j(m) n^{-s})$  and

$$L(1, \chi_j) = L_j, \quad (j = 1, 2, 3).$$

$$\text{Also } E_1 = 10^{-3} x^{\lambda_1} \left( 1160 + \frac{96}{s x^{\frac{1}{2}}} + \frac{192 \log(x+3)}{s^2 x^{\frac{1}{2}}} \right),$$

$$\lambda_1 = \frac{1}{2} - s,$$

$$\text{and } E_2 = 10^{-3} x^{\lambda_2} \left( 1160 + \frac{96}{(4s-3)x^{\frac{1}{2}}} + \frac{768 \log(x+3)}{(4s-3)^2 x^{\frac{1}{2}}} \right),$$

$$\lambda_2 = \frac{7}{8} - s.$$

*Proof:* We have only to put  $\phi = 0$  and  $\frac{3}{4}$  in Theorem 1 and use Lemma 1. This proves the lemma.

From these two lemmas we can deduce in a simple way the sharper version of Siegel's theorem, as follows.

**Lemma 3:** Let  $\rho$  be the greatest real zero of  $L(s, \chi_1)$ . If

there is no real zero at all or if  $\rho \leq \frac{7}{8}$  we take  $s$  in the range  $\frac{7}{8} \leq s < 1$ . In the other case we take  $s$  in the range  $\rho < s < 1$ .

Put  $x = (10000 k_1)^4$ . Then we have

$$1 \leq \frac{x}{1-s} L(1, \chi_1) + (1000 k_1)^{-\frac{1}{8}}.$$

Hence  $L(1, \chi_1) \neq 0$ .

*Proof:* Follows from Lemma 2 on using  $\zeta(s) L(s, \chi_1) \leq 0$ .

**Lemma 4:** Let  $L(1, \chi_1) < (60000)^{-8} (\log k_1)^{-1}$ . Then there exist real zeros and their maximum  $\rho$  satisfies  $1 - \rho < (16 \log k_1)^{-1}$  and further

$$\frac{1}{2} < (10000 k_1)^{4(1-\rho)} \frac{L(1, \chi_1)}{1-\rho}.$$

Hence  $1 - \rho \leq 2(10000 k_1)^{4(1-\rho)} L(1, \chi_1)$   
 $\leq 400 L(1, \chi_1)$ .

*Proof:* Follows from lemma 3 on putting  $s = 1 - (16 \log k)^{-1}$  and assuming  $1 - \rho > (16 \log k_1)^{-1}$ , and next putting  $s = \rho$ .

**Lemma 5:** Let  $L(1, \chi_1) < (60000)^{-8} (\log k_1)^{-1}$ . Then if  $\chi_2$  is a non principal real character mod  $k_2$  ( $3 \leq k_1 \leq k_2$ ) such that  $\chi_3 = \chi_1 \chi_2$  is non principal, we must have

$$\frac{1}{2} \leq \frac{x_0^{1-\rho}}{1-\rho} L(1, \chi_1) L(1, \chi_2) L(1, \chi_1 \chi_2)$$

where  $x_0 = (860000)^{32} (k_1 k_2)^{32}$  and  $\rho$  is given by lemma 4.

*Proof:* In the second inequality of lemma 2 we take  $x = x_0$  and  $s = \rho$ . Plainly  $\zeta(\rho) L(\rho, \chi_1) L(\rho, \chi_2) L(\rho, \chi_1 \chi_2) = 0$ . The term involving  $\theta$  is easily seen to be less than  $\frac{1}{2}$ . Hence lemma 5 is proved.

**Lemma 6:** Let  $L(1, \chi_1) \leq (60000)^{-8} (\log k_1)^{-1}$  and let  $\chi_1$  and  $\chi_2$  be as in lemma 5. Then we have

$$\frac{1}{2} < (4 (k_1 k_2)^{20000} L(1, \chi_1)) \left( \frac{L(1, \chi_1) - L(\rho, \chi_1)}{1 - \rho} \right) L(1, \chi_2) L(1, \chi_1 \chi_2).$$

*Proof:* Follows by lemma 5 on using

$$x_0^{1-\rho} < x_0^{400} L(1, \chi_1).$$

Using  $L(1, \chi_1 \chi_2) < 6 \log(k_1 k_2)$  and  $\frac{L(1, \chi_1) - L(\rho, \chi_1)}{1 - \rho} < 40 (\log k_1)^2$  and  $k_1 < k_2$  we state now our main result.

### Theorem 2.

Let  $3 < k_1 < k_2$  where  $k_1$  and  $k_2$  are two integers. Let  $\chi_1$  and  $\chi_2$  be two real non principal characters mod  $k_1$  and  $k_2$  respectively such that there exists an integer  $n > 0$  for

which  $\chi_1(n)\chi_2(n) = -1$ . Put  $L_1 = \sum_{n=1}^{\infty} (\chi_1(n) n^{-1})$ ,

$$L_2 = \sum_{n=1}^{\infty} (\chi_2(n) n^{-1}).$$

If  $L_1 < 10^{-40} (\log k_1)^{-1}$ , then, we must have necessarily,

$$L_2 > (\log k_2)^{-1} \{ 10^{-4} (\log k_1)^{-2} k_2^{-40000} L_1 \}.$$

As a corollary we have immediately the following result due to T. TATUZAWA, which is an improvement of a result of C. L. SIEGEL.

**Theorem 3.**

Given any  $\epsilon$ ,  $0 < \epsilon < \frac{1}{2}$  the inequality

$$\sum_{n=1}^{\infty} (\chi(n) n^{-1}) < k^{-\epsilon} \text{ where } \chi \text{ is a real non principal}$$

character mod  $k$  has only finitely many solutions. Moreover all exceptions to this inequality "can be determined effectively" with (essentially) at most one possible exception.

**Remark :** The first part of Theorem 3 is due to Siegel and the second part due to Tatzawa. We have not bothered to economize the constants in Theorem 2.

§ 2. **Proof of Theorem 1.** The proof is based on

**Lemma 1 :** 
$$\left| \sum_{n=1}^N b_n c_n \right| \leq$$

$$3 \left( \max_{1 \leq n \leq N} \left| \sum_{m=1}^n b_m \right| \right) \max_{1 \leq n \leq N} |c_n|,$$

where  $\{b_n\}$  is a finite sequence of complex numbers and  $\{c_n\}$  a finite monotonic sequence of real numbers. The constant 3 can be improved to 2 if all the  $c_n$  are of the same sign.

*Proof :* Writing  $B_0 = 0$ ,  $B_n = \sum_{m=1}^n b_m$  we have

$$\begin{aligned} \sum_{n=1}^N b_n c_n &= \sum_{n=1}^N (B_n - B_{n-1}) c_n \\ &= \sum_{n=1}^{N-1} B_n (c_n - c_{n+1}) + B_N c_N. \end{aligned}$$

This proves the lemma.

**Lemma 2** Let  $0 < s < 1$  and  $x > 0$ . Then

$$\sum_{1 \leq n \leq x} n^{-s} = \frac{x^{1-s}}{1-s} + \zeta(s) + E(x),$$

$$\text{where } \zeta(s) = \sum_{n=1}^{\infty} u_n + \frac{1}{s-1}, u_n = \frac{1}{n^s} - \int_n^{n+1} \frac{du}{u^s}$$

$$\text{and } E(x) = E(x, s) = \frac{([\![x]\!] + 1)^{1-s} - x^{1-s}}{1-s}$$

$$- \sum_{n > [\![x]\!] + 1} u_n. \text{ Further } |E(x)| \leq 2x^{-s}.$$

*Proof:* LHS

$$= \sum_{1 \leq n \leq x} \left( \frac{1}{n^s} - \int_n^{n+1} \frac{du}{u^s} \right) + \int_1^{[\![x]\!] + 1} \frac{du}{u^s}$$

$$\text{and here the first term is } \sum_{n=1}^{\infty} u_n - \sum_{n > [\![x]\!] + 1} u_n.$$

$$\text{Note that } \frac{([\![x]\!] + 1)^{1-s} - x^{1-s}}{1-s}$$

$$= \int_x^{[\![x]\!] + 1} v^{-s} dv \leq x^{-s} \text{ and}$$

$$\sum_{n > [\![x]\!] + 1} u_n < \sum_{n > [\![x]\!] + 1} \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \leq x^{-s}.$$

This proves the lemma.



**Lemma 3.** Let  $\phi < s < 1$  and  $x \geq 1$ . Then, we have,

$$\sum_{mn \leq x} a_n (mn)^{-s} = \frac{x^{1-s}}{1-s} \sum_{n=1}^{\infty} \frac{a_n}{n} + \zeta(s) \sum_{n=1}^{\infty} \frac{a_n}{n^s} + \sum_{n=1}^4 T_n,$$

where  $T_1 = \frac{x^{1-s} - 1}{s-1} \sum_{n > x} \frac{a_n}{n},$

$$T_2 = \frac{1}{s-1} \int_s^1 \left( \sum_{n > x} - \frac{a_n \log n}{n^u} \right) du,$$

$$T_3 = \left( -\zeta(s) + \frac{1}{s-1} \right) \sum_{n > x} (a_n n^{-s})$$

and  $T_4 = \sum_{n \leq x} a_n n^{-s} E\left(\frac{x}{n}\right),$

$E(x)$  being defined in lemma 2. Also  $|E(x)| \leq 2x^{-s}.$

*Proof:* LHS =  $\sum_{n \leq x} (a_n n^{-s} \sum_{m \leq x/n} m^{-s})$

$$= \sum_{n \leq x} a_n n^{-s} \left( \frac{(n-1)x}{1-s} \right)^{1-s} + \zeta(s) + E\left(\frac{x}{n}\right)$$

by lemma 2

$$= \frac{x^{1-s}}{1-s} \left( \sum_{n=1}^{\infty} \frac{a_n}{n} - \sum_{n > x} \frac{a_n}{n} \right) + \zeta(s) \left( \sum_{n=1}^{\infty} \frac{a_n}{n^s} - \sum_{n > x} \frac{a_n}{n^s} \right) + T_4$$

$$\begin{aligned} \text{Next, } -\frac{x^{1-s}}{1-s} \sum_{n>x} \frac{a_n}{n} - \zeta(s) \sum_{n>x} \frac{a_n}{n^s} \\ = T_1 + T_2 + T_3. \end{aligned}$$

This proves lemma 3.

**Lemma 4.** *We have,*

$$|T_1| \leq 6C \left( \frac{x^{1-s}-1}{1-s} \right) (1-2^{-(1-\phi)}) x^{\phi-1},$$

$$|T_2| \leq \frac{12C \log(x+3)}{(1-2^{\phi-s})^2} x^{\phi-s},$$

$$|T_3| \leq \frac{6C x^{\phi-s}}{(1-2^{\phi-s})}.$$

*Proof:* Follows from lemma 1 since

$$|T_1| \leq 3 \left( \frac{x^{1-s}-1}{1-s} \right) \sum_{n=0}^{\infty} \frac{C (2^{n-1} x)^{\phi}}{2^n x},$$

$$|T_2| \leq \frac{3}{1-s} \int_s^1 \sum_{n=0}^{\infty} \frac{C (2^{n+1} x)^{\phi}}{(2^n x)^u} \log(2^{n+1} x) du$$

$$\leq 6C \sum_{n=0}^{\infty} \frac{(n+1) \log 2 + \log(x+3)}{(2^n x)^{s-\phi}}$$

$$= 6C \left( \frac{\log(x+3)}{(1-2^{\phi-s})} + \frac{\log 2}{(1-2^{\phi-s})^2} \right) x^{\phi-s},$$

$$|T_3| \leq 3C \sum_{n=0}^{\infty} \frac{(2^{n+1} x)^{\phi}}{(2^n x)^s}$$

$$< \frac{6 C x^{\phi-s}}{(1-2^{\phi-s})},$$

$$\sum_{n=1}^{\infty} u_n < \sum_{n=1}^{\infty} \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right) = 1.$$

**Lemma 5.** We have,

$$|T_4| < 32 \left( 9 + \frac{6}{1-2^{\phi-1}} \right) C x^{\frac{1}{2}(1+\phi-2s)}$$

*Proof.*

$$\begin{aligned} |T_4| &< \sum_{2^m < x} \sum_{2^m < n < 2^{m+1}} \left( a_n n^{-s} E\left(\frac{x}{n}\right) \right) \\ &= \sum_{2^m < x} S(U) \text{ say, where } U = 2^m \end{aligned}$$

We have trivially

$$\begin{aligned} |S(U)| &< \sum_{U < n < 2U} \left( 2(2U)^{\phi} C U^{-s} 2 \left(\frac{x}{2U}\right)^{-s} \right) \\ &= 4 C x^{-s} 2^{s+\phi} U^{1+\phi}. \end{aligned}$$

Also  $E\left(\frac{x}{n}\right)$  is monotonic except for those  $n$  for which  $\left[\frac{x}{n}\right]$

has a change. Hence  $\left| \sum_{U_1 < n < U_2} a_n n^{-s} E\left(\frac{x}{n}\right) \right|$

does not exceed

$$3 \left( \max_{U_1 < n < U_2} \left| E\left(\frac{x}{n}\right) \right| \max_{U_3} \left| \sum_{U_1 < n < U_3 < 2U} a_n n^{-s} \right| \right)$$

where  $U_1 \leq n < U_2$

is an interval contained in  $U \leq n < 2U$  over which  $\left[ \frac{x}{n} \right]$  does not change. This in turn does not exceed (we have used lemma 1 above and we use it again)

$$\begin{aligned} 6 \max_{U \leq n < 2U} \left| E \left( \frac{x}{n} \right) \right| & (2C(2U)^\phi) U^{-s} \\ & \leq 24 (2^{s+\phi} x^{-s} U^\phi) C. \end{aligned}$$

Since there are not more than

$$\frac{x}{U} - \frac{x}{2U} + 1 = \frac{x}{2U} + 1 \text{ intervals we have}$$

$$|S(U)| < \left( \min \left( U^{1+\phi}, \frac{12x}{U^{1-\phi}} \right) \right) 4C 2^{s+\phi} x^{-s}.$$

Hence

$$\begin{aligned} |T_4| & < \left\{ \sum_{2^m < (12x)^{\frac{1}{2}}} 2^m (1+\phi) \right. \\ & \quad + \left. \sum_{2^m > (12x)^{\frac{1}{2}}} \frac{12x}{2^m (1-\phi)} \right\} 4C 2^{s+\phi} x^{-s} \\ & < \left\{ (12x)^{\frac{1+\phi}{2}} \sum_{m=0}^{\infty} 2^{-m(1+\phi)} \right. \\ & \quad + \left. (12x)^{\frac{1+\phi}{2}} \sum_{m=0}^{\infty} 2^{-m(1-\phi)} \right\} 4C 2^{s+\phi} x^{-s} \end{aligned}$$

$$\begin{aligned}
 &< (12) \frac{\phi+1}{2} C \left\{ \frac{1}{1-2^{-(1+\phi)}} + \frac{1}{1-2^{-(1-\phi)}} \right\} \\
 &\quad \times 4C 2^{s+\phi} x^{\frac{1}{2}(1+\phi-2s)} \\
 &= 32 C x^{\frac{1}{2}(1+\phi-2s)} \left\{ \frac{2^{s+1+\phi+\phi-3} 3^{\frac{\phi+1}{2}}}{1-2^{-(1+\phi)}} \right. \\
 &\quad \left. + \frac{2^{s+2\phi-2} 3^{\frac{\phi+1}{2}}}{1-2^{-(1-\phi)}} \right\}
 \end{aligned}$$

**Lemma 6.** We have,

$$\begin{aligned}
 \left| T_1 + T_2 + T_3 + T_4 \right| &\leq \left\{ 1160 + \frac{96(1-\phi)}{(s-\phi)x^{s-\phi+\lambda}} \right. \\
 &\quad \left. + \frac{192(\log(x+3))(1-\phi)}{(s-\phi)^2 x^{s-\phi+\lambda}} \right\} \frac{Cx}{1-\phi},
 \end{aligned}$$

where  $\lambda = \frac{1}{2}(1+\phi-2s)$ .

*Proof:* We have only to verify that

$$\begin{aligned}
 T_4' + 6 \left( \frac{x^{1-s} - 1}{1-s} \right) \left( \frac{x^{\phi-1}}{1-2^{-(1-\phi)}} \right) \\
 + \frac{12 \log(x+3)}{(1+2^{\phi-s})^2} x^{\phi-s} + \frac{6 x^{\phi-s}}{(1-2^{\phi-s})} \\
 < \left\{ 640 + \frac{96(1-\phi)}{(s-\phi)x^{s-\phi+\lambda}} \right. \\
 \left. + \frac{192(\log(x+3))(1-\phi)}{(s-\phi)^2 x^{s-\phi+\lambda}} \right\} \frac{x^\lambda}{1-\phi},
 \end{aligned}$$

where  $T_4' = 32 \left( 9 + \frac{6}{1-2^{-(1-\phi)}} \right) x^\lambda$  and

$$\lambda = \frac{1}{2} (1 + \phi - 2s).$$

(i.e.) (since  $x^{1-s} - 1 \leq 1 - s$ ),

$$\begin{aligned} & 288 + \frac{192}{1-2^{-(1-\phi)}} + \frac{6 x^{\phi-1-\lambda}}{1-2^{-(1-\phi)}} \\ & + \frac{12 \log(x+3) x^{\phi-s-\lambda}}{(1-2^{\phi-s})^2} + \frac{6 x^{\phi-s-\lambda}}{1-2^{\phi-s}} \\ & < \frac{1160}{1-\phi} + \frac{96}{(s-\phi) 2^{s-\phi+\lambda}} + \frac{192 \log(x+3)}{(s-\phi)^2 x^{s-\phi+\lambda}}. \end{aligned}$$

This is true since

$$283 + \frac{192}{1-2^{-(1-\phi)}} + \frac{6}{1-2^{-1+\phi}} \leq \frac{1160}{1-\phi} \quad \text{and}$$

$$\frac{1}{1-2^{\phi-s}} < \frac{3}{s-\phi}.$$

This completes the proof of theorem 1 and its corollary, viz. theorem 2 assuming the truth of Lemma 1.

§ 3. *Proof of lemma 1:*

It is clear that  $\left| \sum_{n=1}^{k_1} X_1(n) \right| = 0$  and so

$$\left| \sum_{n \leq x} X_1(n) \right| < \max_{1 \leq n \leq k_1} \left| \sum_{m=1}^n X_1(m) \right| < \frac{5}{8} k_1$$

(it is well known, due to I. M. V. and G. P., that this sum is in fact  $O(k^{\frac{1}{2}} \log k)$ ). Next  $\left| \sum_{mn \leq x} X_1(m) X_2(n) \right|$  is by a

familiar argument

$$\left| \sum_{m \leq \sqrt{x}} \dots + \sum_{n \leq \sqrt{x}} \dots - \left( \sum_{n \leq \sqrt{x}} \chi_1(n) \right) \left( \sum_{n \leq \sqrt{x}} \chi_2(n) \right) \right|$$

$$\text{and so does not exceed } \frac{5}{3} x^{\frac{1}{2}} \left( \frac{k_1 + k_2}{2} \right) + \frac{25}{36} k_1 k_2.$$

(Here  $\chi_1$  and  $\chi_2$  are any two non-principal characters.)

We can prove lemma 1 by an extension of this argument as follows. We have

$$\begin{aligned} & \left| \sum_{lmn \leq x} \chi_1(l) \chi_2(m) \chi_3(n) \right| \\ &= \left| \sum_{l \leq x^3} \dots \frac{1}{\dots} + \sum_{m \leq x^3} \dots \frac{1}{\dots} + \sum_{n \leq x^3} \dots \frac{1}{\dots} + J \right| \end{aligned}$$

$$\begin{aligned} \text{where } J = & \sum_{l \leq x^3, m \leq x^3} - \left( \dots \right) \frac{1}{\dots} + \sum_{m \leq x^3, n \leq x^3} - \left( \dots \right) \frac{1}{\dots} \\ & + \sum_{n \leq x^3, l \leq x^3} - \left( \dots \right) \frac{1}{\dots} \\ & + \sum_{l \leq x^3, m \leq x^3, n \leq x^3} \dots \frac{1}{\dots} \end{aligned}$$

(This is done by counting the (net) number of times a lattice point  $(l, m, n)$  appears in these sums. For example a lattice point with precisely one co-ordinate say  $l$  satisfying

$l \leq x^{\frac{1}{3}}$ . Next two co-ordinates and next three co-ordinates) and so a bound for the LHS of the second inequality of lemma 1 is

$$\begin{aligned}
 & \sum_{l \leq x^{\frac{1}{3}}} \left( \frac{5}{3} \left( \frac{x}{l} \right)^{\frac{1}{3}} \left( \frac{k_2 + k_3}{2} \right) + \frac{25}{36} k_2 k_3 \right) + 2 \text{ other symmetric terms} \\
 & + \sum_{l \leq x^{\frac{1}{3}}} \sum_{m \leq x^{\frac{1}{3}}} \left( \frac{5}{6} k_3 \right) + 2 \text{ other symmetric terms} \\
 & + \frac{125}{216} k_1 k_2 k_3.
 \end{aligned}$$

For  $x > 1$  we have  $\sum_{n \leq x} n^{-\frac{1}{3}} < 1 + \int_1^x \frac{du}{u^{\frac{4}{3}}} = 2x^{\frac{1}{3}} - 1$

and hence the bound above does not exceed

$$\begin{aligned}
 & \frac{125}{216} k_1 k_2 k_3 + \frac{5}{6} (k_1 + k_2 + k_3) x^{\frac{2}{3}} \\
 & + \frac{25}{36} (k_1 k_2 + k_2 k_3 + k_3 k_1) x^{\frac{1}{3}} \\
 & + \frac{5}{3} x^{\frac{1}{2}} (2x^{\frac{1}{6}} - 1) (k_1 + k_2 + k_3) \\
 & < x^{\frac{2}{3}} (k_1 k_2)^2 \left\{ \frac{125}{216} x^{-\frac{2}{3}} + \frac{5}{2} (k_1 k_2)^{-1} \right. \\
 & \quad \left. + \frac{25}{12} x^{-\frac{1}{3}} k_1^{-1} + 10 (k_1 k_2)^{-1} \right\}
 \end{aligned}$$

provided  $3 \leq k_1 \leq k_2$  and  $k_3 = k_1 k_2$ . Plainly this does

not exceed  $(2k_1 k_2 x^{\frac{1}{3}})^2$ . This proves lemma 1.

Hence the proof of our main theorem 2 is complete.



**References**

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(The results of this paper were known to the author by 1975 - 1976, but due to some reasons it was not possible to publish them earlier. In the meanwhile J. Pintz has published results which are somewhat weaker than ours by nearly the same method.)

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