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ON WARING'S PROBLEM : $g(4) \leq 21$

By R. BALASUBRAMANIAN

(Dedicated to the Memory of Dr. S. S. PILLAI)

§ 1. Developing the ideas of Chen-jing-run [2] we prove the following

Theorem :

Every natural number is expressible as a sum of at most twenty one fourth powers.

In the usual notation, our result reads : $19 < g(4) \leq 21$. This is an improvement of the result of Thomas [9] who proved that $g(4) \leq 22$. We recall that the general problem for the k^{th} powers in the place of fourth powers is nearly complete, which is due to Dickson [5] and Pillai (independent of each other); (see "S. S. Pillai" by K. Chandrasekharan : Jour. of Indian Math. Soc. 15 (1951) (1-10) for the list of complete works of Dr. S. S. Pillai). On the other hand, in the case of $G(4)$, it has been proved by Davenport [4] that $G(4) = 16$. Incidentally, we

remark that Auluck [1] proved that every integer $> 10^{10^{89-39}}$ is a sum of nineteen fourth powers. It was improved by Thomas [8] who proved that every integer $\geq 10^{1408.3}$ is representable as a sum of 19 fourth powers (Theorem 12.1 ; pp. 152 of [8]). Our method improves the bound ; but since it does not prove any thing substantial, we are not including the proof of this fact.

§ 2. Notations :

Let $e(x)$ stand for $e^{2\pi ix}$ and

$$eq(x) \text{ for } e^{\frac{2\pi ix}{q}}$$

Define $S_{a,q} = \sum_{x=1}^q e^{\frac{2\pi i ax^4}{q}}$ where the accent here (and

elsewhere) shows that the summation is restricted to only those x for which $(x, q) = 1$.

Let $N (\geq 10^{560})$ be a given integer to be represented as a sum of 21 biquadratics. Define two integers N_0 and N_1 by

$$N_0 \leq N \leq 2N_0 \text{ and } N_0 - P^{3\frac{1}{2}} \leq N_1 \leq N_0$$

Define $P = [N^{\frac{1}{4}}]$ and $T(a) = \sum_{1 \leq x \leq P} e(ax^4)$

For any real number a , $0 < a \leq 1$, there exist two integers h, q with $1 \leq h < q \leq 8P^3$, $(h, q) = 1$ such that

$$\left| a - \frac{h}{q} \right| < \frac{1}{8qP^3}$$

The major arc $\Omega = \left\{ a \mid \left| a - \frac{h}{q} \right| < \frac{1}{8qP^3} \text{ for some } q < P^{\frac{1}{2}} \right\}$

and the minor arc $\mathfrak{m} = \left\{ a \mid \left| a - \frac{h}{q} \right| < \frac{1}{8qP^3} \text{ for some } q, \right. \\ \left. P^{\frac{1}{2}} < q \leq 8P^3 \right\}$

The singular series $S(n; m)$ is defined by

$$S(n) = S(n, m) = \sum_{q=1}^{\infty} \sum_{a=1}^q \left(\frac{S_{a,q}}{q} \right)^m e^{\frac{-2\pi i an}{q}}$$

The truncated singular series $S_1(n, m)$ is defined by

$$S_1(n) = S_1(n, m) = \sum_{q \leq P^{\frac{1}{2}}} \sum_{a=1}^q \left(\frac{S_{a,q}}{q} \right)^m e^{\frac{-2\pi i a n}{q}}$$

θ is a constant (depending on various parameters), such that $|\theta| < 1$.

$$\psi = \psi(a) = \int_0^P e^{2\pi i a x^4} dx$$

$$W(N') = \int_0^1 T_a^m e^{-2\pi i a N'} da$$

$$W_0(N) = \int_{\Omega} T_a^m e^{-2\pi i a N'} da$$

$$R(N_0) = \int_{-\infty}^{\infty} \psi^m e^{-2\pi i a N_0} da$$

§ 3. An upper bound for $S_{a,q}$

$$\text{Let us recall that } S_{a,q} = \sum_{\substack{x=1 \\ (x,q)=1}}^q e_q(ax^4).$$

We then have,

Lemma 1: $S_{a,q}$ is a multiplicative function of q

Proof: A proof can be found in Davenport [3] (lemma 6 in page 31).

Because of lemma 1, it is sufficient to have the bound for $S_{a,p}$. In this direction, we have

- Lemma 2:**
- (a) For any prime $p \neq 2$, $|S_{a,p}| < (\delta - 1)p^{\frac{1}{2}}$
where $\delta = (4, p - 1)$
 - (b) For any prime $p \neq 2$, $|S_{a,p^\nu}| = p^{\nu-1}$
If $2 \leq \nu < 4$

(c) For any prime $p \geq 2$,

$$|S_{a,p}^{\nu}| = p^3 |S_{a,p}^{\nu-4}|$$

if $\nu > 4$.

Proof: A proof can be found in Davenport [3] (lemma 12 in page 42, lemma 13 in page 43, and lemma 14 in page 44).

One can now get a bound for $|S_{a,q}|$ from lemmas 1 and 2. But, for small primes (namely $p \leq 79$), we find the bound for $S_{a,p}$ by actual computation (and the bound is better than the one given by lemma 2) and deduce

Lemma 3: We have $|S_{a,q}| \leq (4.3)q^{\frac{3}{2}}$ if $(a,q) = 1$.

Proof: A proof of the lemma can be found in Thomas [8] (Theorem 2.1, Page 38).

§ 4: A bound for $T(\alpha)$ in the minor arc:

Let us recall that $T(\alpha)$ is defined by

$$T(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^4)$$

Minor arc is defined by

$$\mathfrak{m} = \left\{ \alpha \mid \left| \alpha - \frac{h}{p} \right| \leq \frac{1}{8qP^3} \text{ for some } q, P^{\frac{1}{2}} < q \leq 8P^3 \right\}$$

Lemma 4: Let $d(m)$ be the number of divisors of m .

$$\text{Then } \sum_{1 \leq i \leq n} (d(i))^j \leq A_j n^{(j + \log n)^{j-1}}$$

where A_j depends only on j . In particular one can take

$$A_1 = 1; \quad A_2 = \frac{1}{3}; \quad A_3 = \frac{1}{2^{\frac{1}{2}}}; \quad A_4 = \frac{1}{24 \times 192}.$$

Proof: The values of A_j for every integer j is given by Mardjanichvili [7]. The values of A_1 , A_2 , A_3 and A_4 are discussed in Chen [2] (lemma 8)

Lemma 5: Let $h_1(n)$ denote the number of solutions of the equation $n = l_1 l_2$, $1 \leq l_1 \leq P$; $1 \leq l_2 \leq P$ in integers.

Then,
$$\sum_{n \leq P^2} (h_1(n))^2 \leq 2P^2 (\log P + 3)$$

Proof: We have

$$\begin{aligned} \sum_{n \leq P^2} (h_1(n))^2 &= \sum_{n \leq P^2} \left(\sum_{\substack{d|n \\ d \leq P; \frac{n}{d} \leq P}} 1 \right)^2 \\ &= \sum_{n \leq P^2} \sum_{\substack{d_1 | n \\ d_1 \leq P; \frac{n}{d_1} \leq P}} \sum_{\substack{d_2 | n \\ d_2 \leq P; \frac{n}{d_2} \leq P}} 1 \\ &\leq 2 \sum_{d_1 \leq P} \sum_{\substack{d_2 \leq d_1 \\ n \equiv 0 \pmod{[d_1, d_2]}}} \sum_{n \leq P d_2} 1 \\ &\leq 2 \sum_{d_1 \leq P} \sum_{d_2 \leq d_1} \frac{P d_2}{[d_1, d_2]} \\ &\leq 2 \sum_{d_1 \leq P} \sum_{l|d_1} \sum_{\substack{d_2 \leq \frac{d_1}{l} \\ d_1 d_2}} \frac{P d_2 l}{d_1 d_2} \\ &\leq 2P^2 (\log P + 3). \end{aligned}$$

Lemma 6: Let $h_2(n)$ denote the number of solutions of the equation $n = l_1 l_2$, $1 \leq l_1 \leq P$; $1 \leq l_2 \leq P^2/4$ in integers.

Then

$$\sum_{n \leq P^2/4} (h_2(n))^4 \leq \frac{P^3}{12} (\log P + 3)^{11}$$

Proof: We have

$$\sum_{n \leq P^2/4} (h_2(n))^4 = \sum_{n \leq P^2/4} \left(\sum_{\substack{l|n \\ l \leq P; \frac{n}{l} \leq P^2/4}} 1 \right)^4$$

$$\leq 24 \sum_{l_1 \leq P} \sum_{l_2 \leq l_1} \sum_{l_3 \leq l_2} \sum_{l_4 \leq l_3} \frac{P^3 l_4}{4 [l_1, l_2, l_3, l_4]}$$

$$\leq 6P^3 \sum_{1 \leq l_1 \leq P} \sum_{d_1 | l_1} \sum_{l_2 \leq \frac{l_1}{d_1}} \sum_{d_2 | l_1 l_2} \sum_{l_3 \leq \frac{d_1 l_2}{d_2}}$$

$$d_3 | l_1 l_2 l_3 \quad d_4 \leq \frac{d_2 l_3}{d_3} \quad \frac{d_3 l_4}{l_1 l_2 l_3 l_4}$$

$$\leq 6P^3 \sum_{1 \leq l_1 \leq P} \sum_{d_1 | l_1} \sum_{l_2 \leq \frac{l_1}{d_1}} \sum_{d_2 | l_1 l_2}$$

$$l_3 \leq \frac{d_1 l_2}{d_2} \quad \frac{d(l_1) d(l_2) d(l_3) d_2}{l_1 l_2}$$

$$\leq 6P^3 \sum_{1 \leq l_1 \leq P} \sum_{d_1 | l_1} \sum_{l_2 \leq \frac{l_1}{d_1}}$$

$$d_2 | l_1 l_2 \quad \frac{d(l_1) d(l_2) \cdot (\log P + 1)}{l_1}$$

$$\leq 6P^3 (\log P + 1) \sum_{1 \leq l_1 \leq P} \sum_{d_1 | l_1}$$

$$l_2 \leq \frac{l_1}{d_1} \quad \frac{d^2(l_1) d^2(l_2) d_1}{l_1}$$

$$\leq 2P^3 (\log P + 1) \sum_{1 \leq l_1 \leq P} \sum_{d_1 | l_1} d^3(l_1) \cdot (\log P + 2)^3$$

$$\leq 2P^3 (\log P + 2)^4 \sum_{1 \leq l_1 \leq P} d^3(l_1)$$

$$\leq \frac{P^3}{12} (\log P + 3)^{11}$$

Lemma 7: Let $\alpha = \frac{a}{q} + \frac{\theta}{8P^3 q}$. Then, if $P > 10^{50}$,

we have

$$\sum_{0 < n < P^3/4} \min \left(P, \frac{1}{2 \| 24 \alpha n \|} \right)^{4/3} < 50P^{1/3} \text{ provided } 8P^3 \geq q > P;$$

Proof: Let $q' = \frac{q}{(24, q)}$. The sum over n can be broken into not more than $\left(\frac{P^3}{4q'} + 1 \right)$ parts, in each of which n runs over at most q' consecutive integers. Let us consider the typical sum say $\sum_{B < n \leq B + q'} \min \left(P, \frac{1}{2 \| 24 \alpha n \|} \right)^{4/3}$.

Then it is easily seen that, there are at most $\left(\frac{q'}{P} + 4 \right)$ values for n for which $\frac{1}{2 \| 24 \alpha n \|} \geq P$ and these values of n contribute at most $\left(\frac{q'}{P} + 4 \right) P^{4/3} = q' P^{1/3} + 4 P^{4/3}$ to the sum. The remaining values of n contribute

$$m \geq \frac{q'}{P} \frac{1}{2} \left(\frac{q'}{m} \right)^{4/3} < 4 q' P^{1/3}.$$

Hence

$$\sum_{B < n \leq B + q'} \min \left(P, \frac{1}{2 \| 24 \alpha n \|} \right)^{4/3} < 5 q' P^{1/3} + 4 P^{4/3}$$

Hence the total sum is at most

$$\left(5 q' P^{1/3} + 4 P^{4/3} \right) \left(\frac{P^3}{4q'} + 1 \right) < 50 P^{1/3}$$

Lemma 8: If $h_2(n)$ denotes the number of solutions of the equation $n = l_1 l_2$, $1 \leq l_1 \leq P$; $1 \leq l_2 \leq P^2/4$, in integers, then

$$\sum_{n \leq P^3/4} h_2(n) \min \left(P, \frac{1}{2 \parallel 24 an \parallel} \right) \leq 11 P^{\frac{13}{2}} (\log P + 3)^{\frac{1}{2}}.$$

where $a = \frac{a}{q} + \frac{\theta}{8P^2 q}$; $P \geq 10^{50}$ and $P \leq q \leq 8P^2$

Proof: If $P \leq q \leq 8P^2$, then

$$\begin{aligned} & \sum_{n \leq P^3/4} h_2(n) \min \left(P, \frac{1}{2 \parallel 24 an \parallel} \right) \\ & \leq \left(\sum_{n \leq P^2} (h_2(n))^4 \right)^{\frac{1}{4}} \left(\sum_{n \leq \frac{P^2}{4}} \left(\min \left(P, \frac{1}{2 \parallel 24 an \parallel} \right) \right)^4 \right)^{\frac{3}{4}} \end{aligned}$$

and the result follows from lemmas 6 and 7.

Lemma 9: Let $a = \frac{a}{q} + z$; $(a, q) = 1$; $P \geq 10^{100}$, $P \leq q \leq 8P^2$

$|z| \leq \frac{1}{8q P^3}$. Then

$$T(a) = \left| \sum_{x=1}^P e^{2\pi i a x^4} \right| \leq (2.86) P^{\frac{3}{2}} (\log P + 3)^{\frac{1}{2}}$$

Proof: Let $l_1 = x - y$; $f(x, y) = 4x^3y + 6x^2y^2 + 4xy^3$

$$h(x; y; z) = 12xyz(x+y+z).$$

$h_1(n)$ and $h_2(n)$ are as defined in lemmas 5 and 6.

Then

$$|T(a)|^2 \leq P + 2 \left| \sum_{x=1}^P \sum_{\substack{y=1 \\ y \neq x}}^P e^{2\pi i a (x^4 - y^4)} \right|$$

$$\leq P+2 \sum_{l=1}^P \left| \sum_{x=1}^{P-l} e^{2\pi i a f(x,l)} \right|$$

$$= P+2S_1, \text{ say}$$

$$|S_1|^2 \leq \left(\sum_{l=1}^P 1 \right) \left(\sum_{l=1}^P \left| \sum_{x=1}^{P-l} e^{2\pi i a f(x,l)} \right|^2 \right)$$

$$\leq P^3 + 2P \sum_{l_1=1}^P \sum_{l_2=1}^{P-l_1} \left| \sum_{x=1}^{P-l_1-l_2} e^{2\pi i a h(x; l_1, l_2)} \right|$$

$$= P^3 + 2PS \text{ say}$$

Now

$$|S| \leq \sum_{1 \leq n \leq \frac{P^2}{4}} h_1(n) \max_{l_1, l_2 = n} \left| \sum_{x=1}^{P-l_1-l_2} e^{2\pi i a . h(x; l_1, l_2)} \right|$$

$$\text{Hence } |S|^2 \leq \left(\sum_{1 \leq n \leq P^2/4} h_1(n) \right)^2$$

$$\left(\sum_{1 \leq n \leq P^2/4} \max_{l_1, l_2 = n} \left| \sum_{x=1}^{P-l_1-l_2} e^{2\pi i a h(x; l_1, l_2)} \right|^2 \right)$$

Now using lemma 5,

$$|S|^2 \leq 2P^3 (\log P + 3) \sum_{1 \leq n \leq P^2/4} \max_{l_1, l_2 = n} \left(P + 2 \sum_{l_3=1}^P \min \left(P, \frac{1}{2 \| 24 a l_1 l_2 l_3 \|} \right) \right)$$

$$\leq 2P^3 (\log P + 3) \sum_{1 \leq i \leq P^2/4}$$

$$\left(P + 2 \sum_{l=1}^P \min \left(P, \frac{1}{2 \| 24 a li \|} \right) \right)$$

$$\leq 2P^{\alpha} (\log P + 3) \times$$

$$\left(\frac{P^{\alpha}}{4} + 2 \sum_{0 < n \leq P^{\alpha}/4} h_2(n) \min \left(P, \frac{1}{2 \| 24 \alpha n \|} \right) \right)$$

and using lemma 8,

$$|S|^{\alpha} \leq 2P^{\alpha} (\log P + 3) (P^{\alpha}/4 + 22P^{\frac{13}{4}} (\log P + 3)^{\frac{11}{4}})$$

and this gives

$$|T(\alpha)| \leq (2.86) P^{\frac{39}{4}} (\log P + 3)^{\frac{15}{4}}$$

Lemma 10: If $\alpha = \frac{a}{q} + \beta$, where $q < \sqrt{P}$, $|\beta| \leq \frac{1}{8q P^3}$, and $P \geq 6$, then

$$|T(\alpha)| \leq q^{-\frac{1}{4}} (1 + \log q) \min(16 P^{\frac{3}{4}} |\beta|^{-1} P^{-3})$$

Proof: This is lemma 9.5 (page 139) of Thomas [8]. A similar result is proved in lemma 9 of Davenport [4]. Even though the result in [8] is proved under the condition

$$|\beta| \leq \frac{1}{64q P^3} \text{ it holds, in fact, for } |\beta| \leq \frac{1}{8q P^3}.$$

Lemma 11: If $\alpha = \frac{a}{q} + \beta$, $P^{\frac{1}{2}} < q \leq P$,

$$|\beta| \leq \frac{1}{8q P^3}, \text{ and } P \geq 10^{100}, \text{ then}$$

$$|T(\alpha)| \leq (2.86) P^{\frac{39}{4}} (\log P + 3)^{\frac{15}{4}}$$

Proof: This follows from lemma 10.

Lemma 12: *In the minor arc, the following estimate holds:*

$$|T(\alpha)| \leq 2.86 P^{\frac{39}{4}} (\log P + 3)^{\frac{15}{4}}$$

Proof: The result follows from lemmas 9 and 11.

§ 5. A lower bound for $R(N_0)$

Let us recall that

$$\psi = \psi(\alpha) = \int_0^P e^{2\pi i \alpha x^4} dx$$

$$R(N_0) = \int_{-\infty}^{\infty} \psi^m e^{-2\pi i \alpha N_0} d\alpha$$

$$\text{Define } B = B(\alpha) = \begin{cases} P & \text{if } |\alpha| \leq P^{-4} \\ \sqrt{2} |\alpha|^{-\frac{1}{4}} & \text{if } |\alpha| > P^{-4} \end{cases}$$

Then it is easily seen that $|\psi| \leq B$.

$$W(N') = \int_0^1 T_{\alpha}^m e^{-2\pi i \alpha N'} d\alpha$$

Lemma 13: (*van der Corput*): *Suppose $f(x)$ is a real function which is twice differentiable for $A \leq x \leq B$. Suppose further that, in this interval $0 < f'(x) < \frac{1}{2}$ and $f''(x) \geq 0$.*

$$\text{Then } \sum_{A < n \leq B} e(f(n)) = \int_A^B e(f(x)) dx + o(1)$$

Proof: This is lemma 16 (page 65) of Davenport [3]. For the O -constants see lemma 13 (page 34) of Vinogradov [10].

Lemma 14: If $N_0 - P^{3\frac{1}{2}} \leq N' \leq N_0$, then

$$W(N') - R(N_0) + \int_{\frac{1}{8P^3}}^{1 - \frac{1}{8P^3}} T_a^m e^{-2\pi i a N'} da$$

$$+ 2\theta \cdot 10^3 \cdot P^{m-5+3/4},$$

where $m = 9$ or 10 and $P > 10^{10}$

Proof: In $-\frac{1}{8P^3} < a < \frac{1}{8P^3}$ we have, by lemma 13,

$$T(a) = \sum_{1 \leq x < P} e^{2\pi i a x^4} = \int_0^P e^{2\pi i a x^4} dx + \theta$$

$$= \psi + \theta$$

hence $|(T(a))^m - \psi^m| < (9m)(B+5)^{m-1}$

Consequently we have

$$|(T(a))^m e^{-2\pi i a N'} - \psi^m e^{-2\pi i a N_0}|$$

$$= |T_a^m e^{-2\pi i a N'} - \psi^m e^{-2\pi i a N'} + \psi^m e^{-2\pi i a N'} - \psi^m e^{-2\pi i a N_0}|$$

$$< 9m(B+5)^{m-1} + B^m(2\pi a)P^{3\frac{1}{2}}$$

Hence

$$\int_{\frac{1}{8P^3}}^{1 - \frac{1}{8P^3}} |T_a^m e^{-2\pi i a N'} - \psi^m e^{-2\pi i a N_0}| da$$

$$\begin{aligned} &\leq 2 \int_0^1 ((9m)(B+3)^{m-1} + B^m (2\pi a) \cdot P^{3\frac{3}{4}}) da \\ &\leq 2 \int_0^1 P^{-4} (9m(P+5)^{m-1} + (2\pi a) \cdot P^{m+3\frac{3}{4}}) da \\ &+ 2 \int_{P^{-4}}^1 (9m(\sqrt{2}a^{-\frac{1}{4}} + 5)^{m-1} \\ &\quad + 2\pi P^{3\frac{3}{4}} (\sqrt{2}a^{-\frac{1}{4}})^m \cdot a) da \\ &\leq 10^3 P^{m-5+3/4} \end{aligned}$$

Now, we have

$$\begin{aligned} W(N') = &\int \frac{1}{8P^3} (T(a))^m \cdot e^{-2\pi iaN'} da \\ &- \frac{1}{8P^3} \\ &+ \int \frac{1 - \frac{1}{8P^3}}{8P^3} (T(a))^m e^{-2\pi iaN'} da \\ &- \frac{1}{8P^3} \end{aligned}$$

Now, replace the integrand

$(T(a))^m e^{-2\pi iaN'}$ of the first integral on the right by $\psi^m e^{-2\pi iaN_0}$ and we have just proved that the error is at most $10^3 P^{m-5+3/4}$. Hence we have

$$\begin{aligned} W(N') = &\int \frac{1}{8P^3} \psi^m e^{-2\pi iaN_0} da + \\ &- \frac{1}{8P^3} \end{aligned}$$

$$\begin{aligned}
 & 1 - \frac{1}{8P^3} \\
 & + \int \frac{1}{8P^3} (T(\alpha))^m e^{-2\pi i \alpha N'} d\alpha \\
 & + 10^3 \theta. P^{m-5+\frac{3}{4}}
 \end{aligned}$$

Now we replace the first integral on the right by $R(N_0)$.

The error involved is at most,

$$\begin{aligned}
 & \frac{2 \int_1^\infty |\psi|^m d\alpha}{8P^3} \\
 & \leq \frac{2 \int_1^\infty (\sqrt{2} \alpha^{-\frac{1}{4}})^m d\alpha}{8P^3} \\
 & \leq 10^3 P^{\frac{3m}{4} - 3} \\
 & \leq 10^3 P^{m-5+\frac{3}{4}}
 \end{aligned}$$

and this proves the lemma.

We now take $M = \left[\frac{P^{3\frac{3}{4}}}{2} \right]$ and apply lemma 14 with $N' = N - N'' - N'''$, $0 \leq N''$, $N''' \leq M$ and add the M^2 equations. This gives

Lemma 15: We have, with $M = \left[\frac{P^{3\frac{3}{4}}}{2} \right]$,

$$\sum_{N''} \sum_{N'''} W(N - N'' - N''') = M^2 R(N_0) +$$

$$\begin{aligned}
 & + \int_{\frac{1}{8P^3}}^{1 - \frac{1}{8P^3}} (T(\alpha))^m e^{-2\pi i \alpha N} \left(\sum_{N''} e^{-2\pi i \alpha N''} \right)^n d\alpha \\
 & + 2\theta \cdot 10^3 \cdot M^* P^{m-5+\frac{3}{4}}
 \end{aligned}$$

Lemma 16 : *The integral on the right of lemma 15 is at most $8 P^{m+3}$.*

Proof : Since $|T(\alpha)| \leq P$, $|e^{-2\pi i \alpha N}| \leq 1$,

$$\left| \sum_{N''} e^{-2\pi i \alpha N''} \right| \leq \frac{1}{\| \alpha \|},$$

the result is clear.

Lemma 17 : *If $K_r(N)$ denotes the number of integer solutions of the equation $x_1^r + x_2^r + \dots + x_r^r \leq N$, then*

$$K_r(N) = T_r \cdot N^{\frac{r}{n}} - \theta \cdot r \cdot N^{\frac{r-1}{n}},$$

where $T_r = \frac{(\Gamma(\frac{r}{4}))^r}{\Gamma(1 + \frac{r}{4})}$ and $0 \leq \theta \leq 1$

Proof : This is lemma 3 in (page 22) in Vinogradov [10].

Lemma 18 : *If $9 \leq m \leq 11$,*

$$\sum_{N''=1}^M \sum_{N'''=1}^M W(N - N'' - N''')$$

$$\geq 0.102 \cdot M^* \cdot N^{\frac{m}{4} - 1}$$

Proof : We have

$$\sum_{N''=1}^M W(N - N'' - N''')$$

$$= \sum_{N''=1}^M (K_m(N - N'' - N''') - K_m(N - N'' - N''' - 1))$$

$$\begin{aligned}
 &= K_m (N - N'') - K_m (N - N'' - M) \\
 &= T_m \left((N - N'')^{\frac{m}{4}} - (N - N'' - M)^{\frac{m}{4}} \right) \\
 &\quad - 2\theta m \cdot N^{\frac{m-1}{4}} \\
 &> T_m \frac{m}{8} MN^{\frac{m}{4}-1} - 2\theta \cdot m \cdot N^{\frac{m-1}{4}} \\
 &\geq T_m \frac{m}{10} M N^{\frac{m}{4}-1} \\
 &\geq 0.102 \cdot M \cdot N^{\frac{m}{4}-1}
 \end{aligned}$$

and hence the result.

Lemma 19: *The following inequality holds*

$$|R(N_0)| > \frac{1}{16} N^{\frac{m}{4}-1}$$

Proof: The inequality follows from lemmas 15 and 18.

§ 6: A lower bound for the singular series

Let us recall that the singular series is defined by

$$S(n) = S(n; m) = \sum_{q=1}^{\alpha} \sum_{a=1}^q \left(\frac{S_{a,q}}{q} \right)^m \cdot e^{\frac{-2\pi i a n}{q}}$$

and the truncated singular series

$$S_1(n) \pm S_1(n; m) = \sum_{q \leq P^{\frac{1}{2}}} \sum_{a=1}^q \left(\frac{S_{a,q}}{q} \right)^m \cdot e^{\frac{-2\pi i a n}{q}}$$

Here (and elsewhere), the accent shows that the summation is restricted to only those a 's, for which $(a, q) = 1$.

$$\text{Define } A_m(n; q) = \sum_{\substack{a=1 \\ (a,q)=1}}^q \left(\frac{S_{a,q}}{q} \right)^m e^{\frac{-2\pi i a n}{q}}$$

Let us define

$$\chi_p(n, m) = \sum_{i=0}^{\infty} A_m(n; p^i)$$

Lemma 20: We have $S(n; 11) \geq 0.5304 \chi_p(n, 11)$

Proof: This is Theorem 4.1 of Thomas [8] (page 98)

Lemma 21: Suppose $1 < n \leq m \leq 15$. Then

$$\chi_p(n; m) = 16 m c_n 2^{-m},$$

$$\text{where } m c_n = \frac{m!}{n! (m-n)!}$$

Proof: This is Theorem 4.2 of Thomas [8] (page 98)

Lemma 22: If $n \equiv 2, 3, 4, 5, 6, 7$ or $8 \pmod{16}$, then

$$\chi_p(n, 11) \geq 0.42$$

Proof: If $n_1 \equiv n_2 \pmod{16}$, then

$\chi_p(n_1, m) = \chi_p(n_2, m)$. Hence it follows from lemma 21, that $\chi_p(n, 11) \geq 16 \cdot 11 c_2 \cdot 2^{-11} \geq 0.42$

Lemma 23: The singular series $|S(n)| \geq 0.222$ if $m = 11$ provided $n \equiv 2, 3, 4, 5, 6, 7$ or $8 \pmod{16}$

Proof. The result follows from lemmas 20 and 22

Lemma 24 : *The truncated singular series*

$$|S_1(n)| \geq 0.22 \quad \text{if } m = 11$$

provided $n \equiv 2, 3, 4, 5, 6, 7$ or $8 \pmod{16}$

Proof : We have

$$\begin{aligned} & |S(n) - S_1(n)| \\ &= \left| \sum_{q > p^{\frac{1}{2}}} \left(\sum_{\substack{a=1 \\ a \neq 1}}^q \left(\frac{S_{a,q}}{q} \right)^{11} e^{-2\pi i \frac{an}{q}} \right) \right| \\ &< \sum_{q \geq p^{\frac{1}{2}}} \left(\sum_{a=1}^q \left| \frac{S_{a,q}}{q} \right|^{11} \right) \\ &< \sum_{q > p^{\frac{1}{2}}} 5^{11} \cdot q^{-\frac{7}{2}} \\ &< 5^{11} \times 2 \times P^{\frac{3}{8}} \\ &< 0.01, \text{ since } P > 10^{39} \end{aligned}$$

and hence the result.

§ 6. The estimate on the major arc

Let us recall that the major arc is defined by

$$\Omega = \left\{ \alpha \mid \left| \alpha - \frac{a}{q} \right| \leq \frac{1}{8qP^3}; 1 \leq a \leq q; (a, q) = 1; q \leq P^{\frac{1}{2}} \right\}$$

and the integral

$$W_0(N_1) = \int_{\Omega} (T(\alpha))^m \cdot e^{-2\pi i N_1 \alpha} d\alpha$$

Lemma 25 : *The following approximation holds :*

If $\alpha = \frac{a}{q} + z$, and $|N_1 - N_0| \leq P^{3\frac{1}{2}}$, then

$$(T(\alpha))^m e^{-2\pi i N_1 \alpha} = \psi^m \left(\frac{S_{a,q}}{q} \right)^m e^{\frac{-2\pi i a N_1}{q} - 2\pi i z N_0} + O \left(5^{m+1} q^{\frac{5-m}{4}} B^{m-1} + 5^m \cdot q^{\frac{5-m}{4}} B^m (2\pi z) P^{3\frac{1}{2}} \right)$$

Proof : We have

$$T(\alpha) = \sum_{y=0}^{q-1} \sum_{-yq^{-1} \leq t \leq (P-y)q^{-1}} e^{2\pi i \left(\frac{ay^4}{q} + z(qt+y)^4 \right)} = \sum_{y=0}^{q-1} e^{\frac{2\pi i ay^4}{q}} D_y(z), \text{ say}$$

since $\frac{d}{dt} (z(qt+y)^4) \leq \frac{1}{2}$, we have, by lemma 13,

$$D_y(z) = \int_{-yq^{-1}}^{(P-y)q^{-1}} \frac{1}{yq^{-1}} e^{2\pi i z (qt+y)^4} dt + 4\theta = \frac{1}{q} \int_0^P e^{2\pi i z x^4} dx + 4\theta$$

Hence $T(\alpha) = \psi \frac{S_{a,q}}{q} + 4\theta q$

Since $|z| \leq \frac{1}{8q P^3}$, $q < P^{\frac{1}{2}}$ we have $zq^{-\frac{1}{2}} \geq 8q$

$$\begin{aligned} \text{Hence } & \left| (T(\mathbf{a}))^m - \left(\psi \frac{S_{\mathbf{a},q}}{q} \right)^m \right| \\ & \leq 4m \cdot q \cdot 4q^{-\frac{1}{4}} B)^{m-1} \\ & \leq 5^{m+1} \cdot q^{\frac{5-m}{4}} \cdot B^{m-1} \end{aligned}$$

$$\begin{aligned} \text{Hence } & \left| (T(\mathbf{a}))^m e^{-2\pi i N_1 \mathbf{a}} - \left(\psi \frac{S_{\mathbf{a},q}}{q} \right)^m \right. \\ & \left. e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_0} \right| \end{aligned}$$

$$\leq \left| T_{\mathbf{a}}^m e^{-2\pi i N_1 \mathbf{a}} - \left(\psi \frac{S_{\mathbf{a},q}}{q} \right)^m \right. \\ \left. e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_1} \right|$$

$$\begin{aligned} & + \left| \left(\psi \frac{S_{\mathbf{a},q}}{q} \right)^m e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_1} \right. \\ & \left. - \left(\psi \frac{S_{\mathbf{a},q}}{q} \right)^m e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_0} \right| \end{aligned}$$

$$\leq 5^{m+1} q^{\frac{5-m}{4}} B^{m-1} + \left| \frac{S_{\mathbf{a},q}}{q} \right|^{m-5} \\ \times B^m (2\pi z) P^{\frac{3}{4}}$$

$$\leq 5^{m+1} q^{\frac{5-m}{4}} B^{m-1} + 5^{m-5} q^{\frac{5-m}{4}} \\ \times B^m (\pi z) P^{\frac{3}{4}}$$

Lemma 26: *We have*

$$W_0(N_1) = \sum_{q \leq p^{\frac{1}{2}}} \sum_{\substack{q \\ a=1}} \int \frac{1}{8qP^3} \left(\psi \frac{S_{a,q}}{q} \right)^m \times e^{-2\pi i \frac{a}{q} N_1 - 2\pi iz N_0} dz + 5^m q^{\frac{5-m}{4}} P^{m-5+\frac{1}{2}} \cdot \theta$$

Proof: From lemma 25, we have

$$\begin{aligned} W_0(N) &= \int_{\Omega} (T(\alpha))^m e^{-2\pi i N_1 \alpha} d\alpha \\ &= \sum_{q \leq p^{\frac{1}{2}}} \sum_{\substack{q \\ a=1}} \int \frac{1}{8qP^3} \left(\psi \frac{S_{a,q}}{q} \right)^m e^{-2\pi i \frac{a}{q} N_1 - 2\pi iz N_0} dz \\ &+ \theta \sum_{q \leq p^{\frac{1}{2}}} \sum_{\substack{q \\ a=1}} \int \frac{1}{8qP^3} \left(5^{m+1} q^{\frac{5-m}{4}} B^{m-1} + 5^{m-5} q^{\frac{5-m}{4}} B^m \cdot (2\pi z) P^{3\frac{1}{4}} \right) dz \end{aligned}$$

Now, using the value of B , the integral in the error term is easily seen to be at most

$$2 \int_0^1 \left(5^{m+1} q^{\frac{5-m}{4}} B^{m-1} + 5^{m-5} q^{\frac{5-m}{4}} B^m \cdot 2\pi z \cdot P^{3\frac{1}{4}} \right) dz$$

$$\leq 2 \int_0^{P^{-4}} (5^{m+1} q^{\frac{5-m}{4}} P^{m-1} + 5^{m-5} q^{\frac{5-m}{4}} P^m 2\pi z \cdot P^{3\frac{1}{4}}) dz$$

$$+ 2 \int_{P^{-4}}^1 (5^{m-1} q^{\frac{5-m}{4}} (\sqrt{2}|z|^{-\frac{1}{4}})^{m-1} + 5^{m-5} q^{\frac{5-m}{4}} (\sqrt{2}|z|^{-\frac{1}{4}})^m \cdot 2\pi z \cdot P^{3\frac{1}{4}}) dz$$

$$\leq 5^m \cdot q^{\frac{5-m}{4}} P^{m-5+\frac{1}{4}}.$$

Lemma 27: We have

$$W_0(N_1) = \sum_{q \leq P^{\frac{1}{2}}} \sum_{a=1}^q \int_{-\infty}^{\infty} \left(\psi \frac{S_{0,q}}{q} \right)^m \times$$

$$e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_0} dz$$

$$+ O(2 \cdot 5^m \cdot P^{m-5+\frac{1}{4}} \cdot A)$$

where

$$A = \begin{cases} P^{\frac{1}{2}} & \text{if } m = 9 \\ \log P & \text{if } m = 10 \\ 1 & \text{if } m = 11 \end{cases}$$

The result follows almost immediately from lemma 26; we have only to prove that the error in extending the range of integration to $[-\infty, \infty]$ is small.

Actually the error is almost

$$2 \sum_{q \leq P^{\frac{1}{2}}} \sum_{a=1}^q \frac{\int_1^{\infty} \left| \left(\psi \frac{S_{0,q}}{q} \right) \right|^m dz}{8q P^s}$$

$$\begin{aligned} &\leq 2 \sum_{q \leq P^{\frac{1}{2}}} \sum_{\substack{q \\ a=1}}^q \frac{\int_1^{\infty} (\sqrt{2}|z| \cdot -\frac{1}{4})^m \cdot (5q^{-\frac{1}{4}})^m dz}{8q P^a} \\ &\leq 5^m \cdot P^{m-5+\frac{1}{4}} A. \end{aligned}$$

Lemma 28 : If $|N_1 - N_0| \leq P^{\frac{3}{4}}$, then

$$W_0(N_1) = S_1(N_1) R(N_0) + 2\theta 5^m P^{m-5+\frac{1}{4}} \text{ where}$$

$$S_1(N_1) = \sum_{q \leq P^{\frac{1}{2}}} \sum_{\substack{q \\ a=1}}^q \left(\frac{S_{a,q}}{q} \right)^m e^{-2\pi i \frac{a}{q} N_1}$$

$$\text{and } R(N_0) = \int_{-\infty}^{\infty} \psi^m e^{-2\pi i z N_0} dz$$

Proof : This follows from lemma 27.

Lemma 29 : If $|N_1 - N_0| \leq P^{\frac{3}{4}}$, and $N_1 \equiv 2, 3, 4, 5, 6, 7,$

$$\text{or } 8 \pmod{16} \text{ then } W_0(N_1) > 0.02 N^{\frac{m-1}{4}} \text{ for } m = 11.$$

Proof : This follows from lemmas 19, 24 and 28

§ 8 : A lower bound for the number of integers less than a given integer which are representable as a sum of five biquadratics.

Lemma 30 : Let P be a positive integer and assume $P > 100$. Let δ and C be fixed positive reals. Let μ be a fixed number in the interval $(0, 1)$ and suppose

$U = \{u_1, u_2, \dots, u_U\}$ is a set of (distinct) integers in the interval $[0, P^{3+\mu}]$ where

$$U \geq C \cdot P^3 (1-\mu) - \delta.$$

Then the number of solutions M of the equation

$$x^4 + u_i = y^4 + u_j$$

where u_h and u_j vary over \mathbf{U} and

$$P \leq x, y \leq 2P$$

and x, y have the same fixed parity modulo 2 does not exceed

$$C^{-\frac{3}{4}} P^{\delta} U^{\delta} P^{3\mu-4+\frac{3\delta}{4}} \left(C^{-\frac{1}{4}} P^{\frac{\delta}{4}} + \frac{1}{2} \cdot P^{\frac{(3+\mu)\varepsilon}{4}} \right) \left\{ \frac{1}{2} K_2(\varepsilon) \cdot (192)^{-\varepsilon} \right\}^{\frac{1}{4}}$$

In particular if we take

$$\varepsilon = \frac{\delta}{3+\mu}, \text{ then}$$

$$M \leq C^{-\frac{3}{4}} P^{\delta} U^{\delta} P^{3\mu-4+\delta} \left\{ C^{-\frac{1}{4}} + \frac{1}{2} \left(\frac{1}{2} K_2(\varepsilon) \cdot (192)^{-\varepsilon} \right)^{\frac{1}{4}} \right\}$$

Here $K_2(\varepsilon)$ is defined by

$$d_4(m) \leq K_2(\varepsilon) m^{\varepsilon} \text{ for all } m > 2$$

and we have

$$\left\{ K_2(0.20) \cdot \left(\frac{1}{192} \right)^{0.20} \right\}^{\frac{1}{4}} \leq (6.2124170) 10^{\delta}$$

Proof: This is lemma (7.1) of Thomas [8] (page 118).

The bound for M when $\varepsilon = \frac{\delta}{3+\mu}$ is given in (7.7) in Thomas [8] (page 119). The bound for $K_2(0.20)$ is given in page 127 of Thomas [8].

Lemma 31: Let $1 \leq l < 16$; Let f be a fixed integer in $(0, 1, \dots, l)$. Let f_1 be a fixed integer in the set $(0, 1, \dots, l+1)$. Let it be given that, for all integers $X > X_0$, the number of integers less than X , which are congruent to $f \pmod{16}$ and which are representable as a sum of l biquadratics is at least $C_l \cdot P^l$. Then the number of integers less than

Y , which are congruent to $f_1 \pmod{16}$ and which are representable as a sum of $(l+1)$ biquadratics is atleast

$C_{l+1} P^{l+1}$ where

$$C_{l+1} = \frac{1}{8^{\frac{1}{4}}} C_l^{\frac{3}{4}} (C_l^{-\frac{1}{4}} + (2 \cdot 3) \times 10^{\mu})^{-1}$$

$$\text{and } \nu_{l+1} = \frac{1}{2} + \frac{3 \nu_l}{3 + \nu_l + \varepsilon}$$

$$\text{Provided } Y \geq 10^{\mu} \cdot X^{\frac{4}{3+\mu}}$$

$$\text{with } \mu = \frac{3(1 - \nu_l - \varepsilon)}{3 + \nu_l + \varepsilon}$$

Here ε is any positive number.

Proof: Define $\delta = (3 + \mu) \varepsilon$

Note that $3(1 - \mu) - \delta = (3 + \mu) \nu_l$

and $-4 \nu_{l+1} = -4 + 3\mu + \delta$

We observe that $[Y^{\frac{3+\mu}{4}}] \geq X$

Choose f_2 from the set $(0, 1)$ such that

$$f_2 + f = f_1.$$

Let $U = \{x; 0 \leq x < [Y^{\frac{3+\mu}{4}}]\};$

$$x = \begin{matrix} 4 & 4 & 4 & 4 \\ x & + x & + x & + x \\ 0 & 1 & 2 & \end{matrix}$$

where $x_i \equiv f_1 \pmod{2}$ }

Let $U = \text{Card } U.$

By the hypothesis,

$$U > C_l Y^4 = C_l Y^4 \frac{3(1-\mu) - \delta}{4}$$

Let $r(m)$ denote the number of solutions of $M = u_h + y^4$, where u_h runs over the set U and

$$z^{\frac{1}{4}} \leq y \leq 2z^{\frac{1}{4}}; \quad y \equiv f_a \pmod{2}$$

Then $\sum_m r(m) \geq \frac{1}{2} P U$; with $P = [z^{\frac{1}{4}}]$

$$m \leq 16z + y^4 \frac{3 + \mu}{4}$$

Also $\sum_m (r(m))^2$ does not exceed the number of solutions of $x^4 + u_h = y^4 + u_j$ subject to the conditions of lemma 30

Hence $\sum (r(m))^2$

$$\leq C_l^{-\frac{3}{4}} \cdot P^2 U^2 P^{3\mu - 4 + \delta} \{ C_l^{-\frac{1}{4}} + (2.3) \times 10^5 \}$$

$$\sum_m 1 \geq \frac{(\sum r(m))^2}{\sum (r(m))^2}$$

$$r(m) \neq 0$$

$$\geq \frac{1}{4} C_l^{\frac{3}{4}} \cdot (C_l^{-\frac{1}{4}} + (2.3) \times 10^5)^{-1} \cdot P^{- (3\mu - 4 + \delta)}$$

$$\geq \frac{1}{4} C_l^{\frac{3}{4}} (C_l^{-\frac{1}{4}} + (2.3) \times 10^5)^{-1} \cdot P^{4\nu l + 1}$$

$$\geq \frac{1}{4} C_l^{\frac{3}{4}} (C_l^{-\frac{1}{4}} + (2.3) \times 10^5)^{-1} \cdot ([Z^{\frac{1}{4}}]^{4\nu l + 1})$$

Choosing Z such that $16Z + Y \frac{3+\mu}{4} = Y$, we have

$$\sum_{m \leq Y} 1 \geq \frac{1}{64} C_l^{\frac{8}{4}} (C_l^{-\frac{1}{4}} + (2.3) \times 10^9)^{-1} Y^{\nu_l + 1}$$

$$r(m) \neq 0$$

and hence the result.

Lemma 32 : Let f be a fixed integer from the set $\{0, 1, 2, 3, 4\}$. Suppose $n \geq 2.10^{44}$. Then the number of non negative $m \equiv f \pmod{16}$ such that $m \leq n$ and such that m is the sum of four fourth powers is greater than

$$(1.1950) 10^{-18} \cdot n^{0.745633624425}$$

Proof : This is lemma 7.4 of Thomas [8] (page 127)

Lemma 33 : Let f_i be a fixed integer from the set $\{0, 1, 2, 3, 4, 5\}$. Suppose $n \geq 10^{80}$. Then the number of non negative $m \equiv f \pmod{16}$ such that $m \leq n$ and such that m is the sum of five fourth powers is greater than

$$(0.8) \times 10^{-21} \cdot n^{0.8168}$$

Proof : The result follows from a direct application of lemmas 31 and 32. In the notation of lemma 31, we have

$$C_l = (1.195) \times 10^{-18}$$

Hence

$$C_{l+1} = \frac{1}{64} \cdot (1.195 \times 10^{-18})^{\frac{8}{4}} \left((1.195 \times 10^{-18})^{-\frac{1}{4}} + (2.3) \times 10^9 \right)^{-1}$$

$$\geq 2.17 \times 10^{-21}.$$

$$\nu_l = 0.745633$$

$$\text{Hence } \nu_{l+1} = 0.25 + \frac{3 \times 0.745633}{3.945633} \geq 0.8168$$

and hence the result.

§ 7: An asymptotic formula

Let N be an integer $> 10^{560}$ and $P = [N^{\frac{1}{4}}]$;

Let N_0 and N_1 be integers satisfying

$$\frac{N}{2} \leq N_0 \leq N; \quad N_0 - P^{\frac{41}{5}} < N_1 \leq N_0.$$

Let f be an integer in the set $(0, 1, 2, 3, 4, 5)$ such that

$$N - 2f \equiv 2, 3, 4, 5, 6, 7 \text{ or } 8 \pmod{16}.$$

Let μ_1, μ_2 go independently over the same sequence of numbers, which is less than $\frac{N}{4}$ and can be represented as a sum of five biquadratics.

Let U denote the number of numbers μ .

Lemma 34: *Every integer $N > 10^{560}$ can be represented as a sum of twenty one biquadratics*

Proof: Let $I(N) =$

$$\int_0^1 (T(\alpha))^{11} \sum_{\mu_1} \sum_{\mu_2} e^{2\pi i(\mu_1 + \mu_2)\alpha} e^{-2\pi i N \alpha} d\alpha$$

It is sufficient to prove that $I(N) > 0$

$$\text{Now } I(N) = \int_{\Omega} + \int_{\mathfrak{M}}$$

where Ω is the major arc and \mathfrak{M} is the minor arc.

By lemma 30

$$\begin{aligned} & \int_{\Omega} (T(\alpha))^{11} \sum_{\mu_1} \sum_{\mu_2} e^{2\pi i(\mu_1 + \mu_2)\alpha} e^{-2\pi i N \alpha} d\alpha \\ &= \sum_{\mu_1} \sum_{\mu_2} W_0(N - \mu_1 - \mu_2) \end{aligned}$$

$$> U^2 \cdot (0.02) \cdot N^{\frac{7}{2}}$$

$$> U^2 \cdot (0.02) \cdot P^7$$

By lemma 13, we have

$$\int_0^1 (T(\alpha))^{11} \cdot \sum_{\mu_1} \sum_{\mu_2} e^{2\pi i(\mu_1 + \mu_2)\alpha} e^{-2\pi i N \alpha} d\alpha$$

$$\leq \max_{\alpha \in \mathfrak{M}} |T(\alpha)|^{11} \int_0^1 \left| \sum_{\mu} e^{2\pi i \mu \alpha} \right|^2 d\alpha$$

$$\leq \max_{\alpha \in \mathfrak{M}} |T(\alpha)|^{11} \cdot U$$

$$\leq (3 P^{\frac{32}{3}} (\log P + 3)^{\frac{15}{3}})^{11} \cdot U$$

We have only to check that

$$U^2 (0.02) P^7 > 3^{11} \cdot P^{10 - \frac{1}{3}} (\log P + 3)^{5 + \frac{5}{3}} U$$

Now using the value

$$U > \frac{1}{4} P^{3.2672} \times (0.8) \times 10^{-21}$$

we have only to check that

$$P^{0.2984} \geq 10^{28.55} (\log P + 3)^{5 + \frac{5}{3}}$$

Taking $P \geq 10^{140}$, we see that the inequality is satisfied and this proves the result.

§ 8: The ascent

Lemma 35: *Let l be an integer ≥ 0 ;*

$$\text{Let } \nu = \frac{1 - \frac{l}{L_0}}{n};$$

$$L_0 > l; \nu L_0)^{\frac{n}{n-1}} \geq L_0.$$

Compute L_t by

$$\log L_t = \left(\frac{n}{n-1} \right)^t (\log L_0 + n \log \nu) - n \log \nu$$

If all integers between l and L_0 inclusive are sums of k integral n^{th} powers, then all integers between l and L_t , inclusive are sums of $(k + t)$ integral n^{th} powers ≥ 0 .

Proof: This is Theorem 12 (page 711) of Dickson [5].

Lemma 36 : *Every natural number in the range $[13793, 10^{143}]$ is a sum of sixteen biquadratics.*

Proof: This is Theorem 3.4 of Thomas [9].

Lemma 37 : *Every natural number less than 10^{560} is a sum of twenty one biquadratics.*

Proof: In the notation of lemma 36, we take $l = 13793$; $L_0 = 10^{143}$; $t = 5$; $n = 4$ and this proves the result.

§ 9 : Completion of the proof

The proof of the main theorem follows from lemmas 34 and 37.

§ 10 : Acknowledgement

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References

1. **Auluck F. C.** : "On Waring's problem for Biquadrates" *Proc. of the Indian Acad. of Sciences, Sec A*, 11 (1940) (437-50).
2. **Chen Jing Run** : "Waring's problem for $g(5) = 37$ " *Chinese Mathematics Acta*, 6 (1965) (105-127).
3. **Davenport H.** : Analytical methods for Diophantine equations and Diophantine Inequalities, *Ann Arbor, Michigan*.
4. **Davenport H.** : "On Waring's problem for fourth powers", *Annals of Mathematics*, 40 (1939) (731-47).
5. **Dickson L. E.** : "Recent progress on Waring's theorem and its Generalisations" *Bull. Amer. Maths. Soc.* 39 (1933), (701-727).
6. **Landau E.** : Vorle Sungen uber Zahlen theorie, *Chelsea Publishing Company*, 1947.
7. **Mardjanichivili C.** : "Estimation d'une somme Arithmetique" *Comptes Rendus (Doklady) de l' Academies des Sciences de l'URSS*, 22 (1939).
8. **Thomas H.** : A numerical approach to Waring's problem for fourth powers, *The University of Michigan*, Ph. D. 1973.
9. **Thomas H.** : "Waring's problem for twenty two biquadrates", *Trans. Amer. Maths. Soc.* , 193 (1974) (427-430).
10. **Vinogradov I. M.** : The method of trigonometric sums in the theory of numbers, *Interscience Publishers*.

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