On warning's problem : g (4) < 21

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ON WARING'S PROBLEM : $g(4) \leq 21$

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(Dedicated to the Memory of Dr. S. S. PILLAI)

§ 1. Developing the ideas of Chen-jing-run [2] we prove the following

Theorem :

Every natural number is expressible as a sum of atmost twenty one fourth powers.

In the usual notation, our result reads : $19 \leq g(4) \leq 21$. This is an improvement of the result of Thomas [9] who proved that $g(4) \leq 22$. We recall that the general problem for the k^{th} powers in the place of fourth powers is nearly complete, which is due to Dickson [5] and Pillai (independent of each other); (see "S. S. Pillai" by K. Chandrasekharan : Jour. of Indian Math. Soc. 15 (1951) (1–10) for the list of complete works of Dr. S. S. Pillai). On the other hand, in the case of $G(4)$, it has been proved by Davenport [4] that $G(4) = 16$. Incidentally, we remark that Auluck [1] proved that every integer $\geq 10^{10^{89 \cdot 39}}$ is a sum of nineteen fourth powers. It was improved by Thomas [8] who proved that every integer $\geq 10^{1408 \cdot 3}$ is representable as a sum of 19 fourth powers (Theorem 12.1; pp. 152 of [8]). Our method improves the bound; but since it does not prove any thing substantial, we are not including the proof of this fact.

§ 2. Notations :

Let $e(x)$ stand for $e^{2\pi ix}$ and

$$eq(x) \text{ for } e^{\frac{2\pi ix}{q}}$$

Define $S_{a,q} = \sum_{x=1}^{q}{}' e^{\frac{2\pi iax^4}{q}}$ where the accent here (and elsewhere) shows that the summation is restricted to only those x for which $(x, q) = 1$.

Let $N (\geqslant 10^{560})$ be a given integer to be represented as a sum of 21 biquadratics. Define two integers N_0 and N_1 by

$$N_0 \leqslant N \leqslant 2 N_0 \text{ and } N_0 - P^{3\frac{3}{4}} \leqslant N_1 \leqslant N_0$$

Define $P = [N^{\frac{1}{4}}]$ and $T(\alpha) = \sum_{1 \leqslant x \leqslant P} e(\alpha x^4)$

For any real number α, $0 < \alpha \leqslant 1$, there exist two integers h, q with $1 \leqslant h \leqslant q \leqslant 8 P^3$, $(h, q) = 1$ such that

$$\left| \alpha - \frac{h}{q} \right| \leqslant \frac{1}{8q P^3}.$$

The major arc $\Omega = \left\{ \alpha \mid \left| \alpha - \frac{h}{q} \right| \leqslant \frac{1}{8q P^3} \text{ for some } q \leqslant P^{\frac{1}{2}} \right\}$

and the minor arc $\mathfrak{m} = \left\{ \alpha \mid \left| \alpha - \frac{h}{q} \right| \leqslant \frac{1}{8q P^3} \text{ for some } q, \right.$

$$\left. P^{\frac{1}{2}} < q \leqslant 8 P^3 \right\}$$

The singular series $S(n ; m)$ is defined by

$$S(n) = S(n, m) = \sum_{q=1}^{\infty} \sum_{a=1}^{q}{}' \left(\frac{S_{a,q}}{q} \right)^m e^{\frac{-2\pi ian}{q}}.$$

The truncated singular series $S_1(n, m)$ is defined by

$$S_1(n) = S_1(n, m) = \sum_{q \leq P^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \left(\frac{S_{a,q}}{q}\right)^m e^{\frac{-2\pi i a n}{q}}.$$

θ is a constant (depending on various parameters), such that $|\theta| \leq 1$.

$$\psi = \psi(a) = \int_0^P e^{2\pi i a x^4}\, dx$$

$$W(N') = \int_0^1 T_a^m\, e^{-2\pi i a N'}\, da$$

$$W_0(N) = \int_\Omega T_a^m\, e^{-2\pi i a N'}\, da$$

$$R(N_0) = \int_{-\infty}^{\infty} \psi^m\, e^{-2\pi i a N_0}\, da$$

§ 3. An upper bound for $S_{a,q}$

Let us recall that $S_{a,q} = \sum_{\substack{x=1 \\ (x,q)=1}}^{q} e_q(ax^4)$.

We then have,

Lemma 1: *$S_{a,q}$ is a multiplicative function of q*

Proof: A proof can be found in Davenport [3] (lemma 6 in page 31).

Because of lemma 1, it is sufficient to have the bound for S_{a,p^α}. In this direction, we have

Lemma 2: (a) *For any prime* $p \neq 2$, $|S_{a,p}| \leq (\delta - 1)p^{\frac{1}{2}}$ where $\delta = (4, p-1)$

(b) *For any prime* $p \neq 2$, $|S_{a,p^\nu}| = p^{\nu-1}$ If $2 \leq \nu \leq 4$

(c) *For any prime* $p \geqslant 2$,

$$|S_{a,p^{\nu}}| = p^3 |S_{a,p^{\nu-4}}|$$

if $\nu > 4$.

Proof: A proof can be found in Davenport [3] (lemma 12 in page 42, lemma 13 in page 43, and lemma 14 in page 44).

One can now get a bound for $|S_{a,q}|$ from lemmas 1 and 2. But, for small primes (namely $p \leqslant 79$), we find the bound for $S_{a,p}$ by actual computation (and the bound is better than the one given by lemma 2) and deduce

Lemma 3: *We have* $|S_{a,q}| \leqslant (4{\cdot}3)\, q^{\frac{3}{4}}$ *if* $(a,q) = 1$.

Proof: A proof of the lemma can be found in Thomas [8] (Theorem 2.1, Page 38).

§ 4: A bound for T (α) in the minor arc:

Let us recall that $T(\alpha)$ is defined by

$$T(\alpha) = \sum_{1 \leqslant x \leqslant P} e(\alpha x^4)$$

Minor arc is defined by

$$\mathfrak{m} = \left\{ \alpha \mid \left| \alpha - \frac{h}{p} \right| \leqslant \frac{1}{8q P^3} \text{ for some } q,\ P^{\frac{1}{2}} < q \leqslant 8P^3 \right\}$$

Lemma 4: *Let* $d(m)$ *be the number of divisors of* m.

Then

$$\sum_{1 \leqslant i \leqslant n} (d(i))^j \leqslant A_j\, n\, (j + \log n)^{2^j - 1}$$

where A_j *depends only on* j *In particular one can take*

$$A_1 = 1;\quad A_2 = \tfrac{1}{3};\quad A_3 = \tfrac{1}{2^4};\quad A_4 = \frac{1}{24 \times 192}.$$

Proof: The values of A_j for every integer j is given by Mardjanichvili [7]. The values of A_1, A_2, A_3 and A_4 are discussed in Chen [2] (lemma 8)

Lemma 5 : *Let $h_1(n)$ denote the number of solutions of the equation $n = l_1 l_2$, $1 \leqslant l_1 \leqslant P$; $1 \leqslant l_2 \leqslant P$ in integers.*

Then, $\sum\limits_{n \leqslant P^2} (h_1(n))^2 \leqslant 2P^2 (\log P + 3)$

Proof : We have

$$\sum_{n \leqslant P^2} (h_1(n))^2 = \sum_{n \leqslant P^2} \Big(\sum_{\substack{d \mid n \\ d \leqslant P;\ \frac{n}{d} \leqslant P}} 1 \Big)^2$$

$$= \sum_{n \leqslant P^2} \sum_{d_1 \mid n} \sum_{d_2 \mid n} 1.$$

$$d_1 \leqslant P;\ \frac{n}{d_1} \leqslant P \qquad d_2 \leqslant P;\ \frac{n}{d_2} \leqslant P$$

$$\leqslant 2 \sum_{d_1 \leqslant P} \sum_{d_2 \leqslant d_1} \sum_{\substack{n \leqslant P d_2 \\ n \equiv 0 (\mathrm{mod}\ [d_1, d_2])}} 1$$

$$\leqslant 2 \sum_{d_1 \leqslant P} \sum_{d_2 \leqslant d_1} \frac{P d_2}{[d_1, d_2]}$$

$$\leqslant 2 \sum_{d_1 \leqslant P} \sum_{l \mid d_1} \sum_{d_2 \leqslant \frac{d_1}{l}} \frac{P d_2 l}{d_1 d_2}$$

$$\leqslant 2P^2 \ (\log P + 3).$$

Lemma 6 : *Let $h_2(n)$ denote the number of solutions of the equation $n = l_1 l_2$, $1 \leqslant l_1 \leqslant P$; $1 \leqslant l_2 \leqslant P^2/4$ in integers.*

Then

$$\sum_{n \leqslant P^2/4} (h_2(n))^4 \leqslant \frac{P^3}{12} (\log P + 3)^{11}$$

Proof : We have

$$\sum_{n \leqslant P^3/4} (h_2(n))^4 = \sum_{n \leqslant P^3/4} \Big(\sum_{\substack{l \mid n \\ l \leqslant P;\ \frac{n}{l} \leqslant P^2/4}} 1 \Big)^4$$

$$\leq 24 \sum_{l_1 \leq P} \sum_{l_2 \leq l_1} \sum_{l_3 \leq l_2} \sum_{l_4 \leq l_3} \frac{P^3 l_4}{4[l_1, l_2, l_3, l_4]}$$

$$\leq 6P^3 \sum_{1 \leq l_1 \leq P} \sum_{d_1 | l_1} \sum_{l_2 \leq \frac{l_1}{d_1}} \sum_{d_2 | l_1 l_2} \sum_{l_3 \leq \frac{d_1 l_2}{d_2}}$$

$$\sum_{d_3 | l_1 l_2 l_3} \sum_{d_4 \leq \frac{d_2 l_3}{d_3}} \frac{d_3 l_4}{l_1 l_2 l_3 l_4}$$

$$\leq 6P^3 \sum_{1 \leq l_1 \leq P} \sum_{d_1 | l_1} \sum_{l_2 \leq \frac{l_1}{d}} \sum_{d_2 | l_1 l_2}$$

$$\sum_{l_3 \leq \frac{d_1 l_2}{d_2}} \frac{d(l_1)\, d(l_2)\, d(l_3)\, d_2}{l_1 l_2}$$

$$\leq 6P^3 \sum_{1 \leq l_1 \leq P} \sum_{d_1 | l_1} \sum_{l_2 \leq \frac{l_1}{d}}$$

$$\sum_{d_2 | l_1 l_2} \frac{d(l_1)\, d(l_2) \cdot (\log P + 1)}{l_1}$$

$$\leq 6P^3 (\log P + 1) \sum_{1 \leq l_1 \leq P} \sum_{d_1 | l_1}$$

$$\sum_{l_2 \leq \frac{l_1}{d}} \frac{d^2(l_1)\, d^2(l_2)\, d_1}{l_1}$$

$$\leq 2P^3 (\log P + 1) \sum_{1 \leq l_1 \leq P} \sum_{d_1 | l_1} d^3(l_1) \cdot (\log P + 2)^3$$

$$\leq 2P^2 (\log P + 2)^4 \sum_{1 \leq l_1 < P} d^3(l_1)$$

$$\leq \frac{P^3}{12} (\log P + 3)^{11}$$

Lemma 7: *Let* $\alpha = \frac{a}{q} + \frac{\theta}{8P^3 q}$. *Then, if* $P > 10^{50}$, *we have*

$$\sum_{0 \leq n \leq P^3/4} \min \left(P, \frac{1}{2 \| 24 \alpha n \|} \right)^{4/3} \leq 50 P^{\frac{10}{3}} \text{ provided } 8P^3 \geqslant q > P;$$

Proof: Let $q' = \frac{q}{(24, q)}$. The sum over n can be broken into not more than $\left(\frac{P^3}{4q'} + 1 \right)$ parts, in each of which n runs over atmost q' consecutive integers. Let us consider the typical sum say $\sum_{B < n \leqslant B + q'} \min \left(P, \frac{1}{2 \| 24 \alpha n \|} \right)^{\frac{4}{3}}$.

Then it is easily seen that, there are atmost $\left(\frac{q'}{P} + 4 \right)$ values for n for which $\frac{1}{2 \| 24 \alpha n \|} \geqslant P$ and these values of n contribute atmost $\left(\frac{q'}{P} + 4 \right) P^{4/3} = q' P^{1/3} + 4 P^{4/3}$ to the sum. The remaining values of n contribute

$$\sum_{m \geqslant \frac{q'}{P}} \frac{1}{2} \left(\frac{q'}{m} \right)^{4/3} < 4 q' P^{1/3}.$$

Hence

$$\sum_{B < n \leqslant B + q'} \min \left(P, \frac{1}{2 \| 24 \alpha n \|} \right)^{4/3} < 5 q' P^{1/3} + 4P^{4/3}$$

Hence the total sum is atmost

$$(5 q' P^{1/3} + 4P^{4/3}) \left(\frac{P^3}{4q'} + 1 \right) < 50 P^{\frac{10}{3}}$$

Lemma 8: *If $h_2(n)$ denotes the number of solutions of the equation $n = l_1 l_2$, $1 \leqslant l_1 \leqslant P$; $1 \leqslant l_2 \leqslant P^2/4$, in integers, then*

$$\sum_{n \leqslant P^3/4} h_2(n) \min\left(P, \frac{1}{2 \| 24 \alpha n \|}\right) \leqslant 11\, P^{\frac{18}{4}} (\log P + 3)^{\frac{11}{4}}.$$

where $\alpha = \frac{a}{q} + \frac{\theta}{8P^3 q}$; $P \geqslant 10^{50}$ *and* $P \leqslant q \leqslant 8P^3$

Proof: If $P \leqslant q \leqslant 8P^3$, then

$$\sum_{n \leqslant P^3/4} h_2(n) \min\left(P, \frac{1}{2 \| 24 \alpha n \|}\right)$$

$$\leqslant \left(\sum_{n \leqslant \frac{P^3}{4}} (h_2(n))^4\right)^{\frac{1}{4}} \left(\sum_{n \leqslant \frac{P^3}{4}} \left(\min\left(P, \frac{1}{2 \| 24 \alpha n \|}\right)\right)^{\frac{4}{3}}\right)^{\frac{3}{4}}$$

and the result follows from lemmas 6 and 7.

Lemma 9: *Let* $\alpha = \frac{a}{q} + z$; $(a,q) = 1$; $P \geqslant 10^{100}$, $P \leqslant q \leqslant 8P^3$

$$|z| \leqslant \frac{1}{8q P^3}.$$ *Then*

$$T(\alpha) = \left| \sum_{x=1}^{P} e^{2\pi i \alpha x^4} \right| < (2.86)\, P^{\frac{39}{43}} (\log P + 3)^{\frac{15}{43}}$$

Proof: Let $l_1 = x - y$; $f(x, y) = 4x^3y + 6x^2y + 4xy^3$

$$h(x\,;\, y\,;\, z) = 12\, xyz\,(x+y+z).$$

$h_1(n)$ and $h_2(n)$ are as defined in lemmas 5 and 6.

Then

$$|T(\alpha)|^8 \leqslant P + 2 \left| \sum_{x=1}^{P} \sum_{\substack{y=1 \\ y \neq x}}^{P} e^{2\pi i \alpha (x^4 - y^4)} \right|$$

$$\leqslant P+2 \sum_{l=1}^{P} \left| \sum_{x=1}^{P-l} e^{2\pi i a f(x,l)} \right|$$

$$= P+2S_1, \quad \text{say}$$

$$|S_1|^2 \leqslant \left(\sum_{l=1}^{P} 1 \right) \left(\sum_{l=1}^{P} \left| \sum_{x=1}^{P-l} e^{2\pi i a f(x,l} \right|^2 \right)$$

$$\leqslant P^3+2P \sum_{l_1=1}^{P} \sum_{l_2=1}^{P-l_1} \left| \sum_{x=1}^{P-l_1-l_2} e^{2\pi i a\, h(x;\, l_1,l_2)} \right|$$

$$= P^3 + 2PS \quad \text{say}$$

Now

$$|S| \leqslant \sum_{1 \leqslant n \leqslant \frac{P^2}{4}} h_1(n) \max_{l_1 l_2 = n} \left| \sum_{x=1}^{P-l_1-l_2} e^{2\pi i a.h(x;l_1,l_2)} \right|$$

Hence $|S|^2 \leqslant \left(\sum_{1 \leq n \leq P^2/4} h_1(n) \right)^2$

$$\left(\sum_{1 \leq n \leq P^2/4} \max_{l_1 l_2 = n} \left| \sum_{x=1}^{P-l_1-l_2} e^{2\pi i a\, h(x;\, l_1,\, l_2)} \right|^2 \right)$$

Now using lemma 5,

$$|S|^2 \leq 2P^2 (\log P + 3) \sum_{1 \leq n \leq P^2/4} \max_{l_1 l_2 = n}$$

$$\left(P + 2 \sum_{l_3=1}^{P} \min \left(P, \frac{1}{2 \| 24\, a\, l_1 l_2 l_3 \|} \right) \right)$$

$$\leq 2P^2 (\log P + 3) \sum_{1 \leq i \leq P^2/4}$$

$$\left(P + 2 \sum_{l=1}^{P} \min \left(P, \frac{1}{2 \| 24\, a\, li \|} \right) \right)$$

$$\leq 2P^2 (\log P + 3) \times$$

$$\left(\frac{P^2}{4} + 2 \sum_{0 < n \leq P^2/4} h_2(n) \min \left(P, \frac{1}{2 \| 24 a n \|} \right) \right)$$

and using lemma 8,

$$|S|^2 \leq 2P^2 (\log P + 3) (P^2/4 + 22P^{\frac{18}{4}} (\log P + 3)^{\frac{11}{4}})$$

and this gives

$$|T(a)| \leq (2.86) P^{\frac{29}{32}} (\log P + 3)^{\frac{15}{32}}$$

Lemma 10: *If* $a = \frac{a}{q} + \beta$, *where* $q < P$, $|\beta| \leq \frac{1}{8q P^3}$, *and* $P \geq 6$, *then*

$$|T(a)| \leq q^{-\frac{1}{4}} (1 + \log q) \min (16P, \tfrac{3}{4} |\beta|^{-1} P^{-3})$$

Proof: This is lemma 9.5 (page 139) of Thomas [8]. A similar result is proved in lemma 9 of Davenport [4]. Even though the result in [8] is proved under the condition $|\beta| \leq \frac{1}{64q P^3}$ it holds, in fact, for $|\beta| \leq \frac{1}{8q P^3}$.

Lemma 11: *If* $a = \frac{a}{q} + \beta$, $P^{\frac{1}{2}} \leq q \leq P$,

$$|\beta| \leq \frac{1}{8q P^2}, \text{ and } P \geqslant 10^{100}, \text{ then}$$

$$|T(a)| \leq (2.86) P^{\frac{29}{32}} (\log P + 3)^{\frac{15}{32}}$$

Proof: This follows from lemma 10.

Lemma 12 : *In the minor arc, the following estimate holds :*

$$| T (\alpha) | \leq 2.86 \, P^{\frac{29}{8}} \, (\log P + 3)^{\frac{15}{8}}$$

Proof : The result follows from lemmas 9 and 11.

§ 5. A lower bound for R (N_0)

Let us recall that

$$\psi = \psi (\alpha) = \int_0^P e^{2\pi i \alpha x^4} \, dx$$

$$R (N_0) = \int_{-\infty}^{\infty} \psi^m \, e^{-2\pi i \alpha N_0} \, d\alpha$$

Define $B = B(\alpha) = \begin{cases} P & \text{if } | \alpha | \leqslant P^{-4} \\ \sqrt{2} \, | \alpha |^{-\frac{1}{4}} & \text{if } | \alpha | > P^{-4} \end{cases}$

Then it is easily seen that $| \psi | \leq B$

$$W (N') = \int_0^1 T_\alpha^m \, e^{-2\pi i \alpha N'} \, d\alpha$$

Lemma 13 : *(van der Corput) : Suppose $f(x)$ is a real function which is twice differentiable for $A \leq x \leq B$ Suppose further that, in this interval $0 \leq f' (x) \leq \frac{1}{2}$ and $f'' (x) \geqslant 0$.*

Then $\sum_{A \leq n \leq B} e (f (n)) = \int_A^B e (f (x)) \, dx + 4\theta$

Proof : This is lemma 16 (page 65) of Davenport [3]. For the O — constants see lemma 13 (page 34) of Vinogradov [10].

Lemma 14: *If* $N_0 - P^{3\frac{3}{4}} \leqslant N' \leqslant N_0$, *then*

$$W(N') = R(N_0) + \int_{-\frac{1}{8P^3}}^{1-\frac{1}{8P^3}} T_\alpha^m e^{-2\pi i\alpha N'}\, d\alpha$$

$$+ 2\theta \cdot 10^3 \cdot P^{m-5+3/4},$$

where $m = 9$ or 10 and $P > 10^{100}$

Proof: In $-\frac{1}{8P^3} \leqslant \alpha \leqslant \frac{1}{8P^3}$ we have, by lemma 13,

$$T(\alpha) = \sum_{1 \leqslant x \leqslant P} e^{2\pi i\alpha x^4} = \int_0^P e^{2\pi i\alpha x^4}\, dx + 9\theta$$

$$= \psi + 9\theta$$

hence $|(T(\alpha))^m - \psi^m| \leqslant (9m)(B+5)^{m-1}$

Consequently we have

$$|(T(\alpha))^m e^{-2\pi i\alpha N'} - \psi^m e^{-2\pi i\alpha N_0}|$$

$$= |T_\alpha^m e^{-2\pi i\alpha N'} - \psi^m e^{-2\pi i\alpha N'}$$

$$+ \psi^m e^{-2\pi i\alpha N'} - \psi^m e^{-2\pi i\alpha N_0}|$$

$$\leqslant 9m(B+5)^{m-1} + B^m (2\pi\alpha) P^{3\frac{3}{4}}$$

Hence

$$\int_{-\frac{1}{8P^3}}^{\frac{1}{8P^3}} |T_\alpha^m e^{-2\pi i\alpha N'} - \psi^m e^{-2\pi i\alpha N_0}|\, d\alpha$$

$$\leqslant 2\int_0^1 \left((9m)\,(B+3)^{m-1} + B^m\,(2\pi a)\,.\,P^{3\frac{3}{4}}\right)\,da$$

$$\leqslant 2\int_0^{P^{-4}} \left(9m\,(P+5)^{m-1} + (2\pi a)\,.\,P^{m+3\frac{3}{4}}\right)\,da$$

$$+ 2\int_{P^{-4}}^1 \left(9m\,(\sqrt{2}\,a^{-\frac{1}{4}} + 5)^{m-1}\right.$$

$$\left. + 2\pi\,P^{3\frac{3}{4}}\,(\sqrt{2}\,a^{-\frac{1}{4}})^m\,.\,a\right)\,da$$

$$\leqslant 10^8\,P^{m-5+\frac{3}{4}}$$

Now, we have

$$W(N') = \int_{-\frac{1}{8P^3}}^{\frac{1}{8P^3}} (T(a))^m.\ e^{-2\pi i a N'}\,da$$

$$= \int_{-\frac{1}{8P^3}}^{1-\frac{1}{8P^3}} (T(a))^m\,e^{-2\pi i a N'}\,da$$

Now, replace the integrand

$(T(a))^m\,e^{-2\pi i a N'}$ of the first integral on the right by $\psi^m\ e^{-2\pi i a N_0}$ and we have just proved that the error is atmost $10^3\ P^{m-5+3/4}$. Hence we have

$$W(N') = \int_{-\frac{1}{8P^3}}^{\frac{1}{8P^3}} \psi^m\,e^{-2\pi i a N_0}\,da +$$

$$+ \int_{\frac{1}{8P^3}}^{1 - \frac{1}{8P^3}} (T(\alpha))^m e^{-2\pi i \alpha N'} \, d\alpha$$

$$+ 10^3 . \theta . P^{m - 5 + \frac{3}{4}}$$

Now we replace the first integral on the right by $R(N_0)$. The error involved is at most,

$$2 \int_{\frac{1}{8P^3}}^{\infty} |\psi|^m \, d\alpha$$

$$\leqslant 2 \int_{\frac{1}{8P^3}}^{\infty} (\sqrt{2}\, \alpha^{-\frac{1}{4}})^m \, d\alpha$$

$$\leqslant 10^3 P^{\frac{3m}{4} - 3}$$

$$\leqslant 10^3 P^{m - 5 + \frac{3}{4}}$$

and this proves the lemma.

We now take $M = \left[\frac{P}{2}^{3\frac{3}{4}} \right]$ and apply lemma 14 with $N' = N - N'' - N'''$, $0 \leqslant N''$, $N''' \leqslant M$ and add the M^2 equations. This gives

Lemma 15 : *We have, with* $M = \left[\frac{P}{2}^{3\frac{3}{4}} \right]$,

$$\sum_{N''} \sum_{N'''} W(N - N'' - N''') = M^2 R(N_0) +$$

$$+\int_{\frac{1}{8P^5}}^{1-\frac{1}{8P^5}} (T(\alpha))^m \, e^{-2\pi i\alpha N} \left(\sum_{N''} e^{-2\pi i\alpha N''} \right)^8 d\alpha$$

$$+ 2\theta \cdot 10^3 \cdot M^4 \quad P^{m-5+\frac{3}{4}}$$

Lemma 16 : *The integral on the right of lemma* 15 *is atmost* $8\,P^{m+3}$.

Proof : Since $|T(\alpha)| \leqslant P$, $|e^{-2\pi i\alpha N}| \leqslant 1$,

$$\left| \sum_{N''} e^{-2\pi i\alpha N''} \right| \leqslant \frac{1}{\|\alpha\|},$$

the result is clear.

Lemma 17 : *If* $K_r(N)$ *denotes the number of integer solutions of the equation* $x_1^n + x_2^n + \ldots\ldots + x_r^n \leqslant N$, *then*

$$K_r(N) = T_r \cdot N^{\frac{r}{n}} - \theta \cdot r \cdot N^{\frac{r-1}{n}},$$

where $T_r = \dfrac{(\Gamma(\frac{5}{4}))^r}{\Gamma\left(1+\frac{r}{4}\right)}$ and $0 \leqslant \theta \leqslant 1$

Proof : This is lemma 3 in (page 22) in Vinogradov [10].

Lemma 18 : *If* $9 \leqslant m \leqslant 11$,

$$\sum_{N''=1}^{M} \sum_{N'''=1}^{M} W(N - N'' - N''') \geqslant 0.102 \cdot M^8 \cdot N^{\frac{m}{4}-1}$$

Proof : We have

$$\sum_{N''=1}^{M} W(N - N'' - N''')$$

$$= \sum_{N''=1}^{M} (K_m(N-N''-N''') - K_m(N-N''-N'''-1))$$

$$= K_m (N - N'') - K_m (N - N'' - M)$$

$$= T_m \left((N - N'')^{\frac{m}{4}} - (N - N'' - M)^{\frac{m}{4}} \right)$$

$$- 2\theta m. N^{\frac{m-1}{4}}$$

$$\geqslant T_m \frac{m}{8} MN^{\frac{m}{4}-1} - 2\theta.m.N^{\frac{m-1}{4}}$$

$$\geqslant T_m \frac{m}{10} M N^{\frac{m}{4}-1}$$

$$\geqslant 0.102.M. N^{\frac{m}{4}-1}$$

and hence the result.

Lemma 19 : *The following inequality holds*

$$|R(N_0)| \geqslant \tfrac{1}{10} N^{\frac{m}{4}-1}$$

Proof : The inequality follows from lemmas 15 and 18.

§ 6 : A lower bound for the singular series

Let us recall that the singular series is defined by

$$S(n) = S(n; m) = \sum_{q=1}^{a} \sum_{a=1}^{q}{}' \left(\frac{S_{a,q}}{q} \right)^m . e^{\frac{-2\pi ian}{q}}$$

and the truncated singular series

$$S_1(n) \pm S_1(n; m) = \sum_{q \leqslant P^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \left(\frac{S_{a,q}}{q} \right)^m . e^{\frac{-2\pi ian}{q}}$$

Here (and elsewhere), the accent shows that the summation is restricted to only those a's, for which $(a, q) = 1$.

Define $A_m(n; q) = \sum_{a=1}^{q}{}' \left(\frac{S_{a,q}}{q} \right)^m e^{\frac{-2\pi i a n}{q}}$

Let us define

$$\chi_s(n, m) = \sum_{i=0}^{\infty} A_m(n; p^i)$$

Lemma 20: *We have* $S(n; 11) \geqslant 0.5304\ \chi_s(n, 11)$

Proof: This is Theorem 4.1 of Thomas [8] (page 98)

Lemma 21: *Suppose* $1 < n \leqslant m \leqslant 15$. *Then*

$$\chi_2(n; m) = 16\ {}^{m}c_{n}\ 2^{-m},$$

where $${}^{m}c_{n} = \frac{m!}{n!\,(m-n)!}$$

Proof: This is Theorem 4.2 of Thomas [8] (page 98)

Lemma 22: *If* $n \equiv 2, 3, 4, 5, 6, 7$ or $8 \pmod{16}$, *then*

$$\chi_s(n, 11) \geqslant 0.42$$

Proof: If $n_1 \equiv n_2 \pmod{16}$, then

$\chi_s(n_1, m) = \chi_s(n_2, m)$. Hence it follows from lemma 21, that $\chi_2(n, 11) \geqslant 16 \cdot {}^{11}c_{2} \cdot 2^{-11} \geqslant 0.42$

Lemma 23: *The singular series* $|S(n)| \geqslant 0.222$ *if* $m = 11$ *provided* $n \equiv 2, 3, 4, 5, 6, 7$ or $8 \pmod{16}$

Proof. The result follows from lemmas 20 and 22

Lemma 24 : *The truncated singular series*

$$| S_1 (n) | \geqslant 0.22 \quad \text{if } m = 11$$

provided $n \equiv 2, 3, 4, 5, 6, 7$ or $8 \pmod{16}$

Proof : We have

$$| S (n) - S_1 (n) |$$

$$= \left| \sum_{q > p^{\frac{1}{2}}} \left(\sum_{a=1}^{q}{}' \left(\frac{S_{a,q}}{q} \right)^{11} e^{-2\pi i \frac{an}{q}} \right) \right|$$

$$\leqslant \sum_{q \geqslant p^{\frac{1}{2}}} \left(\sum_{a=1}^{q}{}' \left| \frac{S_{a,q}}{q} \right|^{11} \right)$$

$$\leqslant \sum_{q > p^{\frac{1}{2}}} 5^{11} \cdot q^{-\frac{7}{4}}$$

$$\leqslant 5^{11} \times 2 \times P^{\frac{3}{8}}$$

$$\leqslant 0.01, \text{ since } P > 10^{30}$$

and hence the result.

§ 6. The estimate on the major arc

Let us recall that the major arc is defined by

$$\Omega = \{ \alpha \mid \left| \alpha - \frac{a}{q} \right| \leqslant \frac{1}{8qP^3} ; \ 1 \leqslant a \leqslant q; (a,q) = 1; q \leqslant P^{\frac{1}{2}}$$

and the integral

$$W_0 (N_1) = \int_{\Omega} (T (\alpha))^m \cdot e^{-2\pi i N_1 \alpha} \, d\alpha$$

Lemma 25 : *The following approximation holds :*

$$\text{If } \alpha = \frac{a}{q} + z, \text{ and } |N_1 - N_0| \leqslant P^{3\frac{1}{4}}, \text{ then}$$

$$(T(\alpha))^m e^{-2\pi i N_1 \alpha} = \psi^m \left(\frac{S_{a,q}}{q}\right)^m e^{\frac{-2\pi i a N_1}{q} - 2\pi i z N_0}$$

$$+ \theta \left(5^{m+1} q^{\frac{5-m}{4}} B^{m-1} + 5^{m-5} \cdot q^{\frac{5-m}{4}} B^m (2\pi z) P^{3\frac{1}{4}} \right)$$

Proof : We have

$$T(\alpha) = \sum_{y=0}^{q-1} \sum_{-yq^{-1} \leqslant t < (P-y)q^{-1}} e^{2\pi i \left(\frac{ay^4}{q} + z(qt+y)^4\right)}$$

$$= \sum_{y=0}^{q-1} e^{\frac{2\pi i a y^4}{q}} D_y(z), \text{ say}$$

since $\frac{d}{dt}(z(qt + y)^4) \leqslant \frac{1}{2}$, we have, by lemma 13,

$$D_s(z) = \int_{yq^{-1}}^{(P-y)q^{-1}} e^{2\pi i z (qt+y)^4} dt + 4\theta$$

$$= \frac{1}{q} \int_0^P e^{2\pi i z x^4} dx + 4\theta$$

Hence $T(\alpha) = \psi \frac{S_{a,q}}{q} + 4\theta q$

Since $|z| \leqslant \frac{1}{8q P^3}$, $q \leqslant p^{\frac{1}{2}}$ we have $zq^{-\frac{1}{4}} \geqslant 8q$

Hence $\left| (T(a))^m - \left(\psi \frac{S_{a,q}}{q} \right)^m \right|$

$$\leqslant 4m.\, q\,(4q^{-\frac{1}{4}} B)^m - 1$$

$$< 5^{m+1} . q^{\frac{5-m}{4}} . B^{m-1}$$

Hence $\left| (T(a))^m e^{-2\pi i N_1 a} - \left(\psi \frac{S_{a,q}}{q} \right)^m \right.$

$$\left. e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_0} \right.$$

$$\leqslant \left| T_a^m e^{-2\pi i N_1 a} - \left(\psi \frac{S_{a,q}}{q} \right)^m \right.$$

$$\left. e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_1} \right.$$

$$+ \left| \left(\psi \frac{S_{a,q}}{q} \right)^m e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_1} \right.$$

$$\left. - \left(\psi \frac{S_{a,q}}{q} \right)^m e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_0} \right.$$

$$\leqslant 5^{m+1} q^{\frac{5-m}{4}} B^{m-1} + \left| \frac{S_{a,q}}{q} \right| m - 5$$

$$\times B^m (2\pi z) P^{3\frac{1}{4}}$$

$$\leqslant 5^{m+1} q^{\frac{5-m}{4}} B^{m-1} + 5^{m-5} q^{\frac{5-m}{4}}$$

$$\times B^m (2\pi z) P^{3\frac{1}{4}}$$

Lemma 26: *We have*

$$W_0(N_1) = \sum_{q \leqslant p^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \int_{-\frac{1}{8qP^3}}^{\frac{1}{8qP^3}} \left(\psi \frac{S_{a,q}}{q}\right)^m \times e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_0} dz + 5^m q^{\frac{5-m}{4}} P^{m-5+\frac{1}{4}} . \theta$$

Proof: From lemma 25, we have

$$W_0(N) = \int_{\Omega} (T(\alpha))^m e^{-2\pi i N_1 \alpha} d\alpha$$

$$= \sum_{q \leqslant p^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \int_{-\frac{1}{8qP^3}}^{\frac{1}{8qP^3}} \left(\psi \frac{S_{a,q}}{q}\right)^m e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_0} dz$$

$$+ \theta \sum_{q \leqslant p^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \int_{-\frac{1}{8qP^3}}^{\frac{1}{8qP^3}} \left(5^{m+1} q^{\frac{5-m}{4}} B^{m-1} + 5^{m-5} q^{\frac{5-m}{4}} B^m . (2\pi z) P^{3\frac{1}{4}}\right) dz$$

Now, using the value of B, the integral in the error term is easily seen to be at most

$$2 \int_0^1 (5^{m+1} q^{\frac{5-m}{4}} B^{m-1} + 5^{m-5} q^{\frac{5-m}{4}} B^m . 2\pi z . P^{3\frac{1}{4}}) dz$$

$$\leqslant 2\int_0^{P^{-4}} (5^{m+1} q^{\frac{5-m}{4}} P^{m-1} + 5^{m-5} q^{\frac{5-m}{4}} P^m 2\pi z . P^{3\frac{1}{4}})\, dz$$

$$+ 2\int_{P^{-4}}^{1} (5^{m-1} q^{\frac{5-m}{4}} (\sqrt{2}\,|z|^{-\frac{1}{4}})^{m-1} + 5^{m-5} q^{\frac{5-m}{4}}$$

$$(\sqrt{2}\,|z|^{-\frac{1}{4}})^m . 2\pi z . P^{3\frac{1}{4}})\, dz$$

$$\leqslant 5^m \cdot q^{\frac{5m}{4}} P^{m-5+\frac{1}{4}}.$$

Lemma 27 : *We have*

$$W_0(N_1) = \sum_{q \leqslant P^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \int_{-\infty}^{\infty} \left(\psi \frac{S_{a,q}}{q}\right)^m \times$$

$$e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_0}\, dz$$

$$+ \theta . 2\, 5^m . P^{m-5+\frac{1}{4}} . A$$

where

$$A = \begin{cases} P^{\frac{1}{2}} & \text{if } m = 9 \\ \log P & \text{if } m = 10 \\ 1 & \text{if } m = 11 \end{cases}$$

The result follows almost immediately from lemma 26 ; we have only to prove that the error in extending the range of integration to $[-\infty, \infty]$ is small.

Actually the error is atmost

$$2 \sum_{q \leqslant P^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \int_{\frac{1}{8qP^5}}^{\infty} \left|\left(\psi \frac{S_{a,q}}{q}\right)\right|^m dz$$

$$\leqslant 2 \sum_{q \leqslant P^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \int_{\frac{1}{8q P^3}}^{\infty} (\sqrt{2}\,|z|^{-\frac{1}{4}})^m \cdot (5q^{-\frac{1}{4}})^m \, dz$$

$$\leqslant 5^m \cdot P^{m-5+\frac{1}{4}} A.$$

Lemma 28 : *If* $|N_1 - N_0| \leqslant P^{3\frac{1}{4}}$, *then*

$$W_0(N_1) = S_1(N_1)\, R(N_0) + 2\theta\, 5^m\, P^{m-5+\frac{1}{4}} \text{ where}$$

$$S_1(N_1) = \sum_{q \leqslant P^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \left(\frac{S_{a,q}}{q} \right)^m e^{-2\pi i \frac{a}{q} N_1}$$

$$\text{and } R(N_0) = \int_{-\infty}^{\infty} \psi^m \, e^{-2\pi i z N_0} \, dz$$

Proof : This follows from lemma 27.

Lemma 29 : If $|N_1 - N_0| \leqslant P^{3\frac{1}{4}}$, *and* $N_1 \equiv 2, 3, 4, 5, 6, 7,$ or 8 (mod 16) *then* $W_0(N_1) \geqslant 0.02\, N^{\frac{m-1}{4}}$ for $m = 11$.

Proof : This follows from lemmas 19, 24 and 28

§ 8 : A lower bound for the number of integers less than a given integer which are representable as a sum of five biquadratics.

Lemma 30 : *Let P be a positive integer and assume* $P \geqslant 100$. *Let* δ *and C be fixed positive reals. Let* μ *be a fixed number in the interval* (0, 1) *and suppose*

$\mathbf{U} = \{ u_1, u_2 \ldots\ldots u_U \}$ *is a set of (distinct) integers in the interval* $[0, P^{3+\mu}]$ *where*

$$U \geqslant C \cdot P^{3(1-\mu)-\delta}.$$

Then the number of solutions M *of the equation*

$$x^4 + u_j = y^4 + u_j$$

where u_h *and* u_j *vary over* $\mathbf{U}$ *and*

$$P \leqslant x,\ y \leqslant 2P$$

and x, y *have the same fixed parity modulo* 2 *does not exceed*

$$C^{-\frac{3}{4}} P^{s}\ U^{s}\ P^{3\mu-4+\frac{3\delta}{4}} (C^{-\frac{1}{4}} P^{\frac{\delta}{4}+\frac{1}{2}} \cdot P^{\frac{(3+\mu)\varepsilon}{4}} \{\tfrac{1}{4} K_2(\varepsilon) \cdot (192)^{-\varepsilon}\}^{\frac{1}{4}})$$

In particular if we take

$$\varepsilon = \frac{\delta}{3+\mu}, \quad \text{then}$$

$$M \leqslant C^{-\frac{3}{4}} P^{s}\ U^{s}\ P^{3\mu-4+\delta} \{ C^{-\frac{1}{4}+\frac{1}{2}} (\tfrac{1}{4} K_2(\varepsilon).(192)^{-\varepsilon})^{\frac{1}{4}} \}$$

Here $K_2(\varepsilon)$ *is defined by*

$$d_4(m) \leqslant K_2(\varepsilon)\, m^{\varepsilon} \quad \textit{for all} \quad m \geqslant 2$$

and we have

$$\{ K_2(0.20) \cdot (\tfrac{1}{192})^{0.20} \}^{\frac{1}{4}} \leqslant (6.2124170)\ 10^6$$

Proof: This is lemma (7.1) of Thomas [8] (page 118). The bound for M when $\varepsilon = \frac{\delta}{3+\mu}$ is given in (7.7) in Thomas [8] (page 119). The bound for $K_2(0.20)$ is given in page 127 of Thomas [8].

Lemma 31: *Let* $1 \leqslant l < 16$; *Let* f *be a fixed integer in* $(0, 1, \ldots\ldots l)$. *Let* f_1 *be a fixed integer in the set* $(0, 1, \ldots\ldots l+1)$. *Let it be given that, for all integers* $X \geqslant X_0$, *the number of integers less than* X, *which are congruent to* $f \pmod{16}$ *and which are representable as a sum of* l *biquadratics is atleast* $C_l \cdot P^l$. *Then the number of integers less than*

Y, which are congruent to f_1 *(mod* 16*) and which are representable as a sum of* $(l+1)$ *biquadratics is atleast*

$$C_{l+1} P^{l+1} \quad \textit{where}$$

$$C_{l+1} = \tfrac{1}{64} C_l^{\frac{3}{4}} \left(C_l^{-\frac{1}{4}} + (2{\cdot}3) \times 10^5\right)^{-1}$$

$$\textit{and} \quad \nu_{l+1} = \tfrac{1}{4} + \frac{3\nu_l}{3 + \nu_l + \varepsilon}$$

$$\textit{Provided} \quad Y \geqslant 10^8 \cdot X^{\frac{4}{3+\mu}}$$

$$\textit{with} \quad \mu = \frac{3(1 - \nu_l - \varepsilon)}{3 + \nu_l + \varepsilon}$$

Here ε *is any positive number.*

Proof: Define $\delta = (3 + \mu)\varepsilon$

Note that $3(1 - \mu) - \delta = (3 + \mu)\nu_l$

and $-4\nu_{l+1} = -4 + 3\mu + \delta$

We observe that $[Y^{\frac{3+\mu}{4}}] \geqslant X$

Choose f_2 from the set $(0, 1)$ such that

$$f_2 + f = f_1.$$

Let $\mathbf{U} = \{x;\ 0 \leqslant x < [Y^{\frac{3+\mu}{4}}];$

$$x = x_0^4 + x_1^4 + x_2^4 + x^4$$

where $x_i \equiv f_1 \pmod 2\}$

Let $U = \text{Card } \mathbf{U}$.

By the hypothesis,

$$U \geqslant C_l Y^{\frac{(3+\mu)\nu_l}{4}} = C_l Y^{\frac{3(1-\mu)-\delta}{4}}$$

Let $r(m)$ denote the number of solutions of $M = u_h + y^4$, where u_h runs over the set $\mathbf{U}$ and

$$z^{\frac{1}{4}} \leqslant y \leqslant 2z^{\frac{1}{4}}; \qquad y \equiv f_3 \pmod 2$$

Then $\sum_m r(m) \geqslant \frac{1}{2} P U$; with $P = [\, z^{\frac{1}{4}} \,]$

$$m \leqslant 16z + y^{\frac{3+\mu}{4}}$$

Also $\sum_m (r(m))^2$ does not exceed the number of solutions of $x^4 + u_h = y^4 + u_j$ subject to the conditions of lemma 30

Hence $\sum (r(m))^2$

$$\leqslant C_l^{-\frac{3}{4}} \cdot P^3 U^3 P^{3\mu - 4 + \delta} \{ C^{-\frac{1}{4}} + (2.3) \times 10^5 \}$$

$$\sum_m 1 \geqslant \frac{(\sum r(m))^2}{\sum (r(m))^2}$$

$r(m) \neq 0$

$$\geqslant \tfrac{1}{4} C_l^{\frac{3}{4}} \cdot (C_l^{-\frac{1}{4}} + (2.3) \times 10^5)^{-1} \cdot P^{-(3\mu - 4 + \delta)}$$

$$\geqslant \tfrac{1}{4} C_l^{\frac{3}{4}} (C_l^{-\frac{1}{4}} + (2.3) \times 10^5)^{-1} \cdot P^{4\nu_l + 1}$$

$$\geqslant \tfrac{1}{4} C_l^{\frac{3}{4}} (C_l^{-\frac{1}{4}} + (2.3) \times 10^5)^{-1} \cdot ([\, Z^{\frac{1}{4}} \,])^{4\nu_l + 1}$$

Choosing Z such that $16Z + Y^{\frac{3+\mu}{4}} = Y$, we have

$$\sum_{\substack{m \leq Y \\ r(m) \neq 0}} 1 \geq \tfrac{1}{64} C_l^{\frac{3}{4}} (C_l^{-\frac{1}{4}} + (2.3) \times 10^6)^{-1} Y^{\nu_{l+1}}$$

and hence the result.

Lemma 32 : *Let f be a fixed integer from the set $\{0, 1, 2, 3, 4\}$. Suppose $n \geq 2.10^{44}$. Then the number of non negative $m \equiv f \pmod{16}$ such that $m \leq n$ and such that m is the sum of four fourth powers is greater than*

$$(1.1950)\, 10^{-18} \cdot n^{0.745633624425}$$

Proof : This is lemma 7.4 of Thomas [8] (page 127)

Lemma 33 : *Let f_1 be a fixed integer from the set $\{0, 1, 2, 3, 4, 5\}$. Suppose $n \geq 10^{60}$. Then the number of non negative $m \equiv f \pmod{16}$ such that $m \leq n$ and such that m is the sum of five fourth powers is greater than*

$$(0.8) \times 10^{-21} \cdot n^{0.8168}.$$

Proof : The result follows from a direct application of lemmas 31 and 32. In the notation of lemma 31, we have

$$C_l = (1.195) \times 10^{-18}$$

Hence

$$C_{l+1} = \tfrac{1}{64} \cdot (1.195 \times 10^{-18})^{\frac{3}{4}} \quad ((1.195 \times 10^{-18})^{-\frac{1}{4}} + (2.3) \times 10^6)^{-1}$$

$$\geq 2.17 \times 10^{-21}.$$

$$\nu_l = 0.745633$$

Hence $\nu_{l+1} = 0.25 + \dfrac{3 \times 0.745633}{3.945633} \geq 0.8168$

and hence the result.

§ 7 : An asymptotic formula

Let N be an integer $> 10^{560}$ and $P = [N^{\frac{1}{4}}]$;

Let N_0 and N_1 be integers satisfying

$$\frac{N}{2} \leqslant N_0 \leqslant N; \quad N_0 - P^{4\frac{1}{5}} \leqslant N_1 \leqslant N_0$$

Let f be an integer in the set (0, 1, 2, 3, 4, 5) such that

$$N - 2f \equiv 2, 3, 4, 5, 6, 7 \text{ or } 8 \pmod{16}.$$

Let μ_1, μ_2 go independently over the same sequence of numbers, which is less than $\frac{N}{4}$ and can be represented as a sum of five biquadratics.

Let U denote the number of numbers μ.

Lemma 34 : *Every integer $N > 10^{560}$ can be represented as a sum of twenty one biquadratics*

Proof : Let $I(N) =$

$$\int_0^1 (T(\alpha))^{11} \sum_{\mu_1} \sum_{\mu_2} e^{2\pi i(\mu_1 + \mu_2)\alpha} \; e^{-2\pi i N\alpha} \, d\alpha$$

It is sufficient to prove that $I(N) > 0$

Now $I(N) = \int_{\Omega} + \int_{\mathfrak{m}}$

where Ω is the major arc and $\mathfrak{m}$ is the minor arc.

By lemma 30

$$\int_{\Omega} (T(\alpha))^{11} \sum_{\mu_1} \sum_{\mu_2} e^{2\pi i(\mu_1+\mu_2)\alpha} \; e^{-2\pi i N\alpha} \, d\alpha$$

$$= \sum_{\mu_1} \sum_{\mu_2} W_0 (N - \mu_1 - \mu_2)$$

$$\geqslant U^2 \cdot (0.02) \cdot N^{\frac{7}{4}}$$

$$\geqslant U^3 \cdot (0.02) \cdot P^7$$

By lemma 13, we have

$$\int_{\mathfrak{M}} (T(\alpha))^{11} \cdot \sum_{\mu_1} \sum_{\mu_2} e^{2\pi i(\mu_1+\mu_2)\alpha} \; e^{-2\pi i N\alpha} \; d\alpha$$

$$\leqslant \max_{\alpha \in \mathfrak{M}} |T(\alpha)|^{11} \int_{\mathfrak{M}} \left| \sum_{\mu} e^{2\pi i \mu \alpha} \right|^2 d\alpha$$

$$\leqslant \max_{\alpha \in \mathfrak{M}} |T(\alpha)|^{11} \cdot U$$

$$\leqslant (3 P^{\frac{29}{8\cdot 2}} (\log P+3)^{\frac{15}{8\cdot 2}})^{11} \cdot U$$

We have only to check that

$$U^2 (0.02) P^7 > 3^{11} \cdot P^{10-\frac{1}{8\cdot 2}} (\log P+3)^{5+\frac{5}{8\cdot 2}} U$$

Now using the value

$$U \geqslant \tfrac{1}{4} \; P^{3.2672} \times (0.8) \times 10^{-21}$$

we have only to check that

$$P^{0.2984} \geqslant 10^{28.65} (\log P+3)^{5\cdot\frac{5}{8\cdot 2}}$$

Taking $P \geqslant 10^{140}$, we see that the inequality is satisfied and this proves the result.

§ 8 : The ascent

Lemma 35 : *Let l be an integer $\geqslant 0$;*

$$\textit{Let} \quad \nu = \frac{1 - \frac{l}{L_0}}{n} \; ;$$

$$L_0 > l; \; (\nu L_0)^{\frac{n}{n-1}} \geqslant L_0.$$

Compute L_t *by*

$$\log L_t = \left(\frac{n}{n-1}\right)^t (\log L_0 + n \log \nu) - n \log \nu$$

If all integers between l *and* L_0 *inclusive are sums of* k *integral* n^{th} *powers, then all integers between* l *and* L_t, *inclusive are sums of* $(k + t)$ *integral* n^{th} *powers* $\geqslant 0$.

Proof : This is Theorem 12 (page 711) of Dickson [5].

Lemma 36 : *Every natural number in the range* [13793, 10^{143}] *is a sum of sixteen biquadratics.*

Proof : This is Theorem 3·4 of Thomas [9].

Lemma 37 : *Every natural number less than* 10^{560} *is a sum of twenty one biquadratics.*

Proof : In the notation of lemma 36, we take $l = 13793$; $L_0 = 10^{143}$; $t = 5$; $n = 4$ and this proves the result.

§ 9 : Completion of the proof

The proof of the main theorem follows from lemmas 34 and 37.

§ 10 : Acknowledgement

It is a great pleasure for the author to express his gratitude to Professor K. Ramachandra for his constant encouragement and the interest shown by him at various stages of this work, but for which the work could not have been completed Part of this work was done at the Institute Mittag-Leffler and the author thanks for the good working condition in the Institute; special thanks are due to Professor L. Carleson and Professor A. Schinzel for their encouragement. The author wishes to thank the international mathematical union and The Royal Swedish Academy for making the visit to Sweden possible.

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