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SOME REMARKS ON THE MEAN VALUE OF THE RIEMANN ZETA-FUNCTION AND OTHER DIRICHLET SERIES—I

By K. RAMACHANDRA

§ 1. Introduction

In the last section of my paper [2], I raised some questions on the mean value of generalised Dirichlet series. It looks too ambitious unless we limit ourselves to Dirichlet series $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ ($s = \sigma + it$) and that too with some restrictions. We do not claim to solve all the problems raised. Let $F(s)$ be convergent absolutely somewhere in the complex plane and let $F(s)$ admit an analytic continuation in $\sigma > \frac{1}{2}$, $t > \tau_0$ and there $F(s) = O(t^A)$. If σ_1 is large enough, $F(s)$ is zero free in $\sigma > \sigma_1$ and we define $(F(s))^{2k}$ (for any positive real number $2k$) as the analytic continuation of $(F(s))^{2k}$ ($\sigma > \sigma_1$) along lines parallel to the real axis. (If such lines contain a zero of $F(s)$ we do not define $(F(s))^{2k}$ on such lines). Let δ be a positive constant not exceeding $\frac{1}{10}$ and m a non-negative integer. Let $T > T_0(\delta)$ be a real variable and H a real variable subject to $(\log T)^\delta < H < T$. Imposing the conditions $\delta < 2k < \delta^{-1}$ and $0 < m < \delta^{-1}$ we define for $\sigma > \frac{1}{2}$,

$$Q(\sigma) = \frac{1}{H} \int_T^{T+H} \left| \frac{d^m}{ds^m} (F(s))^{2k} \right| dt.$$

Our main problem is to study lower bounds for $Q(\sigma)$. The only known progress so far, in this direction is a theorem of Ingham which gives for fixed $\sigma > \frac{1}{2}$ and fixed $2k$ ($0 < 2k < 4$, $m = 0$) an asymptotic formula for $Q(\sigma)$ in the special case $F(s) = \zeta(s)$, $H = T$. Ingham's proof was complicated and

Davenport gave a simpler proof of Ingham's Theorem (for references see [5]). Our first object in this note is to give in § 3, a satisfactory lower bound for $\max Q(\sigma)$ as σ runs over all real numbers $\geq \frac{1}{2} + \frac{\log \log H}{\log H}$. However our proof of lower bound depends very much on the existence of an "Euler product" for $F(s)$. (The Euler product condition is general enough to include the case when $F(s)$ is the zeta-function and Hecke L-series of algebraic number fields). From such a theorem we can also get satisfactory lower bounds for $Q(\frac{1}{2})$ as will be seen. Two simple samples of our general results (subject to the condition $2k > 1$) are

$$(*) \frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k} dt > (\log H)^{k^2} (\log \log H)^{-C}$$

$$(**) \frac{1}{H} \int_T^{T+H} |\zeta'(\frac{1}{2} + it)| dt > (\log H)^{\frac{5}{2}} (\log \log H)^{-C}$$

where C is a constant depending only on δ . Our next object is to deal (in § 4 and § 5) with the case when $F(s)$ has no Euler product. Here we are forced, for lack of better ideas, to limit ourselves to the case $2k = 1$. To compensate for the generality the lower bounds we obtain for $Q(\sigma)$ are not so satisfactory, but they are still, I hope, of some interest.

The problem of upper bounds for $Q(\sigma)$ seems hopelessly difficult. Even with the assumption of Riemann hypothesis it is not clear how to improve the trivial inequality

$$(***) \frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)| dt = O((\log T)^{\frac{1}{2}}).$$

Foot-Note: The publication of this paper which was ready by the middle of 1977 was delayed due to various reasons. In the meanwhile I have replaced $(\log \log H)^{-C}$ in (*) and (**) by $\frac{1}{C}$. Next I have replaced $O((\log T)^{\frac{1}{2}})$ by $O((\log T)^{\frac{1}{2}})$ in (***) unconditionally. These (and other) results will appear in papers II and III with the same title as the present paper.

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§ 2. Notation

The letter C with or without subscripts denotes a positive constant depending only on δ . The letter K with or without subscripts denotes constants to be chosen later appropriately in a proof. Also the constant C may not be the same at each occurrence.

§ 3. The case of Euler product

Let in $\sigma > 1$, $F(s)$ be defined by

$$F(s) = \prod_p \left(1 - \frac{x(p)}{p^s} \right)^{-a_p} \quad (p \text{ runs over all primes})$$

where $\{x(p)\}$ and $\{a_p\}$ are bounded sequences of complex numbers, and $|x(p)| < p^{\frac{1}{2}}$ for every prime p . We assume either of the following conditions on $F(s)$.

(i) There exists a constant p_0 such that whenever $a_p \neq 0$ and $p > p_0$ we have $|a_p| \geq a$, where a is a positive constant.

Further the function $\prod_p \left(1 - \frac{|x(p)|^2}{p^s} \right)^{-|a_p|}$ which is analytic in $\sigma > 1$ can be continued analytically in a neighbourhood of $s = 1$ and has a simple pole at $s = 1$.

(ii) Suppose that as $\sigma \rightarrow \frac{1}{2} + 0$, $\left(\prod_p \left(1 + \frac{|x(p) a_p|^2}{p^{2\sigma}} \right) \right)$

lies between $\left(\log \frac{1}{\sigma - \frac{1}{2}} \right)^{b_1}$ and $\left(\log \frac{1}{\sigma - \frac{1}{2}} \right)^{b_2}$

where $a > 0$ and b_1, b_2 are three real constants.

We can now state

Theorem 1

With the notation explained already,

$$\max_{\sigma \geq \frac{1}{2} + \frac{\log \log H}{\log H}} Q(\sigma) > (\log H)^{ak^a + m_1} (\log \log H)^{-c}$$

where C is a positive constant, and $m_1 = m$ or $\min(m, 1)$ according as we have (ii) or (i).

As a corollary we can deduce the results for $\zeta(\frac{1}{2} + it)$ quoted in the introduction. More generally we have

Theorem 2

Let $L(s)$ denote either the Dedekind zeta-function or the Hecke L -series of an algebraic number field of degree n and according as it is Galois or not we put $a = n$ or 1 and also $m_1 = m$ or $\min(1, m)$. Then we have with $2k \geq 1$,

$$\frac{1}{H} \int_T^{T+H} |L(\frac{1}{2} + it)^{2k} dt| > (\log H)^{ak^a} (\log \log H)^{-c}$$

$$\frac{1}{H} \int_T^{T+H} |L^{(m)}(\frac{1}{2} + it)| dt > (\log H)^{\frac{a}{2} + m_1} (\log \log H)^{-c}$$

where C is a positive constant. Further the condition $2k \geq 1$ is unnecessary if we assume the hypothesis that $L(s) \neq 0$ for $\sigma > \frac{1}{2}$.

Remark: Theorem 2 gives an improvement on theorem 3 of my paper [3]. It will be seen later that m_1 can also be replaced by $m_a = \max(m_1, mn^{-1})$.

Deduction of Theorem 2 from Theorem 1

Let $F(s) = L(s)$. We put $s = \sigma + it$, $w = u + iv$, where σ is the number at which $\max Q(\sigma)$ is attained. We limit ourselves to the case $2k \geq 1$, $m = 0$ or $2k = 1$, $m = 1$. The proof in both the cases are similar and we consider the first case $2k \geq 1$, $m = 0$. We impose $T + \frac{H}{4} \leq t \leq T + \frac{3H}{4}$.

If now b is an odd positive integer which is fixed to be a large integer depending on δ , we have by Cauchy's Theorem

$$F(s) = \frac{1}{2\pi i} \int_R F(W) e^{(W-s)^{2b}} \frac{dW}{W-s}$$

where R is the rectangle with corners $\frac{1}{2} + i(T + H)$, $\frac{1}{2} + iT$, $2 + iT$, $2 + i(T + H)$. Because b is large we can check that the horizontal portions contribute a bounded quantity to the integral. The same is trivially true of the line $u = 2$. Thus

$$\begin{aligned} |F(s)|^{2k} &= O\left(\left(\int_{u=\frac{1}{2}} \left|F(W) e^{(W-s)^{2b}} \frac{dW}{W-s}\right|\right)^{2k} + 1\right) \\ &= O\left(1 + \int_{u=\frac{1}{2}} \left|F(W)^{2k} e^{(W-s)^{2b}} \frac{dW}{W-s}\right|\right. \\ &\quad \left.\left(\int_{u=\frac{1}{2}} \left|e^{(W-s)^{2b}} \frac{dW}{W-s}\right|\right)^{2k-1}\right) \end{aligned}$$

Theorem 2 now follows on integrating with respect to s and also using the fact that $\sigma - \frac{1}{2} > \frac{\log \log H}{\log H}$. We have to remember that if the field is Galois, condition (ii) is satisfied; otherwise the condition (i) is satisfied.

To prove Theorem 1, we assume that it is false with $C = 1$.

We now proceed to prove by a series of lemmas the truth of the theorem for some $C > 1$. We can certainly choose the latter constant and this would prove the Theorem 1. Accordingly we begin with

Lemma 1: *We have*

$$\sigma > \frac{1}{2} + \frac{\max \log \log H}{\log H} \frac{1}{H} \int_T^{T+H} |F(\sigma + it)|^{2k} dt < (\log H)^{ok^2 + m} \log \log H.$$

Proof: Trivial since, for

$$0 \leq j \leq \delta^{-1}, \left| \frac{d^j}{dt^j} (F(2 + it))^{2k} \right| \text{ is bounded.}$$

Lemma 2: *The maximum of $|F(\sigma + it)|$ taken over all*

$$\sigma > \frac{1}{2} + \frac{2 \log \log H}{\log H}, T+1 \leq t \leq T+H-1, \text{ does not exceed } H^2.$$

Proof: Follows from the fact $|F(s)|^{2k}$ is subharmonic. However we supply a proof. If $F(s) \neq 0$ in $|s - s_0| \leq r$

$$\text{we have } \log |F(s_0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(s_0 + r e^{i\theta})| d\theta.$$

The first quantity is less than the second if $F(s_0) = 0$. Assuming now $F(s_0) \neq 0$ and defining

$$\phi(s) = F(s) \pi \left(\frac{r^2 - (\bar{\rho} - \bar{s}_0)(s - s_0)}{(s - \rho)r} \right)$$

where ρ runs over all the zeros of $F(s)$ in $|s - s_0| \leq r$, we see that $\log |F(s_0)| \leq \log |\phi(s_0)|$ and that on $|s - s_0| = r$ we have $|F(s)| = |\phi(s)|$. This proves

$$\log |F(s_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |F(s_0 + r e^{i\theta})| d\theta.$$

This gives easily

$$\log |F(s_0)| \leq \frac{1}{\pi r^2} \int_{|s-s_0| \leq r} \log |F(s)| dv$$

where dv is the element of area of the disc. We multiply this by $2k$ and apply the arithmetico-geometric inequality (in the limiting form to suit integrals) and we obtain

$$|F(s_0)|^{2k} \leq \frac{1}{\pi r^2} \int_{|s-s_0| \leq r} |F(s)|^{2k} dv.$$

By taking a suitable radius say $\frac{\log \log H}{\log H}$ we get the lemma.

Lemma 3: Let $N(\alpha, T_1, T_2)$ denote the number of zeros of $F(s)$ in $\sigma > \alpha$ $\left(\alpha > \frac{1}{2} + \frac{3 \log \log H}{\log H} \right)$ and $T_1 \leq t < T_2$.

Then if $T + (\log H)^8 \leq t < T + H - (\log H)^8$,

$$N(\alpha, t, t+1) \leq (\log H)^4.$$

Proof: Follows from Jensen's inequality

(see page 126 of [6])

$$\int_0^r \frac{n(x)}{x} dx = \frac{1}{2\pi} \int_0^{2\pi} \log | F(2 + it + re^{i\theta}) | d\theta - \log | F(2 + it) |,$$

where $n(x)$ denotes the number of zeros of $F(s)$ in a disc of radius x with centre $2 + it$. We have to select a suitable r and use Lemma 2. Note that if $0 < x_1 < r$ we have

$$\int_0^r \frac{n(x)}{x} dx \geq n(x_1) \log \frac{r}{x_1}.$$

Lemma 4: For

$$\alpha \geq \frac{1}{2} + \frac{3 \log \log H}{\log H}, T_1 = T + (\log H)^8, \\ T_2 = T + H - (\log H)^8,$$

we have $N(\alpha, T_1, T_2) < H^1 - C_1(\alpha - \frac{1}{2})(\log H)^{C_2}$.

Proof: We select a "well-spaced" system of zeros connected in $N(\alpha, T_1, T_2)$ and proceed to estimate their number by the zero detecting function $F(s) M_H(s) - 1$ where $M_H(s)$ is the sum of the first H terms of the Dirichlet series for $(F(s))^{-1}$. Note that if $\phi_1(s)$ is the zero detecting function in question and $Re s = 1$ then

$$\frac{1}{2\pi i} \int_{Re W=2} \phi_1(s+W) \Gamma(W) X^W dW$$

is a very good approximation to $\phi_1(s)$ if we set $X = H^{C_3}$ where C_3 is a large constant. Fairly routine considerations lead to the lemma. (An excellent reference to our ideas of deducing "density estimates" which was also found by Gallagher in a more perfect form is Gallagher's paper [1]). In the proof we have of course to use Lemma 1.

Lemma 5: Let now $\alpha \geq \alpha_0 = \frac{1}{2} + \frac{K_1 \log \log H}{\log H}$ where K_1 is a large constant. Then $N(\alpha, T_1, T_2) < H(\log H)^{-K_2}$ where $K_2 \rightarrow \infty$ as $K_1 \rightarrow \infty$. (Hereafter T_1, T_2 will be as in Lemma 4 and α_0 as in this Lemma).

Proof: This lemma is a Corollary to Lemma 4.

We next divide the t interval $T_1 \leq t \leq T_2$ into equal intervals of length $(\log H)^{K_3}$ ($K_3 \geq 10$) ignoring a small bit at one end. Let I_1 run through those intervals which do not contain a zero in $\sigma \geq \frac{1}{2} + \frac{K_1 \log \log H}{\log H}$ and I_2 the rest of the intervals. Let I_3 run through the intervals I_1 with t intervals of length $(\log H)^2$ removed both above and below. Plainly the intervals I_3 cover the interval $T_1 \leq t \leq T_2$ except certain bits of total length not exceeding $H(\log H)^{-K_2} + H(\log H)^{2-K_3}$. We put $s_0 = \alpha + it$ (α fixed and $\geq \alpha_0 + \frac{\log \log H}{\log H}$) and set out to obtain an asymptotic formula for

$$Q_1(\alpha) = \sum_{I_3} \int_{I_3} |F(s_0)|^{2k} dt.$$

We prove

Lemma 6: Let $K_2 = K_3$ and $\alpha \geq \alpha_0 + \frac{\log \log H}{\log H}$ where $\alpha_0 = \frac{1}{2} + \frac{K_1 \log \log H}{\log H}$. Then we have,

$$Q_1(\alpha) = H \sum_{n=1}^{\infty} |d_k(n)|^2 n^{-2\alpha} + O(H(\log H)^{C_4 - \frac{1}{2}K_2}) \\ + O(H(\log H)^{C_4} H^{-\frac{1}{8}(\alpha - \alpha_0)}),$$

where $d_k(n)$ are defined by the expansion $(F(s))^k = \sum_{n=1}^{\infty} d_k(n) n^{-s}$ valid in $\sigma > 1$. Also $K_2 \rightarrow \infty$ as $K_1 \rightarrow \infty$.

Proof: The proof of this lemma is nearly standard. We will merely sketch the proof. For t in I_3 we have with $X = H^{\frac{1}{2}}$ and the notation introduced already,

$$F(s_0) = P(t) + E, \quad P(t) = \sum_{n=1}^{\infty} d_k(n) n^{-s_0} e^{-\frac{n}{x}}$$

where $E = \frac{1}{2\pi i} \int (F(s_0+W))^k \Gamma(W+1) X^W \frac{dW}{W} + O(H^{-10})$,
 $|ImW| \leq (\log H)^2, \quad Re(W+\alpha) = \alpha_0$

From Lemma 1, $\int_{I_3} |E|^2 dt = O(H(\log H)^{C_5} X^{-\frac{1}{2}(\alpha-\alpha_0)})$

Next we note that $|(F(s_0))^k|^2 = |P(t)|^2 + O(|(F(s_0))^k - P(t)|^2)$
 $+ O(|(F(s_0) - P(t))P(t)|)$.

Hence

$$\int_{I_3} |F(s_0)|^{2k} dt = \int_{I_3} |P(t)|^2 dt + E' \text{ where}$$

$$E' = O\left(\int_{I_3} |E|^2 dt + \left(\int_T^{T+H} |E|^2 dt\right)^{\frac{1}{2}} \left(\int_{I_3} |P(t)|^2 dt\right)^{\frac{1}{2}}\right)$$

It is easy to see that $\int_T^{T+H} |P(t)|^2 dt = O(H(\log H)^{C_5})$, and

$$\int_T^{T+H} |P(t)|^4 dt = O(H(\log H)^{C_5}) \text{ and so the } O\text{-term is}$$

$$O(H(\log H)^{C_4} X^{-\frac{1}{2}(\alpha-\alpha_0)}). \text{ Also}$$

$$\int_{I_3} |P(t)|^2 dt = \int_T^{T+H} |P(t)|^2 dt + O(H(\log H)^{C_4 - \frac{1}{2}K_2}),$$

and

$$\int_T^{T+H} |P(t)|^2 dt = \int_T^{T+H} \left| \sum_{n \leq X(\log X)^2} d_k(n) n^{-s_0} \right|^2 dt + O(H^{-10})$$

and this by standard arguments (see for instance Lemma 9 below) is

$$(H + O(X(\log X)^3)) \sum_{n \leq X(\log X)^2} |d_k(n)|^2 n^{-2\alpha} e^{-\frac{2n}{X}} + O(H^{10})$$

$$\text{Let } S = \sum_{n=1}^{\infty} |d_k(n)|^2 n^{-2\alpha} \text{ and } S_1 = \sum_{n \leq X(\log X)^2} |d_k(n)|^2 n^{-2\alpha} e^{-\frac{2n}{X}}$$

Then using $1 - e^{-\frac{n}{X}} = O\left(\frac{n}{X}\right)$ for $n \leq X$ and $O(1)$ for $n > X$ we have

$$\begin{aligned} S - S_1 &= O\left(\frac{1}{X} \sum_{n < X} |d_k(n)|^2 n^{1-2\alpha} + \sum_{n \geq X} |d_k(n)|^2 n^{-2\alpha}\right) \\ &= O(X^{1-2\alpha} (\log H)^{C_4}). \end{aligned}$$

This proves Lemma 6.

Lemma 7: In case (i) we have $\left| \frac{d^m}{d\alpha^m} S \right| \gg \left(\frac{1}{\alpha - \frac{1}{2}}\right)^{ak^2 + m}$,

and in case (ii) we have $S (\alpha - \frac{1}{2})^{ak^2}$ is $\gg \left(\log \frac{1}{\alpha - \frac{1}{2}}\right)^{b_2 k^2}$ and

$$\ll \left(\log \frac{1}{\alpha - \frac{1}{2}}\right)^{b_1 k^2}.$$

Proof: We leave this as an exercise to the reader.

Lemma 8: Let $f(x)$ be m times continuously differentiable in the interval of the integration below. Then for any positive number d we have,

$$\begin{aligned} f(x) - \binom{m}{1} f(x+d) + \binom{m}{2} f(x+2d) + \dots \\ + (-1)^m \binom{m}{m} f(x+md) \\ = (-1)^m \int_0^d \int_0^d \int_0^d \dots \int_0^d f^{(m)}(x+u_1+u_2+\dots+u_m) du_1 \dots du_m \end{aligned}$$

where $\binom{m}{1}, \binom{m}{2}, \dots$ are binomial coefficients.

Proof: This can be proved by induction on m . Details are left to the reader.

From Lemmas 6, 7 and 8 we can deduce Theorem 1 as follows. We first assume condition (iii). Define

$$\alpha_1 = \frac{1}{2} + \frac{K_4 \log \log H}{\log H}, \text{ and } \alpha_2, \dots, \alpha_{m+1} \text{ by } \alpha_j =$$

$$\alpha_1 + \frac{(j-1) (\log \log H)^{C_6}}{\log H} \quad (j = 2, \dots, m+1) \text{ where } C_6 \text{ is a}$$

large constant. Taking K_4 a large constant we see that

$$\frac{1}{H} Q_1(\alpha_1) > (\log H)^{ak^2+m} (\log \log H)^{-C_7} \text{ and this leads to}$$

Theorem 1 with $C = \max(1, C_7)$, in case $m = 0$. If $m \geq 1$, we see that $Q_1(\alpha_1)$ dominates $Q_1(\alpha_j)$, ($j \geq 2$).

$$\text{So taking } f(\alpha) = (F(\alpha + it))^{2k}, \quad x = \alpha_1, \quad d = \frac{(\log \log H)^{C_6}}{\log H}$$

we see that

$$\begin{aligned} & \sum_{l_3} \int_{l_3} \left| f(\alpha_1) - \binom{m}{1} f(\alpha_2) + \binom{m}{2} f(\alpha_3) + \dots \right. \\ & \qquad \qquad \qquad \left. + (-1)^m \binom{m}{m} f(\alpha_{m+1}) \right| dt \\ & < \int_0^d \dots \int_0^d \left(\sum_{l_3} \int_{l_3} |f(\alpha_1 + u_1 + \dots + u_m)| dt \right) du_1 \dots du_m \end{aligned}$$

This shows that the max $Q_1(\sigma)H^{-1}$ and so of $Q(\sigma)$ in $\sigma \geq \frac{1}{2} + \frac{\log \log H}{\log H}$ exceeds $(\log H)^{ak^2+m} (\log \log H)^{-C_8}$

and this proves Theorem 1 with $C = \max(1, C_8)$.

Next we prove Theorem 1 subject to the condition (i). The case $m = 0$ can be disposed off as before. Let $m > 1$ and the theorem be false with a large constant C . Then in addition to Lemma 1 we also have

$$\begin{aligned} & \max_{\sigma > \frac{1}{2} + \frac{\log \log H}{\log H}} \frac{1}{H} \int_{\tau}^{\tau+H} \left| \frac{d}{ds} (F(s))^{2k} \right| dt = \\ & O((\log H)^{ak^2+m_1} (\log \log H)^{-c}). \end{aligned}$$

We choose \mathfrak{A}_1 and \mathfrak{A}_2 as before and we have

$$\sum_{I_3} \int_{I_3} (|f(\mathfrak{A}_1)| - |f(\mathfrak{A}_2)|) dt \leq \sum_{I_3} \int_{I_3} |f(\mathfrak{A}_1) - f(\mathfrak{A}_2)| dt$$

Here the left side exceeds $H(\log H)^{ak^2+m_1-1} (\log \log H)^{-C_9}$

while the right side is $O(H(\log H)^{ak^2+m_1-1} (\log \log H)^{C_6-C})$

which is a contradiction if we choose $C = C_6 + C_9 + 1$. This proves Theorem 1 with $C = C_6 + C_9 + 1$.

Before leaving this section we remark that if instead of condition (ii) we are given that the product

$$\pi \left(1 + \frac{|\chi(p)a_p|^2}{p^2\sigma} \right) \text{ lies between } (\sigma - \frac{1}{2})^{-a'} \left(\log \frac{1}{\sigma - \frac{1}{2}} \right)^{b_1}$$

and $(\sigma - \frac{1}{2})^{-a} \left(\log \frac{1}{\sigma - \frac{1}{2}} \right)^{b_2}$, where

$a' > a > 0$, b_1, b_2 are four real constants then we can conclude

$$\max Q(\sigma) > (\log H)^{ak^2+ma(a')^{-1}} (\log \log H)^{-C}$$

$$\sigma \geq \frac{1}{2} + \frac{\log \log H}{\log H}$$

This explains the second part of our remark below Theorem 2 since for non-Galois number fields we can take $a' = n$.

§ 4. The case of no Euler product : In this and the next section we assume instead of the Euler product a condition of the type, $\sum_{n \leq x} |a_n|^8 = O(x(\log x)^{k'})$ where k' is a positive

constant and $x \geq 2$. (The condition can be relaxed, but we do not want to go into such questions). The main result of this section is

Theorem 3 Suppose that as $\sigma \rightarrow \frac{1}{2} + 0$, $|F(2\sigma)|$ exceeds $\left(\frac{1}{\sigma - \frac{1}{2}} \right)^a \left(\log \frac{1}{\sigma - \frac{1}{2}} \right)^{b_3}$ where $a > 1$ and b_3 are two real constants. Then with $2k = 1$, $m = 0$, we have,

$$\frac{1}{H} \int_T^{T+H} |F(\frac{1}{2} + it)| dt = Q(\frac{1}{2}) > (\log H)^{a-1} (\log \log H)^{-C}$$

Proof: It suffices to prove that $\max Q(\sigma)$ for $\sigma > \frac{1}{2} + \frac{\log \log H}{\log H}$, exceeds $(\log H)^{a-1} (\log \log H)^{-C_1}$ and

Theorem 3 follows from this just as Theorem 2 follows from Theorem 1. (The constants C, K with or without subscripts should not be confused with the earlier constants. Also to save space the proof will be only sketchy). We assume that this is false. Lemma 2 has its analogue without modification.

We put $\alpha_1 = \frac{1}{2} + \frac{2 \log \log H}{\log H}$. Let $\alpha \geq \alpha_1 + \frac{\log \log H}{\log H}$,

and put $s_0 = \alpha + it$. We divide the interval $T_1 = T + (\log H)^8 \leq t \leq T_2 = T + H - (\log H)^8$ into equal intervals J of length $(\log H)^K$, ignoring a bit at one end. We put

$$\zeta_Y(s) = \sum_{n < Y} n^{-s}, \quad \zeta_Y^{-1}(s) = \sum_{n < Y} \mu(n) n^{-s}$$

where $Y = H^{\frac{1}{2}}$. It is not hard to prove that if $\phi(s) = \zeta_Y(s) \zeta_Y^{-1}(s) - 1$ and $M(J)$ denotes the maximum of $|\phi(s_0)|$ for t in J ,

$$\sum_J (M(J))^2 = O(H(\log H)^{10} Y^{1-2\alpha})$$

We omit those intervals J for which $M(J) \geq \frac{1}{2}$ and denote the rest of the intervals by I . The number of intervals J which are excluded is $O(HY^{1-2\alpha} (\log H)^{10})$. If K is a sufficiently

large constant and $\alpha = \frac{1}{2} + \frac{K \log \log H}{\log H}$ we proceed to prove

that $\sum_I \int_I |F(s_0)| dt > H(\log H)^{a-1} (\log \log H)^{-C_2}$.

We write $F(s_0) = F_Y(s_0) + E_Y(s_0)$ where $F_Y(s_0) = \sum_{n=1}^{\infty} a_n n^{-s_0} e^{-\frac{n}{Y}}$,

and we have $\int_T^{T+H} |E_Y| dt = O(H(\log H)^{C_3} Y^{\alpha-\alpha_1})$.

We can replace the integrand

$$|F(s_0)| \text{ by } |F_Y(s_0) \zeta_Y^{-1}(s_0) \zeta_Y(s_0)| \text{ without}$$

disturbing the left side very much. Then we replace

$\sum_I \int_I |\dots| dt$ by $\int_T^{T+H} |\dots| dt$ without much error. Next we

use the fact that the last integral is bounded below by

$$\left| \int_T^{T+H} F_Y(s_0) \zeta_Y^{-1}(s_0) \overline{\zeta_Y(s_0)} dt \right|.$$

To see that this is $\gg H |F(2\alpha)| (\alpha - \frac{1}{2})$, we use the following lemma and some simple computations and this would complete the proof of Theorem 3.

Lemma 9: *If $\{x_n\}$ and $\{y_n\}$ are two sequences of complex numbers, then,*

$$\begin{aligned} & \int_0^T \left(\sum_{n=1}^{\infty} x_n n^{-it} \right) \left(\sum_{n=1}^{\infty} \overline{y_n} n^{it} \right) dt \\ &= T \sum_{n=1}^{\infty} x_n \overline{y_n} + O \left(\left(\sum_{n=1}^{\infty} n |x_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} n |y_n|^2 \right)^{\frac{1}{2}} \right) \end{aligned}$$

Remark. For a simple proof of this lemma see [4].

§ 5. Balasubramanian's remark

Theorem 4. *In the case $2k = 1$, $m = 0$ as before, we have,*

$$\frac{1}{H} \int_T^{T+H} |F(\frac{1}{2} + it)| dt = O(\frac{1}{2}) \gg \max(1, S_2^{\frac{3}{2}} S_3^{-1} (\log \log H)^{-1})$$

$$\text{where } S_2 = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\alpha}, \quad S_3 = \sum_{n=1}^{\infty} |a_n|^2 d(n) n^{-2\alpha},$$

$\alpha = \frac{1}{2} + \frac{C \log \log H}{\log H}$ (where C is a large positive constant) and $d(n)$ is the usual divisor function.

Proof: As before we assume $\max Q(\sigma)$ for $\sigma \geq \frac{1}{2} + \frac{\log \log H}{\log H}$ does not exceed $S_2^{\frac{3}{2}} S_3^{-1}$ (Note that this is

$O((\log H, C^4))$. Next we write $s_0 = \alpha + it$, $F(s_0) = F_Y(s_0) + E_Y$,

where $Y = H^{\frac{1}{2}}$, $F_Y(s_0) = \sum_{n=1}^{\infty} a_n n^{-s_0} e^{-\frac{n}{Y}}$. We have an

asymptotic formula for $\int_T^{T+H} |F_Y(s_0)|^2 dt$, and also a

good upper bound for $\int_T^{T+H} |F_Y(s_0)|^4 dt$. Using

$$\int_T^{T+H} |F_Y(s_0)|^2 dt \leq \left(\int_T^{T+H} |F_Y(s_0)| dt \right)^{\frac{3}{2}} \left(\int_T^{T+H} |F_Y(s_0)|^4 dt \right)^{\frac{1}{2}}$$

we are led to the theorem. The details are left to the reader.

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