k-tuple chromatic number of the cartesian product of graphs

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Abstract

A k-tuple coloring of a graph G assigns a set of k colors to each vertex of G such that if two vertices are adjacent, the corresponding sets of colors are disjoint. The k-tuple chromatic number of G, $\chi_k(G)$, is the smallest t so that there is a k-tuple coloring of G using t colors. It is well known that $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$. In this paper, we show that there exist graphs G and H such that $\chi_k(G \square H) > \max\{\chi_k(G), \chi_k(H)\}$ for $k \geq 2$. Moreover, we also show that there exist graph families such that, for any $k \geq 1$, the k-tuple chromatic number of their cartesian product is equal to the maximum k-tuple chromatic number of its factors.

Keywords: k-tuple colorings, Cartesian product of graphs, Kneser graphs, Cayley graphs, Hom-idempotent graphs.

1 Introduction

A classic *coloring* of a graph G is an assignment of colors (or natural numbers) to the vertices of G such that any two adjacent vertices are assigned different colors. The smallest number t such that G admits a coloring with t colors (a *t-coloring*) is called the *chromatic number* of G and is denoted by $\chi(G)$. Several generalizations of the coloring problem have been introduced in the literature, in particular, cases in which each vertex is assigned not only a color but a set of colors, under different restrictions. One of these variations is the *k-tuple coloring* introduced independently by Stahl [11] and Bollobás and Thomason [3]. A k-tuple

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coloring of a graph G is an assignment of k colors to each vertex in such a way that adjacent vertices are assigned distinct colors. The *k-tuple coloring problem* consists into finding the minimum number of colors in a k-tuple coloring of a graph G, which we denote by $\chi_k(G)$. The *cartesian product* $G\Box H$ of two graphs G and H has vertex set $V(G)\times V(H)$, two vertices being joined by an edge whenever they have one coordinate equal and the other adjacent. This product is commutative and associative up to isomorphism. There is a simple formula expressing the chromatic number of a cartesian product in terms of its factors:

$$\chi(G\Box H) = \max\{\chi(G), \chi(H)\}. \tag{1}$$

The identity (1) admits a simple proof first given by Sabidussi [10]. The *Kneser graph* $K(m, n)$ has as vertices all n-element subsets of the set $[m] = \{1, \dots, m\}$ and an edge between two subsets if and only if they are disjoint. We will assume in the rest of this work that $m \geq 2n$, otherwise $K(m, n)$ has no edges. The Kneser graph $K(5, 2)$ is the well known *Petersen Graph.* Lovász [9] showed that $\chi(K(m, n)) = m - 2n + 2$. The value of the k-tuple chromatic number of the Kneser graph is the subject of an almost 40-year-old conjecture of Stahl [11] which asserts that: if $k = qn - r$ where $q \geq 0$ and $0 \leq r < n$, then $\chi_k(K(m, n)) = qm - 2r$. Stahl's conjecture has been confirmed for some values of k, n and m [11, 12].

An *homomorphism* from a graph G into a graph H, denoted by $G \to H$, is an edge-preserving map from $V(G)$ to $V(H)$. It is well known that an ordinary graph coloring of a graph G with m colors is an homomorphism from G into the complete graph K_m. Similarly, an n-tuple coloring of a graph G with m colors is an homomorphism from G into the Kneser graph $K(m, n)$.

A graph G is said *hom-idempotent* if there is a homomorphism from $G\Box G \to G$. We denote by $G \not\to H$ if there exists no homomorphism from G to H.

The *clique number* of a graph G, denoted by $\omega(G)$, is the maximum size of a *clique* in G (i.e., a complete subgraph of G). Clearly, for any graphs G and H, we have that $\chi(G) \geq \omega(G)$ (and so, $\chi_k(G) \geq \chi_k(K_{\omega(G)}) = k\omega(G)$) and, if there is an homomorphism from G to H then, $\chi(G) \leq \chi(H)$ (and so, $\chi_k(G) \leq \chi_k(H)$).

A *stable set* $S \subseteq V$ is a subset of pairwise non adjacent vertices of G. The *stability number* of G, denoted by $\alpha(G)$, is the largest cardinality of a stable set in G. Let $m \geq 2n$. An element $i \in [m]$ is called a *centre* of a stable set S of the Kneser graph $K(m, n)$ if it lies in each n-set in S.

Lemma 1 (Erdős-Ko-Rado [4]). *If $m > 2n$, then $\alpha(K(m, n)) = \binom{m-1}{n-1}$. An independent set of $K(m, n)$ with size $\binom{m-1}{n-1}$ has a centre i, for some $i \in [m]$.*

Lemma 2 (Hilton-Milner [7]). *If $m \geq 2n$, then the maximum size of an stable set in $K(m, n)$ with no centre is equal to $1 + \binom{m-1}{n-1} - \binom{m-n-1}{n-1}$.*

A graph $G = (V, E)$ is *vertex transitive* if its automorphism group acts transitively on V, that is, for any pair of distinct vertices of G there is an automorphism mapping one to the other one. It is well known that Kneser graphs are vertex transitive graphs [5].

Lemma 3 (No-Homomorphism Lemma, Albertson-Collins [1]). *Let G, H be graphs such that H is vertex transitive and $G \to H$. Then, $\alpha(G)/|V(G)| \geq \alpha(H)/|V(H)|$.*

In this paper, we show that equality (1) does not hold in general for k-tuple colorings of graphs. In fact, we show that for some values of $k \geq 2$, there are Kneser graphs $K(m,n)$ for which $\chi_k(K(m,n)\Box K(m,n)) > \chi_k(K(m,n))$. Moreover, we also show that there are families of graphs for which equality (1) holds for k-tuple colorings of graphs for any $k \geq 1$. As far as we know, our results are the first ones concerning the k-tuple chromatic number of cartesian product of graphs.

2 Cartesian products of Kneser graphs

Lemma 4. *Let G be a graph and let $k > 0$. Then, $\chi_k(G\Box G) \leq k\chi(G)$.*

Proof. Clearly, $\chi_k(G\Box G) \leq k\chi(G\Box G)$. However, by equality (1) we know that $\chi(G\Box G) = \chi(G)$, and thus the lemma holds. $\Box$

Corollary 1. $\chi_k(K(m,n)\Box K(m,n)) \leq k\chi(K(m,n)) = k(m-2n+2)$.

Larose et al. [8] showed that no connected Kneser graph $K(m,n)$ is hom-idempotent, that is, for any $m > 2n$, there is no homomorphism from $K(m,n)\Box K(m,n)$ to $K(m,n)$.

Lemma 5 ([8]). *Let $m > 2n$. Then, $K(m,n)\Box K(m,n) \not\to K(m,n)$.*

Concerning the k-tuple chromatic number of some Kneser graphs, Stahl [11] showed the following results.

Lemma 6 ([11]). *If $1 \leq k \leq n$, then $\chi_k(K(m,n)) = m - 2(n-k)$.*

Lemma 7 ([11]). $\chi_k(K(2n+1,n)) = 2k+1+\lfloor \frac{k-1}{n} \rfloor$, *for $k > 0$.*

Lemma 8 ([11]). $\chi_{rn}(K(m,n)) = rm$, *for $r > 0$ and $m \geq 2n$.*

By using Lemma 8 we have the following result.

Lemma 9. *Let $m > 2n$. Then, $\chi_n(K(m,n)\Box K(m,n)) > \chi_n(K(m,n))$.*

Proof. By Lemma 8 when $r = 1$, we have that $\chi_n(K(m,n)) = m$. If $\chi_n(K(m,n)\Box K(m,n)) = m$, then there exists an homomorphism from the graph $K(m,n)\Box K(m,n)$ to $K(m,n)$ which contradicts Lemma 5. $\Box$

By Lemma 6, Lemma 9 and by using Corollary 1, we have that,

Corollary 2. *Let $n \geq 2$. Then, $2n+2 \leq \chi_n(K(2n+1,n)\Box K(2n+1,n)) \leq 3n$. In particular, when $n = 2$, we have that $\chi_2(K(5,2)\Box K(5,2)) = 6$.*

In the case $k = 2$ we have by Lemma 9, Lemma 6 and by using Corollary 1, the following result.

Corollary 3. *Let $q > 0$. Then, $q + 4 \leq \chi_2(K(2n + q, n)\Box K(2n + q, n)) \leq 2q + 4$.*

By Corollary 3, notice that in the case when $k = n = 2$ and $q \geq 1$, we must have that $\chi_2(K(q + 4, 2)\Box K(q + 4, 2)) > q + 4$, otherwise there is a contradiction with Lemma 5. This provides a gap of one unity between the 2-tuple chromatic number of the graph $K(q+4, 2)\Box K(q+4, 2)$ and the graph $K(q+4, 2)$. In the next Lemma 11, we will show that such a gap can be as large as desired. However, first we need to introduce the following.

It is well known that the chromatic index of a complete graph K_{2n} (i.e. the minimum number of colors needed to color the edges of K_{2n} such that any two incident edges be assigned different colors) on $2n$ vertices is equal to $2n - 1$ (see [2]), where each color class i (i.e. the subset of pairwise non incident edges colored with color i) has size n. Therefore, using this fact, we obtain the following result.

Lemma 10. *Let $q \geq 1$. Then, the set of vertices of the Kneser graph $K(2q + 4, 2)$ can be partitioned into $2q + 3$ disjoint cliques, each one with size $q + 2$.*

Proof. Notice that there is a natural bijection between the vertex set of $K(2q + 4, 2)$ and the edge set of the complete graph K_{2q+4} with vertex set $[2q + 4]$: each vertex $\{i, j\}$ in $K(2q + 4, 2)$ is mapped to the edge $\{i, j\}$ in K_{2q+4}. Now, there is a $(2q + 3)$-edge coloring of K_{2q+4} where each class color is a set of pairwise non incident edges with size $q + 2$. Notice that two edges $e, e' \in K_{2q+4}$ are non incident edges if and only if $e \cap e' = \emptyset$. Therefore, a class color of the edge-coloring of K_{2q+4} represents a clique of $K(m, n)$. $\square$

Lemma 11. *Let $q > 0$. Then, $\chi_2(K(2q + 4, 2)\Box K(2q + 4, 2)) \geq 2q + \left\lceil \frac{2}{3}q \right\rceil + 5$.*

Proof. First, recall that a stable set X in $K(2q + 4, 2)$ has size at most $2q + 3$ if X has centre (see Lemma 1) and $|X| \leq 1 + (2q + 4 - 1) - (2q + 4 - 2 - 1) = 3$ if X has no centre (see Lemma 2). Besides, by Lemma 10, observe that the vertex set of $K(2q + 4, 2)$ can be partitioned in $2q+3$ sets $S_1, \ldots, S_{2q+3}$ such that each S_i induces a K_{q+2} for $i = 1, \ldots, 2q+3$. Consider the subgraph H_i of $K(2q+4, 2)\Box K(2q+4, 2)$ induced by $S_i \times V(K(2q+4, 2))$ for $i = 1, \ldots, 2q + 3$. Let I be a stable set in $K(2q + 4, 2)$ and $I_i = I \cap H_i$ for $i = 1, \ldots, 2q + 3$. Then $I_i^v = I_i \cap (\{v\} \times V(K(2q + 4, 2)))$ is a stable set in $K(2q + 4, 2)$ for each $v \in S_i$ and $i = 1, \ldots, 2q + 3$.

Now, assume w.l.o.g that r ($r \leq q+2$) stable sets $I_i^1, \ldots, I_i^r$ have distinct centre $j_1, \ldots, j_r$, respectively (the case when two of these stable sets have the same centre can be easily reduced to this case). Let W be the set of subsets with size two of $\{j_1, \ldots, j_r\}$. Therefore, for all $m \in \{1, \ldots, r\}$, $I_i^m - W$ has size at most $2q + 3 - (r - 1) = 2q + 4 - r$ since each centre j_m belongs to $r - 1$ elements in W. Besides, each element of W belongs to exactly one set I_i^m for $m \in \{1, \ldots, r\}$, since S_i induces a complete subgraph. Then, $|I_i^1 \cup \ldots \cup I_i^r| \leq (\sum_{m=1}^{r} |I_i^m - W|) + |W| \leq r(2q + 4 - r) + \frac{r(r-1)}{2}$.

Next, each remaining stable set (if exist) $I_i^{r+1}, \ldots, I_i^{q+2}$ has no centre, then $|I_i^d| \leq 3$ for all $d \in \{r+1, \ldots, q+2\}$. Thus, $|I_i| \leq r(2q+4-r)+\frac{r(r-1)}{2}+3(q+2-r) = -\frac{r^2}{2}+r(2q+\frac{1}{2})+3(q+2)$. Since the last expression is non decreasing for $r \in \{1, \ldots, q + 2\}$, we have that

$|I_i| \leq -\frac{(q+2)^2}{2} + (q+2)(2q+\frac{1}{2}) + 3(q+2) = (q+2)(\frac{3}{2}q+\frac{5}{2})$.

Therefore, $|I_i| \leq (q+2)(\frac{3}{2}q+\frac{5}{2})$ for every $i = 1, \ldots, 2q+3$. Since $|I| = \sum_{i=1}^{2q+3} |I_i|$, it follows that $|I| \leq (2q+3)(q+2)(\frac{3}{2}q+\frac{5}{2})$ and thus,
$\alpha(K(2q+4,2)\Box K(2q+4,2)) \leq (2q+3)(q+2)(\frac{3}{2}q+\frac{5}{2})$.

Let $t < \frac{2q^2+18q+24}{3q+5}$. Assume that exists a 2-tuple coloring of the graph $K(2q+4,2)\Box K(2q+4,2))$ with $2q+t$ colors. Therefore, there exists an homomorphism from $K(2q+4,2)\Box K(2q+4,2)$ to $K(2q+t,2)$.

Now, from the well known no-homomorphism Lemma 3, we have that,

$\alpha(K(2q+4,2)\Box K(2q+4,2)) \geq \frac{\alpha(K(2q+t,2))|V(K(2q+4,2)\Box K(2q+4,2))|}{|V(K(2q+t,2))|}$.

Then, $\alpha(K(2q+4,2)\Box K(2q+4,2)) \geq \frac{(2q+t-1).(2q+4)^2(2q+3)^2 2}{4(2q+t)(2q+t-1)} = \frac{(2q+4)^2(2q+3)^2}{2(2q+t)}$.

Let us see that $\frac{(2q+4)^2(2q+3)^2}{2(2q+t)}$ is greater than $(2q+3)(q+2)(\frac{3}{2}q+\frac{5}{2})$, which is a contradiction.

To this end, observe that if $t < \frac{2q^2+18q+24}{3q+5}$, then

$(2q+3)(q+2)(\frac{3}{2}q+\frac{5}{2})(2q+t) = (2q+3)(q+2)((3q^2+5q)+(\frac{3}{2}q+\frac{5}{2})t) < (2q+3)(q+2)((3q^2+5q)+(q^2+9q+12)) = (2q+3)(q+2)(4q^2+14q+12)) = \frac{(2q+4)^2(2q+3)^2}{2}$.

Therefore, $\chi_2(K(2q+4,2)\Box K(2q+4,2)) \geq 2q+t+1$.

Claim 1. *Let $q > 0$ be an integer. Then, $\frac{2q^2+18q+24}{3q+5}$ is not an integer.*

Proof. By polynomial division, we have that $2q^2+18q+24 = (3q+5)(\frac{2}{3}q+\frac{44}{9}) - \frac{4}{9}$. If $(2q^2+18q+24)/(3q+5)$ is an integer, then $k = (\frac{2}{3}q+\frac{44}{9}) - \frac{4}{9(3q+5)}$ is also an integer. Multiplying by 9 both terms in the last equality we have, $6q+44 = 9k+\frac{4}{3q+5}$. As $q > 0$ then, $0 < \frac{4}{3q+5} < 1$ contradicting the assumption that k is an integer.

Finally, since $\frac{2}{3}q+4 < \frac{2q^2+18q+24}{3q+5}$ and, by previous Claim 1, $\frac{2q^2+18q+24}{3q+5}$ is not an integer, then $\lceil \frac{2}{3}q \rceil + 4 < \frac{2q^2+18q+24}{3q+5}$ and thus, we have finally that,

$2q + \lceil \frac{2}{3}q \rceil + 5 \leq \chi_2(K(2q+4,2))$. □

As a corollary of Lemma 11 and by Corollary 1, we obtain the following result.

Corollary 4. $\chi_2(K(6,2)\Box K(6,2)) = 8$.

Theorem 1. *Let $k > n$ and let $t = \chi_k(K(m,n)\Box K(m,n))$, where $m > 2n$. Then, either $t > m+2(k-n)$ or $t < m+(k-n)$.*

Proof. Suppose that $m+(k-n) \leq t \leq m+2(k-n)$. Therefore, there exists an homomorphism $K(m,n)\square K(m,n) \to K(t,k)$. Now, Stahl [11] showed that there is an homomorphism $K(m,n) \to K(m-2,n-1)$ whenever $n > 1$ and $m \geq 2n$. Moreover, it's easy to see that there is an homomorphism $K(m,n) \to K(m-1,n-1)$. By applying the former homomorphism $t-(m+(k-n))$ times to the graph $K(t,k)$ we obtain an homomorphism $K(t,k) \to K(2(m+k-n)-t, 2k+m-n-t)$. Finally, by applying $2k+m-t-2n$ times the latter homomorphism to the graph $K(2(m+k-n)-t, 2k+m-n-t)$ we obtain an homomorphism $K(2(m+k-n)-t, 2k+m-n-t) \to K(m,n)$. Therefore, by homomorphism composition, $K(m,n)\square K(m,n) \to K(m,n)$ which contradicts Lemma 5. $\square$

We can also obtain a lower bound for the k-tuple chromatic number of the graph $K(m,n)\square K(m,n)$ in terms of the clique number of $K(m,n)$. In fact, notice that $\omega(K(m,n)\square K(m,n)) = \omega(K(m,n)) = \lfloor \frac{m}{n} \rfloor$. Thus, we have the following result.

Theorem 2. *Let $k > n$. Then, $\chi_k(K(m,n)\square K(m,n)) \geq k\omega(K(m,n)) = k\lfloor \frac{m}{n} \rfloor$. In particular, if n divides m then, $\chi_k(K(m,n)\square K(m,n)) \geq m + (k-n)\frac{m}{n}$.*

3 Cases where $\chi_k(G\square H) = \max\{\chi_k(G), \chi_k(H)\}$

Theorem 3. *Let G and H be graphs such that $\chi(G) \leq \chi(H) = \omega(H)$. Then, $\chi_k(G\square H) = \max\{\chi_k(G), \chi_k(H)\}$.*

Proof. Let $t = \omega(H)$ and let $\{h_1, \ldots, h_t\}$ be the vertex set of a maximum clique K_t in H with size t. Clearly, $\chi_k(G) \leq \chi_k(H) = \chi_k(K_t)$. Let ρ be a k-tuple coloring of H with $\chi_k(H)$ colors. By equality (1), there exists a t-coloring f of $G\square H$. Therefore, the assignment of the k-set $\rho(h_{f((a,b))})$ to each vertex (a,b) in $G\square H$ defines a k-tuple coloring of $G\square H$ with $\chi_k(K_t)$ colors. $\square$

Notice that if G and H are both bipartite, then $\chi_k(G\square H) = \chi_k(G) = \chi_k(H)$. In the case when G is not a bipartite graph, we have the following results.

An *automorphism* σ of a graph G is called a *shift* of G if $\{u, \sigma(u)\} \in E(G)$ for each $u \in V(G)$ [8]. In other words, a shift of G maps every vertex to one of its neighbors.

Theorem 4. *Let G be a non bipartite graph having a shift $\sigma \in AUT(G)$, and let H be a bipartite graph. Then, $\chi_k(G\square H) = \max\{\chi_k(G), \chi_k(H)\}$.*

Proof. Let $A \cup B$ be a bipartition of the vertex set of H. Let f be a k-tuple coloring of G with $\chi_k(G)$ colors. Clearly, $\chi_k(G) \geq \chi_k(H)$. We define a k-tuple coloring ρ of $G\square H$ with $\chi_k(G)$ colors as follows: for any vertex (u,v) of $G\square H$ with $u \in G$ and $v \in H$, define $\rho((u,v)) = f(u)$ if $v \in A$, and $\rho((u,v)) = f(\sigma(u))$ if $v \in B$. $\square$

We may also deduce the following direct result.

Theorem 5. *Let G be an hom-idempotent graph an let H be a subgraph of G. Thus, $\chi_k(G\square H) = \max\{\chi_k(G), \chi_k(H)\} = \chi_k(G)$.*

Let A be a group and S a subset of A that is closed under inverses and does not contain the identity. The *Cayley graph* $Cay(A,S)$ is the graph whose vertex set is A, two vertices u,v being joined by an edge if $u^{-1}v \in S$. If $a^{-1}Sa = S$ for all $a \in A$, then $Cay(A,S)$ is called a *normal Cayley graph*.

Lemma 12 ([6]). *Any normal Cayley graph is hom-idempotent.*

Note that all Cayley graphs on abelian groups are normal, and thus hom-idempotents. In particular, the *circulant* graphs are Cayley graphs on cyclic groups (i.e., cycles, powers of cycles, complements of powers of cycles, complete graphs, etc). By Theorem 5 and Lemma 12 we have the following result.

Theorem 6. *Let $Cay(A,S)$ be a normal Cayley graph and let $Cay(A',S')$ be a subgraph of $Cay(A,S)$, with $A' \subseteq A$ and $S' \subseteq S$. Then, $\chi_k(Cay(A,S)\Box Cay(A',S')) = \max\{\chi_k(Cay(A,S)), \chi_k(Cay(A',S'))\}$.*

Definition 1. *Let G be a graph with a shift σ. We define the* order *of σ as the minimum integer i such that σ^i is equal to the identity permutation.*

Theorem 7. *Let G be a graph with a shift σ of minimum odd order $2s+1$ and let C_{2t+1} be a cycle graph, where $t \geq s$. Then, $\chi_k(G\Box C_{2t+1}) = \max\{\chi_k(G), \chi_k(C_{2t+1})\}$.*

Proof. Let $\{0,\dots,2t\}$ be the vertex set of C_{2t+1}, where for $0 \leq i \leq 2t$, $\{i, i+1 \mod n\} \in E(C_{2t+1})$. Let G_i be the i^{th} copy of G in $G\Box C_{2t+1}$, that is, for each $0 \leq i \leq 2t$, $G_i = \{(g,i) : g \in G\}$. Let f be a k-tuple coloring of G with $\chi_k(G)$ colors. We define a k-tuple coloring of $G\Box C_{2t+1}$ with $\chi_k(G)$ colors as follows: let σ^0 denotes the identity permutation of the vertices in G. Now, for $0 \leq i \leq 2s$, assign to each vertex $(u,i) \in G_i$ the k-tuple $f(\sigma^i(u))$. For $2s+1 \leq j \leq 2t$, assign to each vertex $(u,j) \in G_j$ the k-tuple $f(u)$ if j is odd, otherwise, assign to (u,j) the k-tuple $f(\sigma^1(u))$. It's not difficult to see that this is in fact a proper k-tuple coloring of $G\Box C_{2t+1}$. □

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