



HAL
open science

Constrained Backward SDEs with Jumps: Application to Optimal Switching

Romuald Elie, Idris Kharroubi

► **To cite this version:**

Romuald Elie, Idris Kharroubi. Constrained Backward SDEs with Jumps: Application to Optimal Switching. 2009. hal-01103771v1

HAL Id: hal-01103771

<https://hal.science/hal-01103771v1>

Preprint submitted on 19 Mar 2009 (v1), last revised 15 Jan 2015 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Constrained Backward SDEs with Jumps: Application to Optimal Switching

Romuald ELIE
CEREMADE, CNRS, UMR 7534,
Université Paris-Dauphine,
CREST and FIME
elie@ensae.fr

Idris KHARROUBI
LPMA, CNRS, UMR 7599,
Universités Paris 6-Paris 7,
and CREST
kharroubi@ensae.fr

March 19, 2009

Abstract

In this paper, we introduce a new class of BSDE generalizing and offering a unifying framework to represent the constrained ones presented in [16] or [12] as well as the oblique reflected ones studied by [11] and [9]. Via a penalization procedure, we provide an existence and uniqueness result for this new class of so-called constrained BSDEs with jumps. Remarkably, these BSDEs appear to be very convenient to represent the solution to eventually non-Markovian switching problems. As a by-product, we enlarge the class of obliquely reflected BSDE's, allowing to represent switching problems with controlled underlined diffusion.

Résumé

Nous introduisons une nouvelle classe d'EDS rétrograde qui généralise et offre un cadre unifié pour le traitement des contraintes étudiées par [16] et [12], ainsi que des réflexions obliques considérées dans [11] and [9]. Par une méthode de pénalisation, nous obtenons l'existence et l'unicité d'une solution minimale à cette nouvelle classe d'EDSR. De plus, les EDSRs contraintes à sauts offrent une représentation alternative pour la résolution de problèmes de switching éventuellement non markoviens. L'obtention de ce résultat fournit également une généralisation de la notion d'EDSR réflexion oblique, permettant de traiter les problèmes de switching avec diffusion sous-jacente contrôlée.

Key words: Switching problems, BSDE with jumps, Constrained BSDE, Reflected BSDE.

MSC Classification (2000): 93E20, 60H30, 60J75.

1 Introduction

Since its introduction by Pardoux and Peng in [13], the notion of Backward Stochastic Differential Equations (BSDEs in short) has been widely extended. In particular, it appeared as a very powerful tool to solve partial differential equations (PDE) and corresponding stochastic optimization problems. Several generalizations of this notion are based on the addition of new constraints on the solution. First, El Karoui et al. [7] study the case where the component Y is forced to stay above a given process, leading to the notion of reflected BSDEs related to optimal stopping and obstacle problems. Motivated by super replication problems under portfolio constraints, Cvitanic et al. [5] consider the case where the component Z is constrained to stay in a fixed convex set. More recently, Kharroubi et al. [12] introduce a constraint on the jump component U of the BSDE, providing a representation of solutions to a class of PDE, called quasi-variational inequalities, arising in optimal impulse control problems. Generalizing the results of El Karoui et al. [7] in a multi-dimensional framework, Hu and Tang [11] followed by Hamadène and Zhang [9] consider BSDEs with oblique reflections and connect them with systems of variational inequalities and optimal switching problems. Nevertheless, they only consider cases where the switching strategy does not affect the dynamics of the underlying diffusion. Our paper introduces the new notion of constrained BSDEs with jumps, which offers in particular a nice and natural probabilistic representation for these types of switching problems.

Let us introduce an underlying diffusion process whose dynamics are given by

$$X_t^\alpha = X_0 + \int_0^t b(X_u^\alpha, \alpha_u) du + \int_0^t \sigma(X_u^\alpha, \alpha_u) dW_u, \quad 0 \leq t \leq T, \quad (1.1)$$

where α is a switching control process valued in $\{1, \dots, m\}$. We consider the following switching control problem defined by

$$\sup_{\alpha} \mathbf{E} \left[g(X_T^\alpha, \alpha_T) + \int_0^T f(X_s^\alpha, \alpha_s) ds + \sum_{0 < \tau_k \leq T} c(\alpha_{\tau_k^-}, \alpha_{\tau_k}) \right], \quad (1.2)$$

where $(\tau_k)_k$ denotes the jump times of the control α . This type of stochastic control problem is typically encountered by an agent maximizing the production rentability of a given good by switching between m possible modes of production based on different commodities. A switch is penalized by a given cost function c and, since the agent is a large actor on the market, the chosen mode of production influences the dynamics of the corresponding commodities. As observed by Tang and Yong [19] and Bouchard [2], the value function associated to this problem interprets on $[0, T]$ as the unique viscosity solution of a given coupled system of variational inequalities. The difficulty in the derivation of a BSDE representation for this type of problem is, firstly, the dependence of the solution in mode $i \in \{1, \dots, m\}$ with respect to the global solution in all possible modes, and secondly, the dependence in the control of the drift and the volatility of X . In the spirit of [14], we overcome this difficulty by artificially letting the strategy jump randomly between the different modes of production. This allows to retrieve in the jump component of a one-dimensional backward process, some information regarding the solution in the other

modes of production. Indeed, let us introduce a pure jump process $(I_t)_{0 \leq t \leq T}$ based on an independent random measure μ and construct the underlying process $(X_t^{I_t})_{0 \leq t \leq T}$, whose dynamics are based on the random mode of production I according to equation (1.1). Let now consider the following constrained BSDE associated to the two dimensional forward process (I, X^I) (called diffusion-transmutation process in [14]) and defined on $[0, T]$ by:

$$\begin{cases} Y_t &= g(X_T^{I_T}) + \int_t^T f(X_s^{I_s}, I_s) ds + K_T - K_t - \int_t^T \langle Z_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(ds, di), \\ U_t(i) &\geq c(i, I_{t-}), \quad d\mathbf{P} \otimes dt \otimes \lambda(di) \text{ a.e.} \end{cases} \quad (1.3)$$

We prove in this paper that (1.3) has one unique minimal solution which indeed relates directly to the solution of the corresponding switching problem (1.2).

In order to unify our results with the one based on BSDE with oblique reflection considered in [11] or [9], we extend this approach and introduce the notion of constrained BSDE with jumps whose solution (Y, Z, U, K) satisfies

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds + K_T - K_t - \int_t^T \langle Z_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(ds, di), \quad (1.4)$$

a.s, for $0 \leq t \leq T$, as well as the constraint

$$h(t, Y_{t-}, Z_t, U_t(i), i) \geq 0, \quad d\mathbf{P} \otimes dt \otimes \lambda(di) \text{ a.e.}, \quad (1.5)$$

where f and h are given random functions. Through a penalization argument, we provide, in Section 2, existence and uniqueness of the minimal solution to this type of constrained BSDE with jumps (1.4)-(1.5). This extends the results of [12] by allowing the driver and the constraint functions to depend on all the components of the solution. Furthermore, in the case of general non linear switching problems considered in [9], our minimal solution interprets in terms of solution to their corresponding BSDE with oblique reflections. Therefore, through the addition of a jump component, the one dimensional constrained BSDE with jumps is able to provide the solution to a multidimensional obliquely reflected BSDEs. Nicely, our argumentation only relies on probabilistic arguments and allows to obtain results in the non-Markovian case. The study of the link between this new type of BSDEs and the corresponding systems of variational inequalities in a markovian framework is currently under study and will appear in [8]. We believe that this new approach is numerically promising, through an algorithm adapted from the one introduced by [3] and detailed in [8].

The last section of the paper deals with general switching problems in a non-Markovian framework, i.e. problems of the form (1.2) where the functions g , f and c are possibly random. In this case, the dynamics of the forward process depends on the current switching regime, so that the representation of Hu and Tang [11] or Hamadene and Zhang [9] in terms of oblique reflected BSDE does not apply. We generalize this approach and introduce a family of reflected BSDEs which provides the value process of the optimal switching problem. We finally link via a penalization procedure this family of reflected BSDEs with a member of the class of constrained BSDEs with jumps. Therefore, constrained BSDEs with jumps

offer also a nice representation for general switching problems, even in a non-Markovian framework.

Notations. Throughout this paper we are given a finite terminal time T and a probability space $(\Omega, \mathcal{G}, \mathbf{P})$ endowed with a d -dimensional standard Brownian motion $W = (W_t)_{t \geq 0}$, and a Poisson random measure μ on $\mathbb{R}_+ \times \mathcal{I}$, where $\mathcal{I} = \{1, \dots, m\}$, with intensity measure $\lambda(di)dt$ for some finite measure λ on \mathcal{I} with $\lambda(i) > 0$ for all $i \in \mathcal{I}$. We set $\tilde{\mu}(dt, di) = \mu(dt, di) - \lambda(di)dt$ the compensated measure associated to μ . $\sigma(\mathcal{I})$ denotes the σ -algebra of subsets of \mathcal{I} .

For $x = (x_1, \dots, x_\ell) \in \mathbb{R}^\ell$ with $\ell \in \mathbb{N}$, we set $|x| = \sqrt{|x_1|^2 + \dots + |x_\ell|^2}$ the euclidean norm.

We denote by $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ (resp. $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$) the augmentation of the natural filtration generated by W and μ (resp. by W), and by \mathcal{P} the σ -algebra of predictable subsets of $\Omega \times [0, T]$. We denote by $\mathcal{S}_{\mathbb{F}}^2$ (resp. $\mathcal{S}_{\mathbb{F}}^{\mathbf{c}, 2}$, $\mathcal{S}_{\mathbb{G}}^2$) the set of real-valued càd-làg \mathbb{F} -adapted (resp. continuous \mathbb{F} -adapted, càd-làg \mathbb{G} -adapted) processes $Y = (Y_t)_{0 \leq t \leq T}$ such that

$$\|Y\|_{\mathcal{S}^2} := \left(\mathbf{E} \left[\sup_{0 \leq t \leq T} |Y_t|^2 \right] \right)^{\frac{1}{2}} < \infty.$$

where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ (resp. $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$) denotes the completed filtration generated by W (resp. by W and μ). $\mathbf{L}^{\mathbf{P}}(\mathbf{0}, \mathbf{T})$, $p \geq 1$, is the set of real-valued measurable processes $\phi = (\phi_t)_{0 \leq t \leq T}$ such that

$$\mathbf{E} \left[\int_0^T |\phi_t|^p dt \right] < \infty,$$

and $\mathbf{L}_{\mathbb{F}}^{\mathbf{P}}(\mathbf{0}, \mathbf{T})$ (resp. $\mathbf{L}_{\mathbb{G}}^{\mathbf{P}}(\mathbf{0}, \mathbf{T})$) is the subset of $\mathbf{L}^{\mathbf{P}}(\mathbf{0}, \mathbf{T})$ consisting of \mathbb{F} -progressively measurable (resp. \mathbb{G} -progressively measurable) processes.

$\mathbf{L}_{\mathbb{F}}^{\mathbf{P}}(\mathbf{W})$ (resp. $\mathbf{L}_{\mathbb{G}}^{\mathbf{P}}(\mathbf{W})$), $p \geq 1$, is the set of \mathbb{R}^d -valued \mathbb{F} -progressively measurable (resp. \mathcal{P} -measurable) processes $Z = (Z_t)_{0 \leq t \leq T}$ such that

$$\|Z\|_{\mathbf{L}^{\mathbf{P}}(\mathbf{W})} := \left(\mathbf{E} \left[\int_0^T |Z_t|^p dt \right] \right)^{\frac{1}{p}} < \infty.$$

$\mathbf{L}^{\mathbf{P}}(\tilde{\mu})$, $p \geq 1$, is the set of $\mathcal{P} \otimes \mathcal{E}$ -measurable maps $U : \Omega \times [0, T] \times \mathcal{I} \rightarrow \mathbb{R}$ such that

$$\|U\|_{\mathbf{L}^{\mathbf{P}}(\tilde{\mu})} := \left(\mathbf{E} \left[\int_0^T \int_{\mathcal{I}} |U_t(i)|^p \lambda(di) dt \right] \right)^{\frac{1}{p}} < \infty.$$

$\mathbf{A}_{\mathbb{F}}^2$ (resp. $\mathbf{A}_{\mathbb{G}}^2$) is the closed subset of $\mathcal{S}_{\mathbb{F}}^2$ (resp. $\mathcal{S}_{\mathbb{G}}^2$) consisting of nondecreasing processes $K = (K_t)_{0 \leq t \leq T}$ with $K_0 = 0$.

For ease of notation, we omit in all the paper the dependence in $\omega \in \Omega$, whenever it is explicit.

2 Constrained Backward SDEs with jumps

This section is devoted to the presentation of constrained Backward SDEs with jumps in a framework generalizing the one considered in [12] and [16]. Namely we allow the driver function to depend on the jump component of the backward process and we extend the class of possible constraint functions by letting them depend on all the components of the BSDE solution. We provide here an existence and uniqueness result for this type of BSDEs and remark that they are closely related to the notion of BSDEs with oblique reflections studied by [11] and [9].

2.1 Formulation

A constrained BSDE with jumps is characterized by three objects :

- a terminal condition, i.e. a \mathcal{G}_T -measurable random variable ξ ,
- a generator function, i.e. a progressively measurable map $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathcal{I}} \rightarrow \mathbb{R}$,
- a constraint function, i.e. a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \sigma(\mathcal{I})$ -measurable map $h : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}$ such that $h(\omega, t, y, z, \cdot, i)$ is non-increasing for all $(\omega, t, y, z, i) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{I}$.

Definition 2.1. *A solution to the corresponding constrained BSDE with jumps is a quadruple $(Y, Z, U, K) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$ satisfying*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds + K_T - K_t - \int_t^T \langle Z_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(ds, di), \quad (2.1)$$

for $0 \leq t \leq T$ a.s., as well as the constraint

$$h(t, Y_{t-}, Z_t, U_t(i), i) \geq 0, \quad d\mathbf{P} \otimes dt \otimes \lambda(di) \text{ a.e.} \quad (2.2)$$

Furthermore, (Y, Z, U, K) is referred to as the minimal solution to (2.1)-(2.2) whenever, for any other solution $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$ to (2.1)-(2.2), we have $Y \leq \tilde{Y}$ a.s.. In this case, Y naturally interprets in the terminology of Peng [15] as the smallest supersolution to (2.1)-(2.2).

Remark 2.1. In the case where the driver function f does not depend on U and the constraint function h is of the form $h(u + c(t, y, z), i)$, observe that this BSDE exactly fits in the framework considered in [12]. In the Brownian case (i.e. no jump component), this type of BSDEs has been studied in [16].

In order to derive an existence and uniqueness result for solutions to this type of BSDE, we require the classical Lipschitz and linear growth conditions on the coefficients as well as a constraint on the dependence of the driver function in the jump component of the BSDE. We assemble these conditions in the following assumption.

(H0)

- (i) There exists a constant $k > 0$ such that the functions f and h satisfy \mathbf{P} -a.s. the uniform Lipschitz property

$$|f(t, y, z, (u_j)_j) - f(t, y', z', (u'_j)_j)| \leq k|(y, z, (u_j)_j) - (y', z', (u'_j)_j)|, \quad (2.3)$$

$$|h(t, y, z, u_i, i) - h(t, y', z', u'_i, i)| \leq k|(y, z, u_i) - (y', z', u'_i)| \quad (2.4)$$

for all $(t, i, y, z, (u_j)_j, y', z', (u'_j)_j) \in [0, T] \times \mathcal{I} \times [\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathcal{I}}]^2$.

- (ii) The coefficients f and h satisfy the following growth linear condition : there exists a constant C such that \mathbf{P} -a.s.

$$|f(t, y, z, (u_j)_j)| + |h(t, y, z, u_i, i)| \leq C(1 + |y| + |z| + |(u_j)_j|) \quad (2.5)$$

for all $(t, i, y, z, (u_j)_j) \in [0, T] \times \mathcal{I} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathcal{I}}$.

- (iii) There exist two constants $C_1 \geq C_2 > -1$ such that \mathbf{P} - a.s.

$$f(t, y, z, u) - f(t, y, z, u') \leq \int_{\mathcal{I}} (u_i - u'_i) \gamma_t^{y, z, u, u'}(i) \lambda(di),$$

for all $(y, z, u, u') \in \mathbb{R} \times \mathbb{R}^d \times [\mathbb{R}^{\mathcal{I}}]^2$, where $\gamma^{y, z, u, u'} : \Omega \times [0, T] \times \mathcal{I} \rightarrow \mathbb{R}$ is $\mathcal{P} \otimes \sigma(\mathcal{I})$ -measurable and satisfies $C_2 \leq \gamma^{y, z, u, u'} \leq C_1$.

Remark 2.2. Under Assumption **(H0)** (i) and (ii), existence and uniqueness of a solution (Y, Z, U, K) to the BSDE (2.1) with $K = 0$ follows from classical results on BSDEs with jumps, see [1] or [18] for example. In order to add the h -constraint (2.2), one needs as usual to relax the dynamics of Y by adding the non decreasing process K in (2.1). In mathematical finance, the purpose of this new process K is to increase the super replication price Y of a contingent claim, under additional portfolio constraints. In order to find a minimal solution to the constrained BSDE (2.1)-(2.2), the nondecreasing property of h is crucial for stating comparison principles needed in the penalization approach. The simpler example of constraint function to keep in mind is $h(\cdot, u, i) = c(\cdot, i) - u$, i.e. upper-bounded jumps constraint.

Remark 2.3. Part (iii) of Assumption **(H0)** constrains the form of the dependence of the driver in the jump component of the BSDE. It is inspired from [17] and will ensure comparison results for BSDEs driven by this type of driver.

2.2 Existence, uniqueness and approximation by penalization

In this section, we provide an existence and uniqueness result for solutions to constrained BSDEs with jumps of the form (2.1)-(2.2). This type of BSDEs generalizes the one presented in [12] since they allow the generator function f to depend on the jump component U of the backward process and the constraint involves all the components of the solution. The

proof relies on a classical penalization argument and we introduce the following sequence of BSDEs with jumps

$$\begin{aligned} Y_t^n &= \xi + \int_t^T f(s, Y_s^n, Z_s^n, U_s^n) ds + n \int_t^T \int_{\mathcal{I}} h^-(s, Y_s^n, Z_s^n, U_s^n(i), i) \lambda(di) ds \\ &\quad - \int_t^T \langle Z_s^n, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s^n(i) \mu(ds, di), \quad 0 \leq t \leq T, n \in \mathbb{N}, \end{aligned} \quad (2.6)$$

where $h^-(t, y, z, u, i) := \max(-h(t, y, z, u, i), 0)$ is the negative part of the function h . Under Assumption **(H0)**, the Lipschitz property of the coefficients f and h ensures existence and uniqueness of a solution $(Y^n, Z^n, U^n) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu})$ to (2.6), see [1] or [18]. Let first state two comparison results ensuring a monotonic convergence of the sequence $(Y^n)_n$ under the additional assumption

(H1) There exists a quadruple $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{U}) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$ solution of (2.1)-(2.2).

We present in Section 2.3 some examples where **(H1)** is satisfied. Let us first state a general comparison theorem for BSDEs with jumps.

Lemma 2.1. *Let $f_1, f_2 : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathcal{I}} \rightarrow \mathbb{R}$ two generators satisfying Assumption **(H0)** and $\xi_1, \xi_2 \in \mathbf{L}^2(\Omega, \mathcal{G}_T, \mathbf{P})$. Let $(Y^1, Z^1, U^1) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu})$ satisfying*

$$Y_t^1 = \xi^1 + \int_t^T f_1(s, Y_s^1, Z_s^1, U_s^1) ds - \int_t^T \langle Z_s^1, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s^1(i) \mu(ds, di) \quad (2.7)$$

and $(Y^2, Z^2, U^2, K^2) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$ satisfying

$$Y_t^2 = \xi^2 + \int_t^T f_2(s, Y_s^2, Z_s^2, U_s^2) ds - \int_t^T \langle Z_s^2, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s^2(i) \mu(ds, di) + K_T^2 - K_t^2. \quad (2.8)$$

If $\xi^1 \leq \xi^2$ and $f_1(t, Y_t^1, Z_t^1, U_t^1) \leq f_2(t, Y_t^1, Z_t^1, U_t^1)$ for all $t \in [0, T]$ then we have $Y_t^1 \leq Y_t^2$ for all $t \in [0, T]$.

Proof. Let us denote $\bar{Y} := Y^2 - Y^1$, $\bar{Z} := Z^2 - Z^1$, $\bar{U} := U^2 - U^1$, $\bar{f} = f_2(\cdot, Y^2, Z^2, U^2) - f_1(\cdot, Y^1, Z^1, U^1)$ and $\bar{\xi} = \xi^2 - \xi^1$ so that

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{f}_s ds - \int_t^T \langle \bar{Z}_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} \bar{U}_s(i) \mu(ds, di) + \bar{K}_T - \bar{K}_t, \quad 0 \leq t \leq T, \text{ a.s.} \quad (2.9)$$

Let now define the process a by

$$a_t = \frac{f_2(t, Y_t^2, Z_t^2, U_t^2) - f_2(t, Y_t^1, Z_t^2, U_t^2)}{\bar{Y}_t} \mathbf{1}_{\{\bar{Y}_t \neq 0\}},$$

and b the \mathbb{R}^d -valued process defined component by component by

$$b_t^k = \frac{f_2(t, Y_t^1, Z_t^{(k-1)}, U_t^2) - f_2(t, Y_t^1, Z_t^{(k)}, U_t^2)}{V_t^k} \mathbf{1}_{\{V_t^k \neq 0\}}, \quad k = 1, \dots, d,$$

where $Z_t^{(k)}$ is the \mathbb{R}^d -valued random vector whose k first components are those of Z^1 and whose $(d-k)$ lasts are those of Z^2 , and V_t^k is the k -th component of $Z_t^{(k-1)} - Z_t^{(k)}$.

Notice that the processes a, b are bounded since f is Lipschitz continuous. Observe also that the process \hat{K} defined by

$$\hat{K}_t = K_t^2 - \int_0^t \int_{\mathcal{I}} \gamma_s^{Y_s^1, Z_s^1, U_s^1} \bar{U}_s(i) \lambda(di) ds + \int_0^t (f_2(s, Y_s^1, Z_s^1, U_s^1) - f_1(s, Y_s^1, Z_s^1, U_s^1)) ds$$

is an increasing process according to **(H0)** (iii) and $f_1(t, Y_t^1, Z_t^1, U_t^1) \leq f_2(t, Y_t^1, Z_t^1, U_t^1)$ for all $t \in [0, T]$. With these notations, we rewrite (2.9) as:

$$\begin{aligned} \bar{Y}_t = \bar{\xi} + \int_t^T & \left(a_s \bar{Y}_s + b_s \bar{Z}_s + \int_{\mathcal{I}} \gamma_s^{Y_s^1, Z_s^1, U_s^1} \bar{U}_s(i) \lambda(di) \right) ds \\ & - \int_t^T \langle \bar{Z}_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} \bar{U}_s(i) \mu(ds, de) + \hat{K}_T - \hat{K}_t. \end{aligned}$$

Consider now the positive process Γ solution to the s.d.e.:

$$d\Gamma_t = \Gamma_{t-} \left(a_t dt + \langle b_t, dW_t \rangle + \int_{\mathcal{I}} \gamma_s^{Y_s^1, Z_s^1, U_s^1} \mu(di, ds) \right), \quad \Gamma_0 = 1.$$

Notice that Γ lies in $\mathcal{S}_{\mathbb{G}}^2$ since a, b and γ are bounded, and Γ is positive according to **(H3)** (iii). A direct application of Itô's formula leads to

$$d[\Gamma \bar{Y}]_t = \langle \Gamma_{t-} \bar{Z}_t + \bar{Y}_t \Gamma_{t-} b_t, dW_t \rangle + \Gamma_{t-} \int_{\mathcal{I}} \gamma_s^{Y_s^1, Z_s^1, U_s^1} \bar{U}_s(i) \tilde{\mu}(ds, di) - \Gamma_{t-} d\hat{K}_t,$$

so that the process $\Gamma \bar{Y}$ is a supermartingale since $\Gamma > 0$. Hence

$$\Gamma_t \bar{Y}_t \geq \mathbb{E}[\Gamma_T \bar{Y}_T | \mathcal{G}_t] = \mathbb{E}[\Gamma_T \bar{\xi} | \mathcal{G}_t] \geq 0, \quad 0 \leq t \leq T,$$

leading to $\bar{Y} \geq 0$. □

We can now state our comparison results for the sequence $(Y^n)_n$.

Proposition 2.1. *Under **(H0)**, the sequence $(Y^n)_n$ is nondecreasing, and, for any quadruple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$ satisfying (2.1)-(2.2), we have $Y^n \leq \tilde{Y}$ a.s., $n \in \mathbb{N}$. Under additional Assumption **(H1)**, the sequence of processes (Y^n) converges increasingly and in $\mathbf{L}_{\mathbb{G}}^2(\mathbf{0}, \mathbf{T})$ to a process $Y \in \mathcal{S}_{\mathbb{G}}^2$.*

Proof. The monotonic property of the sequence (Y^n) follows from a direct application of Lemma 2.1 with $f_1 = f + n \int_{\mathcal{I}} h^- d\lambda$, $f_2 = f + (n+1) \int_{\mathcal{I}} h^- d\lambda$ and $K^2 \equiv 0$.

For any quadruple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$ satisfying (2.1)-(2.2), we obtain $Y^n \leq \tilde{Y}$ a.s., $n \in \mathbb{N}$, taking $f_1 = f_2 = f + n \int_{\mathcal{I}} h^- d\lambda$ in the previous lemma.

Under additional Assumption **(H1)**, the sequence (Y^n) is therefore monotonic and upper bounded, which ensures its monotone and in $\mathbf{L}_{\mathbb{G}}^2(\mathbf{0}, \mathbf{T})$ convergence. □

We now turn to the convergence of the triplet $(Z^n, U^n, K^n)_n$ where, for any $n \in \mathbb{N}$, the increasing process $K^n \in \mathbf{A}_{\mathbb{G}}^2$ is defined by

$$K_t^n = n \int_0^t \int_{\mathcal{I}} h^-(s, Y_s^n, Z_s^n, U_s^n(i), i) \lambda(di) ds, \quad 0 \leq t \leq T.$$

Since the convergence of the sequence $(Y^n)_n$ does not hold in $\mathcal{S}_{\mathbb{G}}^2$, we can not obtain the convergence of $(Z^n, U^n, K^n)_n$ in their respective spaces and we use a weak convergence given by the Hilbertian structure of $\mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{0}, \mathbf{T})$. The convergence of the penalized solutions to the minimal solution of (2.1)-(2.2) follows directly from an adaptation of Peng's monotonic theorem (see [15]) to BSDEs with jumps. Since the proof is essentially similar to the one detailed in Section 3 of [12], we prefer to omit it here for ease of presentation.

Theorem 2.1. *Under (H0)-(H1), there exists a unique minimal solution $(Y, Z, U, K) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$ to (2.1)-(2.2), with K predictable. Furthermore Y is the increasing limit of $(Y^n)_n$, K is the weak limit of $(K^n)_n$ in $\mathbf{L}_{\mathbb{G}}^2(\mathbf{0}, \mathbf{T})$, and*

$$\|Z^n - Z\|_{\mathbf{L}^p(\mathbf{W})} + \|U^n - U\|_{\mathbf{L}^p(\tilde{\mu})} \longrightarrow 0,$$

for any $p \in [1, 2)$, as n goes to infinity.

Remark 2.4. Observe that the only purpose of Assumption (H1) is to ensure an upper bound on the sequence $(Y^n)_n$ of solutions to the penalized BSDEs. If such an upper bound already exists, this assumption is not required anymore but will be automatically satisfied from the existence of a minimal solution to the constrained BSDE with jumps. Cases where Assumption (H1) is satisfied are presented in the next Section.

Remark 2.5. Notice that the convergence of Y^n to Y , which is obtained in a weak sense (i.e. in $\mathbf{L}_{\mathbb{G}}^2(\mathbf{0}, \mathbf{T})$) could be improved in the Markovian case (i.e. $Y^n \rightarrow Y$ in $\mathcal{S}_{\mathbb{G}}^2$). In this case we get the convergence (Z^n, U^n) to (Z, U) in $\mathbf{L}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu})$. See [8] for more details.

2.3 Link with multi-dimensional reflected Backward SDEs

In this section, we prove that multidimensional reflected BSDEs introduced by [11] and generalized by [9] are closely related to constrained BSDEs with jumps. The arguments presented here are purely probabilistic and therefore apply in the non Markovian framework considered in [11]. Furthermore, they require precise comparison results based on viability properties that are reported in the Appendix.

Recall that solving a multidimensional reflected BSDE consists in finding m triplets $(Y^i, Z^i, K^i)_{i \in \mathcal{I}} \in (\mathcal{S}_{\mathbb{F}}^{\mathbf{c}, 2} \times \mathbf{L}_{\mathbb{F}}^2(\mathbf{W}) \times \mathbf{A}_{\mathbb{F}}^2)^{\mathcal{I}}$ satisfying

$$\begin{cases} Y_t^i = \xi^i + \int_t^T \psi_i(s, Y_s^1, \dots, Y_s^m, Z_s^i) ds - \int_t^T \langle Z_s^i, dW_s \rangle + K_T^i - K_t^i \\ Y_t^i \geq \max_{j \in A_i} h_{i,j}(t, Y_t^j) \\ \int_0^T [Y_t^i - \max_{j \in A_i} \{h_{i,j}(t, Y_t^j)\}] dK_t^i = 0 \end{cases} \quad (2.10)$$

where, for all $i \in \mathcal{I}$, $\psi_i : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^d \rightarrow \mathbb{R}$ is an \mathbb{F} -progressively measurable map, $\xi^i \in \mathbf{L}^2(\Omega, \mathcal{F}_T, \mathbf{P})$, A_i is a nonempty subset of $\mathcal{I} \setminus \{i\}$, and, for any $j \in A_i$, $h_{i,j} : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function. As detailed in [11], existence and uniqueness of a solution to (2.10) is ensured by the following assumption.

(H2)

- (i) For any $i \in \mathcal{I}$ and $j \in A_i$, we have $\xi^i \geq h_{i,j}(T, \xi^j)$.
- (ii) For any $i \in \mathcal{I}$, $\mathbf{E} \int_0^T \sup_{y \in \mathbb{R}^m} |\psi_i(t, y, 0)|^2 dt + \mathbf{E} |\xi^i|^2 < +\infty$, and ψ_i is Lipschitz continuous: there exists a constant $k_\psi \geq 0$ such that

$$|\psi_i(t, y, z) - \psi_i(t, y', z')| \leq k_\psi (|y - y'| + |z - z'|), \quad \forall (i, y, z, y', z') \in \mathcal{I} \times [\mathbb{R} \times \mathbb{R}^d]^2.$$

- (iii) For any $i \in \mathcal{I}$, and $j \neq i$, ψ_i is increasing in its $(j+1)$ -th variable i.e. for any $(t, y, y', z) \in \mathcal{I} \times [\mathbb{R}^m]^2 \times \mathbb{R}^d$ such that $y_k = y'_k$ for $k \neq j$ and $y_j \leq y'_j$ we have

$$\psi_i(t, y, z) \leq \psi_i(t, y', z) \quad \mathbf{P} - a.s.$$

- (iv) For any $(i, t, y) \in \mathcal{I} \times [0, T] \times \mathbb{R}$ and $j \in A_i$, $h_{i,j}$ is continuous, $h_{i,j}(t, \cdot)$ is a 1-Lipschitz and increasing function satisfying $h_{i,j}(t, y) \leq y$. Furthermore, for any $l \in A_j$, we have $l \in A_i \cup \{i\}$ and $h_{i,l}(t, y) > h_{i,j}(t, h_{j,l}(t, y))$.

Remark 2.6. Part (ii) and (iii) of Assumption **(H2)** are classical Lipschitz and monotony properties of the driver. Part (iv) ensures a tractable form for the domain of \mathbb{R}^m where $(Y^i)_{i \in \mathcal{I}}$ lies, and (i) implies that the terminal condition is indeed in the domain,

Consider now the following constrained BSDE with jump: find a minimal quadruple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$ satisfying

$$\begin{aligned} \tilde{Y}_t &= \xi^{I_T} + \int_t^T \psi_{I_s}(s, \tilde{Y}_s + \tilde{U}_s(1)\mathbf{1}_{I_s \neq 1}, \dots, \tilde{Y}_s + \tilde{U}_s(m)\mathbf{1}_{I_s \neq m}, \tilde{Z}_s) ds + \tilde{K}_T - \tilde{K}_t \quad (2.11) \\ &\quad - \int_t^T \langle \tilde{Z}_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} \tilde{U}_s(i) \mu(ds, di), \quad 0 \leq t \leq T, \quad a.s. \end{aligned}$$

with

$$\mathbf{1}_{A_{I_t^-}}(i) \left[\tilde{Y}_{t^-} - h_{I_t^-, i}(t, \tilde{Y}_{t^-} + \tilde{U}_t(i)) \right] \geq 0, \quad d\mathbf{P} \otimes dt \otimes \lambda(di) \quad a.e. \quad (2.12)$$

where the process I is a pure jumps process defined by

$$I_t = I_0 + \int_0^t \int_{\mathcal{I}} (i - I_{s^-}) \mu(ds, di).$$

Remark that, if $\mu = \sum_{n \geq 0} \delta_{(\tau_n, A_n)}$, the process I is simply the pure jump process which coincides with A_n on each $[\tau_n, \tau_{n+1})$. This BSDE enters obviously into the class of constrained BSDEs with jumps of the form (2.1)-(2.2) studied above, with the following correspondence

$$\xi = \xi^{I_T}, \quad f(t, y, z, u) = \psi_{I_t}(t, (y + u_i \mathbf{1}_{I_t \neq i})_{i \in \mathcal{I}}, z) \quad \text{and} \quad h(t, y, z, v, i) = y - h_{I_t^-, i}(t, y + v).$$

Observe further that Assumption **(H0)** is automatically satisfied under **(H2)**, and, as detailed in the next proposition, **(H2)** also implies **(H1)** for (2.11)-(2.12) and its minimal solution can be directly related to the solution of the BSDE with oblique reflections (2.10).

Proposition 2.2. *Let Assumption **(H2)** hold and $((Y^1, Z^1, K^1), \dots, (Y^m, Z^m, K^m))$ be the solution to (2.10) then **(H1)** holds true for (2.11)-(2.12) and, if we denote $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$ the minimal solution to (2.11)-(2.12), the following equality holds*

$$\tilde{Y}_t = Y_t^{I_t}, \quad \tilde{Z}_t = Z_t^{I_t^-} \quad \text{and} \quad U_t(\cdot) = Y_t^i - Y_t^{I_t^-}.$$

In order to derive this result, we need to introduce and discuss the corresponding penalized BSDEs. For $n \in \mathbb{N}$, consider the system of penalized BSDEs: find m couples $(Y^{i,n}, Z^{i,n})_{i \in \mathcal{I}} \in (\mathcal{S}_{\mathbb{F}}^{\mathbf{c},2} \times \mathbf{L}_{\mathbb{F}}^2(\mathbf{W}))^{\mathcal{I}}$ satisfying

$$\begin{aligned} Y_t^{i,n} &= \xi^i + \int_t^T \psi_i(s, Y_s^{1,n}, \dots, Y_s^{m,n}, Z_s^{i,n}) ds - \int_t^T \langle Z_s^{i,n}, dW_s \rangle \\ &\quad + n \int_t^T \sum_{j \in A_i} [Y_s^{i,n} - h_{i,j}(s, Y_s^{j,n})]^- \lambda(j) ds, \quad 0 \leq t \leq T, \text{ a.s.} \end{aligned} \quad (2.13)$$

Lemma 2.2. *Under (H2), the sequence $(Y^{i,n})_n$ is increasing and converges to Y^i $d\mathbf{P} \otimes dt$ a.e. and in $\mathbf{L}_{\mathbb{F}}^2(\mathbf{0}, \mathbf{T})$, and the sequence $(Z^{i,n})_n$ converges weakly to Z^i in $\mathbf{L}_{\mathbb{F}}^2(\mathbf{W})$, for all $i \in \mathcal{I}$.*

Proof. For any $t \in [0, T]$, $y \in \mathbb{R}^m$ and $z \in [\mathbb{R}^d]^m$, set

$$\psi_i^n(t, y, z) := \psi_i(t, y, z_i) + n \sum_{j \in A_i} [y_i - h_{i,j}(t, y_j)]^- \lambda(j).$$

From (H2) (iii) and (iv), we have $\psi_i^n(t, y + y', z) \geq \psi_i^n(t, y, z)$ for any $y' \in (\mathbb{R}^+)^m$ such that $y'_i = 0$. Since ψ_i^n depends only on z_i , it satisfies inequality (4.23) in the Appendix. Therefore, Theorem 4.2 leads to

$$Y_t^i \geq Y_t^{i,n+1} \geq Y_t^{i,n} \quad \text{for all } (t, i, n) \in [0, T] \times \mathcal{I} \times \mathbb{N}. \quad (2.14)$$

By Peng's monotonic limit theorem, there exist m càdlàg processes $\hat{Y}^1, \dots, \hat{Y}^m \in \mathcal{S}_{\mathbb{F}}^2$, $\hat{Z}^1, \dots, \hat{Z}^m \in \mathbf{L}_{\mathbb{F}}^2(\mathbf{W})$ and $\hat{K}^1, \dots, \hat{K}^m \in \mathbf{A}_{\mathbb{F}}^2$, such that $Y^{i,n} \uparrow \hat{Y}^i$ a.e., $Y^{i,n} \rightarrow \hat{Y}^i$ in $\mathbf{L}_{\mathbb{F}}^2(\mathbf{0}, \mathbf{T})$, $Z^{i,n} \rightarrow \hat{Z}^i$ in $\mathbf{L}_{\mathbb{F}}^2(\mathbf{0}, \mathbf{T})$ weakly and

$$\hat{Y}_t^i = \xi^i + \int_t^T \psi_i(s, \hat{Y}_s^1, \dots, \hat{Y}_s^m, \hat{Z}_s^i) ds - \int_t^T \langle \hat{Z}_s^i, dW_s \rangle + \hat{K}_T^i - \hat{K}_t^i,$$

with $Y_t^i \geq \max_{j \in A_i} h_{i,j}(t, Y_t^j)$. Then, using the same arguments as in the proof of Theorem 2.4 in [9], we prove that $(\hat{Y}, \hat{Z}, \hat{K})$ is the unique solution to (2.10). \square

Proof of Proposition 2.2. For $i \in \mathcal{I}$ and $n \in \mathbb{N}$, define the processes $Y^{I,n} \in \mathcal{S}_{\mathbb{G}}^2$, $Z^{I,n} \in \mathbf{L}_{\mathbb{G}}^2(\mathbf{W})$ and $U^{I,n} \in \mathbf{L}^2(\tilde{\mu})$ by

$$Y_t^{I,n} := Y_t^{I_t, n}, \quad Z_t^{I,n} := Z_t^{I_t, n} \quad \text{and} \quad U_s^{I,n}(i) := Y_s^{i,n} - Y_{s-}^{I,n}, \quad 0 \leq t \leq T. \quad (2.15)$$

We deduce from (2.13) that $(Y^{I,n}, Z^{I,n}, U^{I,n})$ is the solution to the penalized BSDE associated to (2.11)-(2.12) given by

$$\begin{aligned} Y_t^{I,n} &= \xi^{IT} + \int_t^T \psi_{I_s}(s, Y_{s-}^{I,n} + U_s^{I,n}(1)\mathbf{1}_{I_s \neq 1}, \dots, Y_{s-}^{I,n} + U_s^{I,n}(m)\mathbf{1}_{I_s \neq m}, Z_s^{I,n}) ds \\ &\quad - \int_t^T \langle Z_s^{I,n}, dW_s \rangle + n \int_t^T \int_{\mathcal{I}} h^-(s, Y_{s-}^{I,n}, Z_s^{I,n}, U_s^{I,n}(i)) \lambda(i) ds + \int_t^T \int_{\mathcal{I}} U_s^{I,n}(i) \mu(di, ds). \end{aligned}$$

From (2.14), the sequence $(Y^{I,n})_n$ is bounded in $\mathcal{S}_{\mathbb{G}}^2$ and, proceeding as in Section 2, we prove that

$$\|Y^{I,n} - \tilde{Y}\|_{\mathbf{L}^2(0,T)} + \|Z^{I,n} - \tilde{Z}\|_{\mathbf{L}^{\mathbf{P}}(\mathbf{W})} + \|U^{I,n} - \tilde{U}\|_{\mathbf{L}^{\mathbf{P}}(\tilde{\mu})} \longrightarrow 0,$$

where $(\tilde{Y}, \tilde{Z}, \tilde{U})$ is the minimal solution to (2.11)-(2.12). Combined with (2.15) and Lemma 2.2, this concludes the proof. \square

The major interest of the previous result is the unification of the notions of BSDEs with constrained jumps introduced by [12] with the notion of BSDEs with oblique reflections considered in [9] and [11]. In particular, constrained BSDEs with jump allow to solve switching problems which are either Markovian or such that the dynamics of the forward process is not influenced by the control strategy. The purpose of the next Section is to extend this link to a more general class of non-Markovian switching problems, where in particular the dynamics of the forward process depend on the value of the current impulse control.

3 Constrained Backward SDEs with jumps and non-Markovian switching

This section is devoted to the interpretation of non-Markovian switching problems in terms of solutions to BSDEs with constrained jumps. In particular, we consider cases where the impulse strategy influences the dynamics of the forward process. To our knowledge, no BSDE representation has yet been established in this framework. We first introduce an extension to the notion of BSDEs with oblique reflections which allows for a probabilistic representation of the solution to the switching problem. Then, we relate this new class of BSDEs with oblique reflections to the class of constrained BSDEs with jumps.

3.1 Non-Markovian optimal switching

Given the set $\mathcal{I} = \{1, \dots, m\}$ and a terminal time $T < +\infty$, an impulse strategy α consists in a sequence $\alpha := (\tau_k, \zeta_k)_{k \geq 1}$, where $(\tau_k)_{k \geq 1}$ is an increasing sequence of \mathbb{F} -stopping times, and ζ_i are \mathcal{F}_{τ_i} -measurable random variables valued in \mathcal{I} . To a strategy $\alpha = (\tau_k, \zeta_k)_{k \geq 1}$ and an initial regime i_0 , we naturally associate the process $(\alpha_t)_{t \leq T}$ defined by

$$\alpha_t := \sum_{k \geq 0} \zeta_k \mathbf{1}_{[\tau_k, \tau_{k+1})}(t).$$

with $\tau_0 = 0$ and $\zeta_0 = i_0$. We denote by \mathcal{A} the set of admissible strategies. Given a strategy $\alpha \in \mathcal{A}$ and an initial condition (i_0, X_0) , we define the controlled process X^α by

$$X_t^\alpha = X_0 + \int_0^t b(s, \alpha_s, X_s^\alpha) ds + \int_0^t \sigma(s, \alpha_s, X_s^\alpha) dW_s, \quad (3.1)$$

and we consider the total profit at horizon T defined by

$$J(\alpha) := \mathbf{E} \left[g(\alpha_T, X_T^\alpha) + \int_0^T \psi(s, \alpha_s, X_s^\alpha) ds + \sum_{0 < \tau_k \leq T} c(\tau_k, \zeta_{k-1}, \zeta_k) \right]. \quad (3.2)$$

We suppose here that the functions $b, \sigma, \psi : \Omega \times [0, T] \times \mathcal{I} \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \mathbb{R}^{d \times d}, \mathbb{R}$ are progressively measurable functions and $g : \Omega \times \mathcal{I} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is $\mathcal{F}_T \otimes \sigma(\mathcal{I}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable, and that $c : \Omega \times [0, T] \times \mathcal{I} \times \mathcal{I} \rightarrow \mathbb{R}$ is a progressively measurable function.

Given an initial data (X_0, i_0) , the switching problem consists in finding a strategy $\alpha^* \in \mathcal{A}$ such that

$$J(\alpha^*) = \sup_{\alpha \in \mathcal{A}} J(\alpha)$$

Such a strategy is called optimal and we shall work under the following assumption.

(H3)

(i) b and σ satisfy the Lipschitz property: there exists a constant k such that \mathbf{P} -a.s.

$$|b(\omega, t, i, x) - b(\omega, t, i, x')| + |\sigma(\omega, t, i, x) - \sigma(\omega, t, i, x')| \leq k |x - x'|,$$

for all $(\omega, t, i, x, x') \in \Omega \times [0, T] \times \mathcal{I} \times \mathbb{R}^d \times \mathbb{R}^d$.

(ii) The terminal condition g satisfies the following structural condition

$$g(\omega, x, i) \geq \max_{j \in \mathcal{I}} \{g(\omega, x, j) + C(\omega, T, i, j)\} \quad \forall (\omega, i, x) \in \Omega \times \mathbb{R}^d \times \mathcal{I},$$

(iii) The functions g and ψ are bounded, i.e. there exists two constants \bar{g} and $\bar{\psi}$ satisfying

$$\sup_{(\omega, i, x) \in \Omega \times \mathcal{I} \times \mathbb{R}^d} \{|g(\omega, i, x)|\} \leq \bar{g} \quad \text{and} \quad \sup_{(\omega, t, i, x) \in \Omega \times [0, T] \times \mathcal{I} \times \mathbb{R}^d} \{|\psi(\omega, t, i, x)|\} \leq \bar{\psi}.$$

(iv) The cost function c is upper-bounded, i.e. there exists a constant $\bar{c} > 0$ such that

$$\max_{(\omega, t, i, j) \in \Omega \times [0, T] \times \mathcal{I} \times \mathcal{I}} c(\omega, t, i, j) \leq -\bar{c}.$$

Furthermore $c(\cdot, i, j)$ is continuous, for all $i, j \in \mathcal{I}$, and it satisfies the structural condition

$$c(\omega, t, i, l) > c(\omega, t, i, j) + c(\omega, t, j, l), \quad \forall (\omega, t, i, j, l) \in \Omega \times [0, T] \times [\mathcal{I}]^3 \text{ s.t. } j \neq i, j \neq l.$$

Remark 3.1. Part (i) of Assumption **(H3)** provides existence and uniqueness of a solution to (3.1). Part (ii) ensures the non-optimality of a switching at maturity, (iv) makes indirect switching strategy irrelevant and (iii)-(iv) ensures the problem is well posed.

Let define the set of finite strategies \mathcal{D} by

$$\mathcal{D} := \{\alpha = (\tau_k, \zeta_k)_{k \geq 1} \in \mathcal{A} \mid \mathbf{P}(\tau_k < T, k \geq 0) = 0\}$$

Let first observe the following property:

Proposition 3.1. *Under **(H3)**, the supremum of J over $\mathcal{A}_{0,i}$ coincides with the one over $\mathcal{D}_{0,i}$, that is*

$$\sup_{\alpha \in \mathcal{A}} J(\alpha) = \sup_{\alpha \in \mathcal{D}} J(\alpha), \quad \forall i \in \mathcal{I}. \quad (3.3)$$

Proof. Fix $i \in \mathcal{I}$. Consider a strategy $\alpha = (\tau_k, \zeta_k)_{k \geq 0} \in \mathcal{A} \setminus \mathcal{D}$ and define $B := \{\omega \in \Omega \mid \tau_n(\omega) < T, \forall n \in \mathbb{N}^*\}$ so that $\mathbf{P}(B) > 0$. Such a strategy is not optimal since we derive from **(H3)** (iii) and (iv) that

$$J(\alpha) \leq \bar{g} + T\bar{\psi} + \mathbf{E} \left[\sum_{0 < \tau_k \leq T} c(\tau_k, \zeta_{k-1}, \zeta_k) \mathbf{1}_B \right] + \mathbf{E} \left[\sum_{0 < \tau_k \leq T} c(\tau_k, \zeta_{k-1}, \zeta_k) \mathbf{1}_{B^c} \right] = -\infty.$$

□

3.2 Reflected BSDEs and optimal switching

Following the approach of [6], we consider in this section a family of reflected BSDE. For any couple (ν, η) with ν a stopping time valued in $[0, T]$ and η a \mathcal{F}_ν -measurable random variable taking values in \mathbb{R}^d , we consider the following reflected BSDE

$$\begin{cases} (Y^{\nu, i, \eta}, Z^{\nu, i, \eta}, K^{\nu, i, \eta})_{i \in \mathcal{I}} \in (\mathcal{S}_{\mathbb{R}}^2 \times \mathbf{L}_{\mathbb{R}}^2(\mathbf{W}) \times \mathbf{A}_{\mathbb{R}}^2)^{\mathcal{I}}, \\ Y_t^{\nu, i, \eta} = g(i, X_T^{\nu, i, \eta}) + \int_t^T \psi(s, i, X_s^{\nu, i, \eta}) \mathbf{1}_{s \geq \nu} ds - \int_t^T \langle Z_t^{\nu, i, \eta}, dW_s \rangle + K_T^{\nu, i, \eta} - K_t^{\nu, i, \eta}, \\ Y_t^{\nu, i, \eta} \geq \max_{j \in \mathcal{I}} \{Y_t^{\nu, j, \eta} + c(t, i, j)\}, \\ \int_0^T [Y_t^{\nu, i, \eta} - \max_{j \in \mathcal{I}} \{Y_t^{\nu, j, \eta} + c(t, i, j)\}] dK_t^{\nu, i, \eta} = 0, \end{cases} \quad (3.4)$$

where $X^{\nu, i, \eta}$ is a diffusion defined by

$$X_t^{\nu, i, \eta} = \eta \mathbf{1}_{t \geq \nu} + \int_0^t b(s, i, X_s^{\nu, i, \eta}) \mathbf{1}_{s \geq \nu} ds + \int_0^t \sigma(s, i, X_s^{\nu, i, \eta}) \mathbf{1}_{s \geq \nu} dW_s, \quad \forall t \geq 0. \quad (3.5)$$

Under **(H3)**, we know from [9] that (3.4) has a unique solution for any stopping time ν and any \mathcal{F}_ν -measurable random variable η , and we denote by $\mathcal{O}^{\nu, \cdot, \eta}$ its barrier defined by

$$\mathcal{O}_t^{\nu, i, \eta} := \max_{j \in \mathcal{I}} \{Y_t^{\nu, j, \eta} + c(t, i, j)\}, \quad i \in \mathcal{I}, \quad t \leq T. \quad (3.6)$$

We aim at relating the solutions to this class of reflected BSDEs to the solution of the optimal non Markovian switching problem presented in (3.2). The next proposition relates a stability property, a Snell envelope representation and a global estimate on the family of processes $(Y^{\nu, \cdot, \eta})_{(\nu, \eta)}$.

Proposition 3.2. *Assume that **(H3)** holds and take ν, ν' two \mathbb{F} -stopping times such that $\nu \leq \nu'$ and η an \mathcal{F}_ν -measurable random variable valued in \mathbb{R}^d .*

(i) *For all $i \in \mathcal{I}$ and $t \geq \nu'$, we have $Y_t^{\nu, i, \eta} = Y_t^{\nu', i, X_{\nu'}^{\nu, i, \eta}}$, \mathbf{P} -a.s.*

(ii) *For all $i \in \mathcal{I}$ and $t \geq \nu'$, we have the following representation*

$$Y_t^{\nu, i, \eta} = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbf{E} \left[\int_t^\tau \psi(s, i, X_s^{\nu, i, \eta}) \mathbf{1}_{s \geq \nu} ds + \mathcal{O}_\tau^{\nu, i, \eta} \mathbf{1}_{\tau < T} + g(i, X_T^{\nu, i, \eta}) \mathbf{1}_{\tau = T} \middle| \mathcal{F}_t \right]. \quad (3.7)$$

(iii) There exists a constant \bar{Y} such that

$$\sup_{(t,\nu,i,\eta)} |Y_t^{\nu,i,\eta}| \leq \bar{Y}. \quad (3.8)$$

Proof. (i) Notice first that $X^{\nu,i,\eta}$ and $X^{\nu',i,X_{\nu'}^{\nu,\zeta,\eta}}$ solve the same SDE, namely

$$X_{\nu'} = X_{\nu'}^{\nu,i,\eta} \quad \text{and} \quad dX_t = b(t,i,X_t)dt + \sigma(t,i,X_t)dW_t \quad \text{for } t \geq \nu'. \quad (3.9)$$

Under **(H3)** (i), equation (3.9) admits a unique solution and we have $X^{\nu',i,X_{\nu'}^{\nu,\zeta,\eta}} = X^{\nu,i,\eta}$ on $[\nu', T]$. We deduce that $(Y^{\nu',i,X_{\nu'}^{\nu,\zeta,\eta}}, Z^{\nu',i,X_{\nu'}^{\nu,\zeta,\eta}}, K^{\nu',i,X_{\nu'}^{\nu,\zeta,\eta}})_{i \in \mathcal{I}}$ satisfies the same BSDE as $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})_{i \in \mathcal{I}}$ on $[\nu', T]$. Uniqueness of solution to this BSDE is given by [9].

(ii) Regarding of (3.4), $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})$ interprets as the solution to a reflected BSDE with single barrier $\mathcal{O}^{\nu,i,\eta}$. We deduce from [7] that $Y^{\nu,i,\eta}$ admits the snell envelope representation (3.7).

(iii) For fixed ν and η , the family $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})_{i \in \mathcal{I}}$ is the solution to the BSDE with oblique reflection (3.4). We know from [9] that $(Y^{\nu,i,\eta,n}, Z^{\nu,i,\eta,n}, K^{\nu,i,\eta,n})$ converges to $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})$, where the sequence $(Y^{\nu,i,\eta,n}, Z^{\nu,i,\eta,n}, K^{\nu,i,\eta,n})_n$ is defined recursively by

$$Y_t^{\nu,i,\eta,0} = g(i, X_T^{\nu,i,\eta}) + \int_t^T \psi(s, i, X_s^{\nu,i,\eta}) \mathbf{1}_{s \geq \nu} ds - \int_t^T \langle Z_s^{\nu,i,\eta,0}, dW_s \rangle \quad \text{and} \quad K_t^{\nu,i,\eta,0} = 0,$$

and, for $n \geq 1$,

$$\begin{cases} Y_t^{\nu,i,\eta,n} = g(i, X_T^{\nu,i,\eta}) + \int_t^T \psi(s, i, X_s^{\nu,i,\zeta}) \mathbf{1}_{s \geq \nu} ds - \int_t^T \langle Z_t^{\nu,i,\eta,n}, dW_s \rangle + K_T^{\nu,i,\eta,n} - K_t^{\nu,i,\eta,n}, \\ Y_t^{\nu,i,\eta,n} \geq \max_{j \in \mathcal{I}} \{Y_t^{\nu,j,\eta,n-1} + c(t, i, j)\}, \\ \int_0^T [Y_t^{\nu,i,\eta,n} - \max_{j \in \mathcal{I}} \{Y_t^{\nu,j,\eta,n-1} + c(t, i, j)\}] dK_t^{\nu,i,\eta,n} = 0. \end{cases} \quad (3.10)$$

To derive (3.8), it suffices to prove by induction on n that

$$|Y_t^{\nu,i,\eta,n}| \leq (T - t + 1) \max\{\bar{\psi}, \bar{g}\}, \quad i \in \mathcal{I}, \quad 0 \leq t \leq T, \quad n \in \mathbb{N}.$$

First, rewriting $Y^{\nu,i,\eta,0}$ as a conditional expectation, we derive

$$|Y_t^{\nu,i,\eta,0}| \leq (T - t) \bar{\psi} + \bar{g} \leq (T - t + 1) \max\{\bar{\psi}, \bar{g}\}, \quad 0 \leq t \leq T, \quad i \in \mathcal{I}.$$

Fix $n \in \mathbb{N}$ and suppose the result is true for $Y^{\cdot,n}$. Using the representation of $Y^{\nu,i,\eta,n+1}$ as a Snell envelope, we derive

$$Y_t^{\nu,i,\eta,n+1} = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbf{E} \left[\int_t^\tau \psi(s, i, X_s^{\nu,i,\eta}) \mathbf{1}_{s \geq \nu} ds + \mathcal{O}_\tau^{\nu,i,\eta,n+1} \mathbf{1}_{\tau < T} + g(i, X_T^{\nu,i,\eta}) \mathbf{1}_{\tau = T} \middle| \mathcal{F}_t \right],$$

where $\mathcal{O}_\tau^{\nu,i,\eta,n+1} := \max_{j \in \mathcal{I}} \{Y_\tau^{\nu,j,\eta,n} + c(t, i, j)\}$. Combining this representation with Assumption **(H3)** leads to $|Y_t^{\nu,i,\eta,n+1}| \leq (T - t + 1) \max\{\bar{g}, \bar{\psi}\}$ and concludes the proof.

□

For any stopping time ν , any \mathcal{F}_ν -random variable η and any \mathcal{I} -valued random variable ζ , we naturally introduce the processes $Y^{\nu, \zeta, \eta}$ and $\mathcal{O}^{\nu, \zeta, \eta}$ by

$$Y_t^{\nu, \zeta, \eta} = \sum_{i \in \mathcal{I}} Y_t^{\nu, i, \eta} \mathbf{1}_{\zeta=i} \quad \text{and} \quad \mathcal{O}_t^{\nu, \zeta, \eta} = \sum_{i \in \mathcal{I}} \mathcal{O}_t^{\nu, i, \eta} \mathbf{1}_{\zeta=i}. \quad (3.11)$$

We are now able to state the main results of this section characterizing the optimal solution to the switching problem (3.2) in terms of reflected BSDEs.

Theorem 3.1. *Let $\alpha^* = (\tau_n^*, \zeta_n^*)_{n \geq 0}$ be the strategy given by $\alpha_0^* = (0, i_0)$ and defined recursively for $n \geq 1$ by*

$$\tau_n^* := \inf \left\{ s \geq \tau_{n-1}^* ; Y_s^{\tau_{n-1}^*, \zeta_{n-1}^*, X_{\tau_{n-1}^*}^*} = \mathcal{O}_s^{\tau_{n-1}^*, \zeta_{n-1}^*, X_{\tau_{n-1}^*}^*} \right\}, \quad (3.12)$$

$$\zeta_n^* \text{ is s.t. } \mathcal{O}_{\tau_n^*}^{\tau_{n-1}^*, \zeta_{n-1}^*, X_{\tau_{n-1}^*}^*} = Y_{\tau_n^*}^{\tau_n^*, \zeta_n^*, X_{\tau_n^*}^*} + c(\tau_n^*, \zeta_{n-1}^*, \zeta_n^*), \quad (3.13)$$

with X^* the diffusion defined by

$$X_t^* = x_0 + \sum_{n \geq 1} \int_{\tau_{n-1}^*}^{\tau_n^*} b(s, \zeta_{n-1}^*, X_s^*) \mathbf{1}_{s \leq t} ds + \int_{\tau_{n-1}^*}^{\tau_n^*} \sigma(s, \zeta_{n-1}^*, X_s^*) \mathbf{1}_{s \leq t} ds, \quad t \geq 0. \quad (3.14)$$

Under Assumption **(H3)**, the strategy α^* is optimal for the switching problem (3.2) and we have

$$Y_0^{i_0}(0, x_0) = J(\alpha^*). \quad (3.15)$$

Proof. The proof is performed in two steps.

Step 1. *The strategy $\alpha^* \in \mathcal{D}$ and satisfies $Y_0^{0, i_0, x_0} = J(\alpha^*)$.*

The representation (3.7) rewrites

$$Y_0^{0, i_0, x_0} = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_0} \mathbf{E} \left[\int_0^\tau \psi(s, i_0, X_s^{0, i_0, x_0}) ds + \mathcal{O}_\tau^{0, i_0, x_0} \mathbf{1}_{\tau < T} + g(i_0, X_T^{0, i_0, x_0}) \mathbf{1}_{\tau = T} \right]. \quad (3.16)$$

Since the boundary $\mathcal{O}^{0, i_0, x_0}$ is continuous, the stopping time τ_1^* is optimal for (3.16) and we get

$$Y_0^{0, i_0, x_0} = \mathbf{E} \left[\int_0^{\tau_1^*} \psi(s, i_0, X_s^{0, i_0, x_0}) ds + \mathcal{O}_{\tau_1^*}^{0, i_0, x_0} \mathbf{1}_{\tau_1^* < T} + g(i_0, X_T^{0, i_0, x_0}) \mathbf{1}_{\tau_1^* = T} \right].$$

If $\tau_1^* = T$, the proof is over. Let suppose that $\tau_1^* < T$ and, according to the definition of ζ_1^* , we derive

$$Y_0^{0, i_0, x_0} = \mathbf{E} \left[\int_0^{\tau_1^*} \psi(s, i_0, X_s^{0, i_0, x_0}) ds + Y_{\tau_1^*}^{\tau_1^*, \zeta_1^*, X_{\tau_1^*}^{0, i_0, x_0}} + c(\tau_1^*, i_0, \zeta_1^*) \right].$$

Similarly, we can use the representation of $Y_{\tau_1^*}^{\tau_1^*, \zeta_1^*, X_{\tau_1^*}^{0, i_0, x_0}}$ given by (3.7), and we deduce recursively that

$$Y_0^{0, i_0, x_0} = \mathbf{E} \left[\int_0^{\tau_n^*} \psi(s, i_0, X_s^*) ds + Y_{\tau_n^*}^{\tau_n^*, \zeta_n^*, X_{\tau_n^*}^*} + \sum_{0 < k \leq n} c(\tau_k, \zeta_{k-1}^*, \zeta_k^*) \right], \quad (3.17)$$

for all $n \in \mathbb{N}$ satisfying $\tau_n < T$. We now prove $\alpha^* \in \mathcal{D}$ and assume on the contrary that $p := \mathbf{P}(\tau_n^* < T, \forall n \in \mathbb{N}) > 0$. Combining **(H3)**, (3.8) and (3.17), we derive

$$Y_0(0, i_0, x_0) \leq \bar{\psi}T + \mathbf{E} \left[\sup_{s \leq T} \left| Y_s^{\tau_n^*, \zeta_n^*, X_{\tau_n^*}^*} \right| \right] - n\bar{c}\mathbf{P}(\tau_k^* < T, \forall k \geq 0) \leq \bar{\psi}T + \bar{Y} - n\bar{c}p.$$

Sending n to $-\infty$ leads to $Y_0^{0, i_0, x_0} = -\infty$ which contradicts $Y_0^{0, i_0, x_0} \in \mathcal{S}_{\mathbb{F}}^2$. Therefore $\mathbf{P}(\tau_k^* < T, \forall k \geq 0) = 0$ i.e. $\alpha^* \in \mathcal{D}$. Finally, taking the limit as $n \rightarrow \infty$ in (3.17) leads to $Y_0^{0, i_0, x_0} = J(\alpha^*)$.

Step2. *The strategy α^* is optimal.*

According to Proposition 3.1, it suffices to consider finite strategies and we pick any $\alpha = (\tau_n, \zeta_n)_{n \geq 0} \in \mathcal{D}$. Since τ_1^* is optimal, we deduce from Part (i) of Proposition 3.2 that

$$\begin{aligned} Y_0^{0, i_0, x_0} &= \mathbf{E} \left[\int_0^{\tau_1} \psi(s, i_0, X_s^{0, i_0, x_0}) ds \right] \\ &\geq \mathbf{E} \left[\mathcal{O}_{\tau_1}^{0, i_0, x_0} \mathbf{1}_{\tau_1 < T} + g(i_0, X_T^{0, i_0, x_0}) \mathbf{1}_{\tau_1 = T} \right] \\ &\geq \mathbf{E} \left[(Y_{\tau_1}^{0, \zeta_1, x_0} + c(\tau_1, i_0, \zeta_1)) \mathbf{1}_{\tau_1 < T} + g(i_0, X_T^{0, i_0, x_0}) \mathbf{1}_{\tau_1 = T} \right] \\ &\geq \mathbf{E} \left[\left(Y_{\tau_1}^{\tau_1, \zeta_1, X_{\tau_1}^{0, \zeta_1, x_0}} + c(\tau_1, i_0, \zeta_1) \right) \mathbf{1}_{\tau_1 < T} + g(i_0, X_T^{0, i_0, x_0}) \mathbf{1}_{\tau_1 < T} \right]. \end{aligned}$$

Proceeding as in step 1, an induction argument leads to

$$Y_0^{0, i_0, x_0} \geq \mathbf{E} \left[\int_0^{\tau_n} \psi(i_0, X_s^\alpha) ds + Y_{\tau_n}^{\tau_n, \zeta_n, X_{\tau_n}^\alpha} \mathbf{1}_{\tau_n < T} + g(\zeta_n, X_T^\alpha) \mathbf{1}_{\tau_n = T} + \sum_{0 < k \leq n} c(\tau_k, \zeta_{k-1}, \zeta_k) \right],$$

for all n satisfying $\tau_n < T$. Since the strategy α is finite, we deduce $Y_0^{0, i_0, x_0} \geq J(\alpha)$ by sending $n \rightarrow \infty$. \square

3.3 Approximation by penalisation and link with constrained BSDEs with jumps

We finally present in this paragraph the link between constrained Backward SDEs with jumps and optimal switching in non-Markovian cases. Consider the constrained BSDE with jumps: find a quadruple $(Y, Z, U, K) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$ satisfying

$$\begin{aligned} Y_t &= g(I_T, X_T^I) + \int_t^T \psi(s, I_s, X_s^I) ds + K_T - K_t \\ &\quad - \int_t^T \langle Z_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(ds, di), \quad 0 \leq t \leq T, \text{ a.s.} \end{aligned} \quad (3.18)$$

with

$$-U_t(i) - c(t, I_{t-}, i) \geq 0, \quad d\mathbf{P} \otimes dt \otimes \lambda(di) \text{ a.e.} \quad (3.19)$$

where the process (I, X^I) is defined by

$$\begin{aligned} I_t &= i_0 + \int_0^t \int_{\mathcal{I}} (i - I_{t-}) \mu(dt, di) \\ X_t^I &= x_0 + \int_0^t b(s, I_s, X_s^I) ds + \int_0^t \sigma(s, I_s, X_s^I) dW_s \end{aligned}$$

The link between (3.18)-(3.19) and the optimal switching problem is given by the following result which extends the link between BSDEs with constrained jumps and BSDEs with oblique reflections presented in Proposition 2.2.

Proposition 3.3. *Under **(H3)**, **(H1)** holds for (3.18)-(3.19) and if we denote (Y, Z, U, K) its minimal solution we have*

$$Y_t = Y_t^{t, I_t, X_t^I}, \quad Z_t = Z_t^{t, I_{t-}, X_t^I} \quad \text{and} \quad U_t(i) = Y_t^{t, I_t, X_t^I} - Y_{t-}^{t, I_{t-}, X_t^I}. \quad (3.20)$$

In particular, we deduce $Y_0 = J(\alpha^*) = \sup_{\alpha \in \mathcal{A}} J(\alpha)$.

Proof. For any stopping time ν and any random variable η , let define the processes $(\tilde{Y}^{\nu, i, \eta, n}, \tilde{Z}^{\nu, i, \eta, n}, \tilde{K}^{\nu, i, \eta, n})_{i \in \mathcal{I}} \in (\mathcal{S}_{\mathbb{F}}^2 \times \mathbf{L}_{\mathbb{F}}^2(\mathbf{W}) \times \mathbf{A}_{\mathbb{F}}^2)^{\mathcal{I}}$ as the solution to the penalized BSDE

$$\begin{aligned} \tilde{Y}_t^{\nu, i, \eta, n} &= g(i, X_T^{\nu, i, \eta}) + \int_t^T \psi(s, i, X_s^{\nu, i, \eta}) ds - \int_t^T \langle \tilde{Z}_s^{\nu, i, \eta, n}, dW_s \rangle \\ &\quad + n \int_t^T \left\{ \sum_{j \in \mathcal{I}} [\tilde{Y}_s^{\nu, j, \eta, n} + c(s, i, j) - \tilde{Y}_s^{\nu, i, \eta, n}]^- \lambda(j) \right\} ds \end{aligned}$$

Under **(H3)**, we know from Lemma 2.2 (or [11]) that $(Y^{\nu, i, \eta, n}, Z^{\nu, i, \eta, n}, K^{\nu, i, \eta, n})_{i \in \mathcal{I}}$ converges to $(Y^{\nu, i, \eta}, Z^{\nu, i, \eta}, K^{\nu, i, \eta})_{i \in \mathcal{I}}$ as n goes to ∞ , for each (ν, η) . Proceeding as in the proof of Proposition 2.2, one easily checks that the quadruple $(\tilde{Y}^n, \tilde{Z}^n, \tilde{U}^n)$ defined by

$$\tilde{Y}_t^n = \tilde{Y}_t^{t, I_t, X_t^I, n}, \quad \tilde{Z}_t^n = \tilde{Z}_t^{t, I_{t-}, X_t^I, n} \quad \text{and} \quad \tilde{U}_t^n(i) = \tilde{Y}_t^{t, I_t, X_t^I, n} - \tilde{Y}_{t-}^{t, I_{t-}, X_t^I, n}$$

is solution to the penalized BSDE associated to (3.18)-(3.19), namely

$$\begin{aligned} Y_t &= g(I_T, X_T^I) + \int_t^T \psi(s, I_s, X_s^I) ds + n \int_t^T \int_{\mathcal{I}} [U_s(i) + c(s, I_{s-}, i)]^- \lambda(di) ds \\ &\quad - \int_t^T \langle Z_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(ds, di), \quad 0 \leq t \leq T. \text{ a.s.} \end{aligned}$$

From Proposition 3.2 (iii), we know that the monotone sequence $(\tilde{Y}^n)_n$ is bounded, and we derive from Remark 2.4 that Assumption **(H1)** is satisfied and that $(\tilde{Y}^n, \tilde{Z}^n, \tilde{U}^n)$ converges to (Y, Z, U) , which concludes the proof. \square

Remark 3.2. The optimal strategy can be described by the constrained BSDE with jumps (3.18)-(3.19). Indeed, using the definition of $(\tau_n^*, \zeta_n^*)_{n \geq 0}$ and the identification (3.20) we get

$$\tau_{n+1}^* = \inf \left\{ t \geq \tau_n^* ; \max_{j \in \mathcal{I}} \mathbf{E} \left[U_t(j) - c(t, \zeta_n^*, j) \middle| I_s = \zeta_n^* \quad \forall s \geq \tau_n^* \right] = 0 \right\}$$

and ζ_{n+1} such that

$$\mathbf{E} \left[U_{\tau_{n+1}^*}(\zeta_{n+1}^*) - c(\tau_{n+1}^*, \zeta_n^*, \zeta_{n+1}^*) \middle| I_s = \zeta_n^* \quad \forall s \geq \tau_n \right] = 0.$$

4 Appendix

4.1 Viability property for BSDEs

We extend here the viability property of [4] for a closed convex cone \mathcal{C} of \mathbb{R}^m . Let $(Y, Z) \in (\mathcal{S}_{\mathbb{F}}^{c,2} \times \mathbf{L}_{\mathbb{F}}^2(\mathbf{W}))^m$ satisfying

$$Y_t = Y_T + \int_t^T F(s, Y_s, Z_s) ds - \int_t^T \langle Z_s, dW_s \rangle + K_T - K_t$$

where $F : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^m$ is a progressively measurable function satisfying **(H2)** (i) and (ii) and K is an \mathbb{R}^m -valued finite variation process such that

$$K_t = \int_0^t k_s d|K|_s,$$

with $k_t \in \mathcal{C}$ and $|K|_s$ the variation of K on $[0, s]$. Denote $d_{\mathcal{C}}$ the distance to \mathcal{C} (i.e. $d_{\mathcal{C}}(x) = \min_{y \in \mathcal{C}} |x - y|$) and $\Pi_{\mathcal{C}}$ the projection operator onto \mathcal{C} , then we have the following result:

Proposition 4.4. *Suppose $Y_T \in \mathcal{C}$ and there exists a constant C^0 such that F satisfies*

$$4 \langle y - \Pi_{\mathcal{C}}(y), F(t, y, z) \rangle \leq \langle D^2 |d_{\mathcal{C}}|^2(y) z, z \rangle + 2C^0 |d_{\mathcal{C}}|^2(y) \quad (4.21)$$

for any point $y \in \mathbb{R}^m$ where $|d_{\mathcal{C}}|^2$ is twice differentiable. Then, we have

$$Y_t \in \mathcal{C}, \quad \text{for all } t \in [0, T] \quad \mathbf{P} - a.s.$$

Proof. Let $\eta \in C^\infty(\mathbb{R}^d)$ be a non-negative function with support in the unit ball and such that $\int_{\mathbb{R}^d} \eta(x) dx = 1$. For $\delta > 0$ and $x \in \mathbb{R}^d$ we put

$$\begin{aligned} \eta_\delta(x) &:= \frac{1}{\delta^d} \eta\left(\frac{x}{\delta}\right) \\ \phi_\delta(x) &:= |d_{\mathcal{C}}|^2 \star \eta_\delta(x) = \int_{\mathbb{R}^d} |d_{\mathcal{C}}(x - x')|^2 \eta_\delta(x') dx' \end{aligned}$$

Notice (see part (b) of the proof of Theorem 2.5 in [4]) that $\phi_\delta \in C^\infty(\mathbb{R}^d)$ and

$$\begin{cases} 0 \leq \phi_\delta(x) \leq (d_{\mathcal{C}}(x) + \delta)^2 \\ D\phi_\delta(x) = \int_{\mathbb{R}^d} D|d_{\mathcal{C}}(x')|^2 \eta_\delta(x - x') dx' \quad \text{and} \quad |D\phi_\delta(x)| \leq 2(d_{\mathcal{C}}(x) + \delta) \\ D^2\phi_\delta(x) = \int_{\mathbb{R}^d} D^2|d_{\mathcal{C}}(x')|^2 \eta_\delta(x - x') dx' \quad \text{and} \quad 0 \leq |D^2\phi_\delta(x)| \leq 2I_d \end{cases} \quad (4.22)$$

Applying Itô's formula to $\phi_\delta(Y_t)$, this leads to

$$\begin{aligned}
\mathbf{E}\phi_\delta(Y_t) &= \mathbf{E}\phi_\delta(Y_T) + \mathbf{E} \int_t^T \langle D\Phi_\delta(Y_s), F(s, Y_s, Z_s) \rangle ds - \frac{1}{2} \mathbf{E} \int_t^T \langle \Phi_\delta(Y_s) Z_s, Z_s \rangle ds \\
&\quad + \mathbf{E} \int_t^T \langle D\Phi_\delta(Y_s), k_s \rangle d|K|_s \\
&\leq \delta^2 + \mathbf{E} \int_t^T \int_{\mathbb{R}^d} \left[\langle D|d_{\mathcal{C}}(y)|^2, F(s, y, Z_s) \rangle - \frac{1}{2} \langle D^2|d_{\mathcal{C}}(y)|^2 Z_s, Z_s \rangle \right] \eta_\delta(Y_s - y) dy ds \\
&\quad - \mathbf{E} \int_t^T \int_{\mathbb{R}^d} \langle D|d_{\mathcal{C}}(y)|^2, F(s, y, Z_s) - F(s, Y_s, Z_s) \rangle \eta_\delta(Y_s - y) dy ds \\
&\quad + \mathbf{E} \int_t^T \int_{\mathbb{R}^d} \langle D|d_{\mathcal{C}}(y)|^2, k_s \rangle \eta_\delta(Y_s - y) dy d|K|_s,
\end{aligned}$$

for $0 \leq t \leq T$, $\delta > 0$. Since $k_t \in \mathcal{C}$ and \mathcal{C} is a closed convex cone, we have $\langle D|d_{\mathcal{C}}(y)|^2, k_s \rangle \leq 0$. Then, combining (4.21) with inequality $2d_{\mathcal{C}}(\cdot) \leq 1 + d_{\mathcal{C}}(\cdot)^2$, we get

$$\begin{aligned}
\mathbf{E}\phi_\delta(Y_t) &\leq \delta^2 + C^0 \mathbf{E} \int_t^T \int_{\mathbb{R}^d} |d_{\mathcal{C}}(y)|^2 \eta_\delta(y - Y_s) dy ds \\
&\quad + 2\mathbf{E} \int_t^T \int_{\mathbb{R}^d} d_{\mathcal{C}}(y) \eta_\delta(Y_s - y) \max_{y': |y' - Y_s| \leq \delta} |F(s, y', Z_s) - F(s, Y_s, Z_s)| dy ds \\
&\leq \delta^2 + C^0 \int_t^T \mathbf{E}\phi_\delta(Y_s) ds + \mathbf{E} \int_t^T (1 + \phi_\delta(Y_s)) \max_{y': |y' - Y_s| \leq \delta} |F(s, y', Z_s) - F(s, Y_s, Z_s)| ds.
\end{aligned}$$

Using the Lipschitz property of F , we deduce

$$\mathbf{E}\phi_\delta(Y_t) \leq C(\delta^2 + \delta + \int_t^T \mathbf{E}\phi_\delta(Y_s) ds),$$

and Gronwall's lemma leads to

$$\mathbf{E}\phi_\delta(Y_t) \leq C(\delta^2 + \delta), \quad 0 \leq t \leq T, \quad \delta > 0.$$

Finally, from Fatou's Lemma, we have

$$\mathbf{E}|d_{\mathcal{C}}(Y_t)|^2 \leq \liminf_{\delta \rightarrow 0} \mathbf{E}\phi_\delta(Y_t) = 0, \quad 0 \leq t \leq T,$$

which concludes the proof. \square

4.2 A multi-dimensional comparison theorem for BSDEs

We now turn to the obtention of a multi-dimensional comparison result. Consider $(Y^1, Z^1, K) \in (\mathcal{S}_{\mathbb{F}}^{c,2} \times \mathbf{L}_{\mathbb{F}}^2(\mathbf{W}) \times \mathbf{A}_{\mathbb{F}}^2)^m$ satisfying

$$Y_t^1 = Y_T^1 + \int_t^T F_1(s, Y_s^1, Z_s^1) ds - \int_t^T \langle Z_s^1, dW_s \rangle + K_T - K_t$$

and $(Y^2, Z^2) \in (\mathcal{S}_{\mathbb{F}}^{c,2} \times \mathbf{L}_{\mathbb{F}}^2(\mathbf{W}))^m$ satisfying

$$Y_t^2 = Y_T^2 + \int_t^T F_2(s, Y_s^2, Z_s^2) ds - \int_t^T \langle Z_s^2, dW_s \rangle$$

Then we have the following comparison theorem generalizing the one in [10].

Theorem 4.2. *Suppose that $Y_T^1 \geq Y_T^2$ and that, for any $(y, y', z, z') \in [\mathbb{R}^m]^2 \times [\mathbb{R}^{m \times d}]^2$, we have*

$$-4\langle y^-, F_1(t, y^+ + y', z) - F_2(t, y', z') \rangle \leq 2 \sum_{i=1}^m \mathbf{1}_{y_i < 0} |z_i - z'_i|^2 + 2C^0 |y^-|^2 \mathbf{P} - a.s. (4.23)$$

where $C^0 > 0$ is a constant. Then $Y_t^1 \geq Y_t^2$, for all $t \in [0, T]$.

Proof. As in Theorem 2.1 in [10], it suffices to remark that F^1 is Lipschitz and apply Proposition 4.23 to the couple $(Y^1 - Y^2, Y^2)$ and the closed convex cone $\mathcal{C} = (\mathbb{R}^+)^m \times \mathbb{R}^m$. \square

References

- [1] BARLES G., BUCKDAHN R. AND E. PARDOUX (1997) Backward stochastic differential equations and integral-partial differential equations, *Stochastics and Stochastics Reports*, **60**, 57-83.
- [2] BOUCHARD B. (2006) A stochastic target formulation for optimal switching problems in finite horizon, *To appear in Stochastics*.
- [3] Bouchar B. and R. Elie (2008) : “Discrete-time approximation of decoupled forward-backward SDE with jumps”, *Stochastic Processes and their Applications*, **118**, 53-75.
- [4] BUCKDAHN R., M. QUINCAMPOIX AND A. RASCANU (2000) Viability property for a backward stochastic differential equation and applications to partial differential equations, *Probab. Theory. Relat. Fields*, **116**, 485-504.
- [5] CVITANIC J., KARATZAS I. AND M. SONER (1998) Backward stochastic differential equations with constraints on the gain-process, *Annals of Probability*, **26**, 1522-1551.
- [6] DJEHICHE B., HAMADÈNE S. AND I. HDHIRI (2008) Stochastic Impulse Control of Non-Markovian Processes. Preprint.
- [7] EL KAROUI N., KAPOUDJIAN C., PARDOUX E., PENG S. AND M.C. QUENEZ (1997) Reflected solutions of Backward SDE’s, and related obstacle problems for PDEs, *Annals of Probability*, **25**, 702-737.
- [8] ELIE R. AND I. KHARROUBI (2009) Constrained BSDEs with jumps and systems of variational inequalities, *In preparation*.
- [9] HAMADÈNE S. AND J. ZHANG (2008) The Starting and Stopping Problem under Knightian Uncertainty and Related Systems of Reflected BSDEs. Preprint.
- [10] HU Y. AND S. PENG (2006) On comparison theorem for multi-dimensional BSDEs’, *C. R. Acad. Sci. Paris*, **343**, 135-140.

- [11] HU Y. AND S. TANG (2007) Multi-dimensional BSDE with oblique Reflection and optimal switching, To appear in *Prob. Theory and Related Fields*.
- [12] KHARROUBI I., MA J., PHAM H. AND J. ZHANG (2008) Backward SDEs with constrained jumps and Quasi-Variational Inequalities, Preprint.
- [13] PARDOUX E. AND S. PENG (1990) Adapted solution of a backward stochastic differential equation, *Systems and Control Letters*, **14**, 55-61.
- [14] PARDOUX E., F. PRADEILLES AND Z. RAO (1997) Probabilistic interpretation of a system of semi-linear parabolic partial differential equations, *Ann. Inst. Henri Poincaré*, section B, **33** (4), 467-490.
- [15] PENG S. (1999) Monotonic limit theory of BSDE and nonlinear decomposition theorem of Doob-Meyer's type, *Prob. Theory and Related Fields*, **113**, 473-499.
- [16] PENG S. AND M. XU (2007) Constrained BSDE and viscosity solutions of variation inequalities, Preprint.
- [17] ROYER M. (2006) Backward stochastic differential equations with jumps and related nonlinear expectations, *Stochastic Processes and their Applications*, **116**, 1358-1376.
- [18] TANG S. AND X. LI (1994) Necessary conditions for optimal control of stochastic systems with jumps, *SIAM J. Control and Optimization*, **32**, 1447-1475.
- [19] TANG S. AND J. YONG (1993) Finite horizon stochastic optimal switching and impulse controls with a viscosity solution approach, *Stochastics and Stochastic Reports*, **45**, 145-176.