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# BIDIMENSIONAL RANDOM EFFECT ESTIMATION IN MIXED STOCHASTIC DIFFERENTIAL MODEL

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## ABSTRACT

In this work, a mixed stochastic differential model is studied with two random effects in the drift. We assume that  $N$  trajectories are continuously observed throughout a time interval  $[0, T]$ . Two directions are investigated. First we estimate the random effects from one trajectory and give a bound of the  $L^2$ -risk of the estimators. Secondly, we build a nonparametric estimator of the common bivariate density of the random effects. The mean integrated squared error is studied. The performances of the density estimator are illustrated on simulations.

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**Keywords** Adaptive bandwidth, density estimation, kernel estimator, mean integrated squared error, mixed-effects models, stochastic differential equations.

## 1. INTRODUCTION

Mixed-effects models are used to analyse repeated measurements with similar functional form but with some variability between experiments (see *e.g.* Davidian and Giltinan, 1995; Pinheiro and Bates, 2000; Diggle *et al.*, 2002). The advantage is that a single estimation procedure is used to fit the overall data simultaneously. In the present work the model of interest is a mixed-effects stochastic differential equation (SDE). Each equation represents the behaviour of one subject and corresponds to one realization of the random effects. Hence the random effects represent the particularity of each process. Mixed-effects SDEs have various applications such as neuronal or pharmacokinetic modelling (see *e.g.* Picchini *et al.*, 2008; Donnet and Samson, 2013).

Estimation methods in SDEs with random effects have been proposed in literature. The main purposes are the estimation of the common distribution of the random effects in a parametric or nonparametric way. The estimation of the common density of the random effects is mainly parametric. Most methods assume normality of the random effects and estimate the population parameters (see *e.g.* Picchini *et al.*, 2008; Delattre *et al.*, 2013; Delattre and Lavielle, 2013). However, one can ask if this assumption is reasonable in some application contexts. Nonparametric estimation can allow us to get around this problem. To the best of our knowledge, the first reference in this context is Comte *et al.* (2013) who propose different nonparametric estimators and then Dion (2014) who develops two adaptive nonparametric estimators for the Ornstein-Uhlenbeck model with an application to a neuronal database. But these two references focus on a one-dimensional random effect.

In the present work we study the case of two random effects or, in other words, of one bidimensional random effect. We want to deal with two points: the estimation of the random effects and the nonparametric estimation of their common density. This bivariate context makes the study more complex. In fact the estimation of the random effects is done using matrix norm and operator, and

anisotropy appears in the density estimation part. We consider  $N$  trajectories, observed on the interval  $[0, T]$  where  $T$  is given. For  $j = 1, \dots, N$  the dynamics of each process is described by the stochastic differential equation

$$\begin{cases} dX_j(t) &= {}^t b(X_j(t))\phi_j dt + \sigma(X_j(t))dW_j(t) \\ X_j(0) &= \gamma_j \end{cases} \quad (1)$$

where  $\phi_j = {}^t(\phi_{j,1}, \phi_{j,2}) \in \mathbb{R}^2$  is the bidimensional random effect,  $b(\cdot) = {}^t(b_1(\cdot), b_2(\cdot))$ ,  $\sigma(\cdot)$  are known functions defined on  $\mathbb{R}$ ,  $(W_j)_{1 \leq j \leq N}$  are  $N$  independent Wiener processes and  $\gamma_j$  is a real valued random variable. The random variables  $((\phi_1, \gamma_1), \dots, (\phi_N, \gamma_N))$  are *i.i.d.* and the sequences  $((\phi_1, \gamma_1), \dots, (\phi_N, \gamma_N))$  and  $(W_1, \dots, W_N)$  are independent. This model is more general than the ones investigated in Comte *et al.* (2013) where the drift has the form  $b(\cdot)\phi$  or  $b(\cdot) + \phi$  with  $\phi$  a real valued random effect. Notice that the  $N$  trajectories  $(X_j(t), 0 \leq t \leq T)_{1 \leq j \leq N}$  are *i.i.d.*

We assume that the  $\phi_j = {}^t(\phi_{j,1}, \phi_{j,2})$ 's have a common bivariate density  $f$ . Our goal is twofold: first estimate the random effects  $\phi_j$ 's and then their density  $f$ , from the observations  $(X_j(t), 0 \leq t \leq T)_{1 \leq j \leq N}$ , with large  $T$ .

The estimation of the random variables  $\phi_j$ 's follows the steps of Genon-Catalot and Larédo (2014) where only one multiplicative random effect in the drift is considered. We build an estimator  $\hat{A}_{j,T}$  of  $\phi_j$  based on the trajectory  $(X_j(t), 0 \leq t \leq T)$  and study its  $L^2$ -risk. This leads to a bound of order  $1/T$ . Then we propose a kernel estimator of the density  $f$  which uses the sample  $(\hat{A}_{j,T})_j$ . When  $f$  is in a Nikolskii space a bound of the mean integrated squared error is established and the rate of convergence is evaluated. Finally a data-driven choice of the bandwidth based on a Goldenshluger and Lepski's criterion for anisotropic multi-index is proposed (see *e.g.* Kerkycharian *et al.*, 2007; Goldenshluger and Lepski, 2011) and leads to an adaptive estimator.

Section 2 is dedicated to assumptions and definitions of some useful quantities for the estimation of  $\phi_j$ . In Section 3 the estimator of the random effects is built and its  $L^2$ -risk is bounded. In particular we deal with two main examples: the Ornstein-Uhlenbeck model and the Cox-Ingersoll-Ross model. In Section 4 the estimator of the density  $f$  is studied. Finally, Section 5 is devoted to numerical simulations to illustrate estimators. Proofs are relegated in Section 7.

## 2. NOTATION AND ASSUMPTIONS

**2.1. General assumptions on the model.** Consider real valued processes  $(X_j(t))_{j=1, \dots, N}$  given by (1). We assume that  $(W_j)_{j=1, \dots, N}$  and  $(\phi_j, \gamma_j)_{j=1, \dots, N}$  are defined on a common probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Consider the following assumptions (see *e.g.* Kutoyants, 2004; Kessler *et al.*, 2012, for details).

**(A1)** The real valued functions  $x \mapsto b_1(x)$ ,  $x \mapsto b_2(x)$  and  $x \mapsto \sigma(x)$  are continuous on  $\mathbb{R}$  and  $b_1, b_2$  Lipschitz and  $\sigma$  Hölder with exponent belonging to  $[1/2, 2]$ .

**(A2)** There exists an open set  $\Phi$  of  $\mathbb{R}^2$  and an interval  $(l, r) \subset \mathbb{R}$  such that  $\sigma^2(x) > 0$  for  $x \in (l, r)$ , and for all  $\varphi \in \Phi$  the function  $s_\varphi : x \mapsto \exp(-2 \int_{x_0}^x \frac{{}^t b(u)\varphi}{\sigma^2(u)} du)$ ,  $x_0 \in (l, r)$  satisfies

$$\int_l s_\varphi(y) dy = +\infty, \quad \int^r s_\varphi(y) dy = +\infty.$$

The function  $m_\varphi : x \mapsto 1/(\sigma^2(x)s_\varphi(x))$  satisfies  $M(\varphi) = \int_l^r m_\varphi(x) dx < +\infty$ . We set

$$\pi_\varphi(x) := \mathbf{1}_{(l,r)}(x) \frac{m_\varphi(x)}{M(\varphi)}. \quad (2)$$

**(A3)**  $\phi_j$  takes values in  $\Phi$ , has distribution  $f(\varphi)d\varphi$  and  $(\phi_j, \gamma_j)$  has distribution on  $\Phi \times (l, r)$

$$\pi(d\varphi, dx) = f(\varphi)d\varphi \otimes \pi_\varphi(x)dx.$$

Assumption **(A1)** ensures the existence and uniqueness of a strong solution  $X_j(\cdot)$  to (1) for all random variables  $(\phi_j, \gamma_j) \in \mathbb{R}^3$ , adapted to the filtration  $\mathcal{F}_t = \sigma((\phi_j, \gamma_j)_{j=1, \dots, N}, W_j(s), s \leq t, j = 1, \dots, N), t \geq 0$ . Then  $(\phi_j, \gamma_j)$  is  $\mathcal{F}_0$ -measurable and  $W_j$  is a  $(\mathcal{F}_t, t \geq 0)$  Brownian motion.

In the following, for  $\varphi = {}^t(\varphi_1, \varphi_2)$  a fixed value in  $\Phi$  we denote by  $X^\varphi$  the process solution of the stochastic differential equation with fixed  $\varphi$ :

$$\begin{cases} dX_j^\varphi(t) = {}^t b(X_j^\varphi(t))\varphi dt + \sigma(X_j^\varphi(t))dW_j(t) \\ X_j^\varphi(0) \sim \pi_\varphi, X_j^\varphi(0) \text{ independent of } W_j. \end{cases} \quad (3)$$

Under **(A1)**-**(A2)**, for  $\varphi \in \Phi$ , the process defined by (3) is strictly stationary and ergodic with scale density  $s_\varphi$ , speed density  $m_\varphi$  and marginal distribution  $\pi_\varphi(x)dx$ . Under **(A1)**-**(A3)**, according to Genon-Catalot and Larédo (2014), the conditional distribution of  $X_j$  given  $\phi_j = \varphi$  is identical to the distribution of (3), the process  $((\phi_j, X_j(t)), t \geq 0)$  is strictly stationary with marginal distribution  $\pi$  and  $\mathbb{P}(X_j(t) \in (l, r), \forall t > 0) = 1$ .

Finally under **(A1)**-**(A3)** as  $X_j(t) \in (l, r), \forall t, j = 1, \dots, N, \forall T > 0$ :

$$\int_0^T \frac{b_1^2}{\sigma^2}(X_j(s))ds < +\infty, \int_0^T \frac{b_2^2}{\sigma^2}(X_j(s))ds < +\infty, \text{ a.s.}$$

**2.2. Key examples.** We investigate two classical examples: the mixed Ornstein-Uhlenbeck model and the mixed Cox-Ingersoll-Ross model.

**Example. [1]** *The Ornstein Uhlenbeck model (O-U) with two random effects is defined as*

$$\begin{cases} dX_j(t) = (\phi_{j,1} - \phi_{j,2}X_j(t))dt + \sigma dW_j(t) \\ X_j(0) = \gamma_j \end{cases} \quad (4)$$

with  $\sigma(x) = \sigma > 0, (l, r) = \mathbb{R}$ , and  $b(x) = {}^t(1, -x)$ . Assumption **(A2)** requires  $\Phi = \mathbb{R} \times (0, +\infty)$  and leads to  $\pi_\varphi = \mathcal{N}(\varphi_1/\varphi_2, \sigma^2/(2\varphi_2))$  as invariant distribution for fixed  $\varphi$ .

**Example. [2]** *The Cox-Ingersoll-Ross model (C-I-R) with two random effects is defined as*

$$\begin{cases} dX_j(t) = (\phi_{j,1} - \phi_{j,2}X_j(t))dt + \sigma\sqrt{X_j^+(t)}dW_j(t) \\ X_j(0) = \gamma_j \end{cases} \quad (5)$$

with  $\sigma(x) = \sigma\sqrt{x^+}$  with  $\sigma > 0, x^+ = \max(0, x)$ , and  $b(x) = {}^t(1, -x)$ . Assumption **(A2)** requires  $2\varphi_1/\sigma^2 \geq 1$  and  $2\varphi_2/\sigma^2 > 0$  (in particular the process is always positive). This leads to consider  $\Phi = (\sigma^2/2, +\infty) \times (0, +\infty)$  and  $\pi_\varphi = \Gamma(2\varphi_1/\sigma^2, \sigma^2/(2\varphi_2))$  is the invariant distribution with  $2\varphi_1/\sigma^2$  the shape parameter and  $\sigma^2/(2\varphi_2)$  the scale parameter.

**2.3. Specific assumptions and notations for estimation.** Our estimation is based on the following quantities. As in Comte *et al.* (2013) we define for  $j = 1, \dots, N$ ,

$$U_j(T) := \int_0^T \frac{b}{\sigma^2}(X_j(s))dX_j(s), \quad (6)$$

which is a column vector with size  $2 \times 1$  and, the  $2 \times 2$  symmetric matrix:

$$V_j(T) := \int_0^T \frac{b {}^t b}{\sigma^2}(X_j(s))ds = \begin{pmatrix} \int_0^T \frac{b_1^2}{\sigma^2}(X_j(s))ds & \int_0^T \frac{b_1 b_2}{\sigma^2}(X_j(s))ds \\ \int_0^T \frac{b_1 b_2}{\sigma^2}(X_j(s))ds & \int_0^T \frac{b_2^2}{\sigma^2}(X_j(s))ds \end{pmatrix}. \quad (7)$$

Using (1),  $U_j(T)$  can be decomposed as follows

$$U_j(T) = V_j(T)\phi_j + M_j(T) \text{ with } M_j(T) := \int_0^T \frac{b}{\sigma}(X_j(s))dW_j(s). \quad (8)$$

Note that

$$V_j(T) = \langle M_j \rangle_T \quad (9)$$

where  $\langle . \rangle$  is the quadratic variation of a continuous local martingale. For all measurable function  $h : (l, r) \rightarrow \mathbb{R}$   $\pi_\varphi$ -integrable for all  $\varphi \in \Phi$ , we define the random variable  $\pi_{\phi_j}(h) := \int_l^r h(x)\pi_{\phi_j}(x)dx$ .

**(A4)** For  $\varphi \in \Phi$  and  $i \in \{1, 2\}$ , we assume

$$\pi_\varphi \left( \frac{b_i^2}{\sigma^2} \right) = \int_l^r \frac{b_i^2}{\sigma^2}(x)\pi_\varphi(x)dx < +\infty.$$

We define the random matrix

$$L_j := \pi_{\phi_j} \left( \frac{b^t b}{\sigma^2} \right).$$

**(A5)** For  $j = 1, \dots, N$ ,  $\mathbb{P}(L_j \text{ invertible}) = 1$ .

Under **(A1)-(A4)**, Theorem 3.1. of Genon-Catalot and Larédo (2014) gives:

$$\frac{\langle M_j \rangle_T}{T} = \frac{V_j(T)}{T} \xrightarrow{T \rightarrow +\infty} L_j, \text{ a.s.} \quad (10)$$

One can notice that assumption **(A5)** implies that  $V_j(T)$  is invertible for  $T$  large enough.

In the study we denote  $\|\cdot\|$  the  $L^2(\mathbb{R})$ -norm,  $\|\cdot\|_2$  the euclidean norm of  $\mathbb{R}^2$  and  $\|\cdot\|_F$  the Frobenius norm of matrices defined for  $A \in M_2(\mathbb{R})$  by  $\|A\|_F^2 = \sum_{i,j} A_{i,j}^2 = \text{Tr}({}^t AA)$ . We denote  $S_2(\mathbb{R})$  the subset of symmetric matrices of  $M_2(\mathbb{R})$ .

### 3. RANDOM EFFECT ESTIMATION

We define first for each  $j$  an estimator of the random variable  $\phi_j$  based on the trajectory  $X_j(t)$ ,  $t \in [0, T]$ . Then these estimators are used to build an estimator of the density  $f$ .

**3.1. Definition of the estimator of the random effects.** Let us define  $N$  bidimensional random variables

$$A_j(T) := {}^t(A_{j,1}(T), A_{j,2}(T)) = V_j(T)^{-1}U_j(T), \quad (11)$$

which corresponds to the maximum likelihood estimator of  $\phi_j$  when  $\phi_j = \varphi$  is deterministic. Note that

$$A_j(T) = \phi_j + V_j(T)^{-1}M_j(T). \quad (12)$$

Thus  $A_j(T)$  is a consistent estimator of  $\phi_j$  as  $T$  tends to infinity according to convergence (10). Because of the presence of the inverse matrix  $V_j(T)^{-1}$  in formulae (11)-(12), it is difficult to prove that  $A_j(T)$  has finite moments and to compute any of them. To overcome this theoretical difficulty, we consider a truncated estimator of  $\phi_j$ . Let

$$\widehat{A}_j(T) := A_j(T)\mathbf{1}_{B_j(T)}, \quad B_j(T) := \{V_j(T) \geq \kappa\sqrt{T}I_2\} = \{\min(\lambda_{1,j}(T), \lambda_{2,j}(T)) \geq \kappa\sqrt{T}\} \quad (13)$$

where  $\lambda_{i,j}(T)$ ,  $i = 1, 2$  are the eigenvalues of  $V_j(T)$  and  $I_2$  is the identity matrix of  $M_2(\mathbb{R})$ . The inequality in the definition of  $B_j(T)$  has a matrix sense: for two matrices  $(A, B) \in S_2(\mathbb{R})$ ,  $A \leq B$  if and only if  $A - B$  is a non negative matrix (see Appendix 7.5).

Relations (7) and (9) show that  $V_j(T)$  is a non negative symmetric matrix, thus its eigenvalues are non negative. We are able to bound the  $L^2$ -risk of the estimator  $\widehat{A}_j(T)$  of  $\phi_j$ . This bound is needed to evaluate the mean integrated squared error of the nonparametric estimator of the density  $f$ .

**3.2. Main result.** We denote  $\mathcal{L}_\varphi$  the infinitesimal generator of the process (3), given for  $F \in \mathcal{C}^2((l, r))$ , by  $\mathcal{L}_\varphi F(x) := (\sigma^2(x)/2)F''(x) + ({}^t b(x)\varphi)F'(x)$ . Its domain is included in  $L^2_{\pi_\varphi}$  which is the space of functions  $f$  such that  $\int_l^r f^2 d\pi_\varphi < \infty$  (for details see *e.g.* Genon-Catalot *et al.* (2000)). When  $F$  is a matrix, the notation  $\mathcal{L}_\varphi F$  indicates that we apply the operator on each coefficient of the matrix. We also define analogously  $F'$  as the matrix of derivatives. For all  $g = (g_{i,k})_{1 \leq i,k \leq 2} \in M_2(\mathbb{R})$  such that  $\pi_\varphi(g_{i,k}^2) < +\infty$  for all  $\varphi \in \Phi$ , we associate the matrix

$$F_\varphi^g = \begin{pmatrix} F_\varphi^{g_{1,1}} & F_\varphi^{g_{1,2}} \\ F_\varphi^{g_{2,1}} & F_\varphi^{g_{2,2}} \end{pmatrix}$$

satisfying  $-\mathcal{L}_\varphi F_\varphi^g = g - \pi_\varphi g$  for all  $\varphi \in \Phi$ . We denote for simplicity

$$H_{\phi_j} := F_{\phi_j}^{b^t b / \sigma^2}. \quad (14)$$

Examples below show how  $H_{\phi_j}$  can be constructed. We are now able to announce the main result on the estimator on the random effects.

**Proposition 1.** *Consider the processes  $(X_j(t), j = 1, \dots, N)$  given by (1) under (A1)-(A5). Assume that for  $j = 1, \dots, N$ ,*

$$\begin{aligned} \mathbb{E} \left[ \frac{\|L_j\|_F^2}{[\det(L_j)]^2} \text{Tr}(L_j) \right] < \infty, \quad \mathbb{E} \left[ \frac{\|L_j\|_F^4}{[\det(L_j)]^4} \text{Tr} \left( \pi_{\phi_j} \left( \left( \frac{b^t b}{\sigma^2} \right)^2 \right) \right) \right] < \infty, \\ \mathbb{E} \left[ \|\phi_j\|_2^2 \left( \frac{1}{L_{j,1,1}^2} + \frac{1}{L_{j,2,2}^2} \right) \pi_{\phi_j} (\|H_{\phi_j}\|_F^2) \right] + \mathbb{E} \left[ \|\phi_j\|_2^2 \left( \frac{1}{L_{j,1,1}^2} + \frac{1}{L_{j,2,2}^2} \right) \pi_{\phi_j} (\|H'_{\phi_j}\|_F^2 \sigma^2) \right] < \infty \end{aligned}$$

and

$$\mathbb{E} [\pi_{\phi_j} (\|H_{\phi_j}\|_F^4)] + \mathbb{E} [\pi_{\phi_j} (\|H'_{\phi_j}\|_F^4 \sigma^4)] < \infty.$$

Then, there exists a constant  $C > 0$  such that

$$\mathbb{E} [\|\hat{A}_j(T) - \phi_j\|_2^2] \leq \frac{C}{T}. \quad (15)$$

The two last assumptions of Proposition 1 correspond to the application of Proposition 7 given in Section 7.1 with  $g = b^t b / \sigma^2$ ,  $p = 1$  and  $p = 2$ . This proposition is the key of the nonparametric estimation procedure set up in Section 4.

**3.3. Key examples continued.** Let us investigate the two examples given in Section 2.2.

**Example. [1](continued)** In this case

$$\frac{b^t b}{\sigma^2}(x) = \frac{1}{\sigma^2} \begin{pmatrix} 1 & -x \\ -x & x^2 \end{pmatrix}, \quad V_j(T) = \frac{1}{\sigma^2} \begin{pmatrix} T & -\int_0^T X_j(s) ds \\ -\int_0^T X_j(s) ds & \int_0^T X_j(s)^2 ds \end{pmatrix}.$$

For simplicity we set here  $\phi_j = \phi = {}^t(\phi_1, \phi_2)$ . The limit matrix  $L$  of  $V(T)/T$  is given by

$$L = \pi_\phi \left( \frac{b^t b}{\sigma^2} \right) = \frac{1}{\sigma^2} \begin{pmatrix} 1 & -\frac{\phi_1}{\phi_2} \\ -\frac{\phi_1}{\phi_2} & \frac{\sigma^2}{2\phi_2} + \frac{\phi_1^2}{\phi_2^2} \end{pmatrix}, \quad \det(L) = \frac{\sigma^2}{2\phi_2} > 0.$$

We can check that

$$H_\phi(x) = \begin{pmatrix} 1 & -\frac{1}{\phi_2 \sigma^2} (x - \frac{\phi_1}{\phi_2}) \\ -\frac{1}{\phi_2 \sigma^2} (x - \frac{\phi_1}{\phi_2}) & \frac{2\phi_1}{\sigma^2 \phi_2^2} (x - \frac{\phi_1}{\phi_2}) + \frac{1}{2\sigma^2 \phi_2} \left[ \left( x - \frac{\phi_1}{\phi_2} \right)^2 - \frac{\sigma^2}{2\phi_2} \right] \end{pmatrix}.$$

Details are in Appendix. The assumptions of Proposition 1 are fulfilled if  $\mathbb{E}[\phi_1^6 + \phi_2^{-10}] < \infty$ .

**Example. [2](continued)** In this case,

$$\frac{b^t b}{\sigma^2}(x) = \frac{1}{\sigma^2} \begin{pmatrix} \frac{1}{x} & -1 \\ -1 & x \end{pmatrix}, \quad V_j(T) = \frac{1}{\sigma^2} \begin{pmatrix} \int_0^T \frac{1}{X_j(s)} ds & -T \\ -T & \int_0^T X_j(s) ds \end{pmatrix}.$$

For  $\Phi = (\sigma^2, +\infty) \times (0, +\infty)$ ,  $\pi_\varphi(x \mapsto 1/x^2) < \infty$  for all  $\varphi \in \Phi$  and the limit matrix  $L$  is given by:

$$L = \pi_\phi \left( \frac{b^t b}{\sigma^2} \right) = \frac{1}{\sigma^2} \begin{pmatrix} \frac{2\phi_2}{2\phi_1 - \sigma^2} & -1 \\ -1 & \frac{\phi_1}{\phi_2} \end{pmatrix}, \quad \det(L) = \frac{\sigma^2}{2\phi_1 - \sigma^2} > 0.$$

One can verify (see Appendix) that

$$H_\phi(x) = \begin{pmatrix} \frac{2}{\sigma^2(2\phi_1 - \sigma^2)} \log(x) & 1 \\ 1 & \frac{1}{\phi_2 \sigma^2} \left(x - \frac{\phi_1}{\phi_2}\right) \end{pmatrix}.$$

The details are relegated in Appendix. For the assumptions of Proposition 1 we must have  $\pi_\phi(x \mapsto |\log(x)|)$ ,  $\pi_\phi(x \mapsto 1/x^2)$  and  $\pi_\phi(x \mapsto 1/x^4)$  finite. The last condition imposes a reduction of  $\Phi$  to  $\Phi = (2\sigma^2, +\infty) \times (0, +\infty)$ . Moreover the other assumptions are fulfilled if  $\mathbb{E}[\log^2(\phi_2) + \phi_1^5 + \phi_2^8 + \phi_2^{-4} + \psi^2(2\phi_1/\sigma^2 - 1)] < \infty$ , where  $\psi(x) := \Gamma'(x)/\Gamma(x)$  is the digamma function.

#### 4. NONPARAMETRIC ESTIMATION

In this Section we assume that  $f \in \mathbb{L}^2(\mathbb{R}^2)$  and set up a nonparametric estimation procedure based on a kernel estimator. We study the obtained estimator and determinate its speed of convergence.

**4.1. Nonparametric estimator of the density of the random effects.** By Proposition 1,  $\widehat{A}_{j,T}$  is a consistent estimator of  $\phi_j$  when  $T$  tends to infinity. It is therefore natural to define a kernel estimator based on the  $\widehat{A}_j(T)$ 's. Let us denote by  $K$  a kernel in  $\mathcal{C}^1(\mathbb{R}^2)$  such that the partial derivatives  $\frac{\partial K}{\partial u}$  and  $\frac{\partial K}{\partial v}$  are in  $L^2(\mathbb{R}^2)$ ,  $K$  is integrable,  $\iint K(u, v) du dv = 1$  and  $\|K\|^2 = \iint K^2(u, v) du dv < +\infty$ . For all  $h = (h_1, h_2)$ ,  $h_1 > 0$ ,  $h_2 > 0$ , for all  $(u, v) \in \mathbb{R}^2$ , we denote

$$K_h(u, v) = \frac{1}{h_1 h_2} K\left(\frac{u}{h_1}, \frac{v}{h_2}\right).$$

For example one can consider the Gaussian kernel  $K(u, v) = K_1(u) \times K_1(v)$ , with  $K_1(u) = (1/\sqrt{2\pi}) \exp(-u^2/2)$ . We define the estimator of the density  $f$  for  $x = (x_1, x_2) \in \mathbb{R}^2$  by

$$\widehat{f}_h(x) = \frac{1}{N} \sum_{j=1}^N K_h(x - \widehat{A}_j(T)). \quad (16)$$

Denoting  $f_h(x) := K_h \star f(x) = \iint f(y_1, y_2) K_h(x_1 - y_1, x_2 - y_2) dy_1 dy_2$ , the following result holds.

**Proposition 2.** *Under (A1)-(A5) and under the assumptions of Proposition 1,*

$$\mathbb{E}[\|\widehat{f}_h - f\|^2] \leq 2\|f - f_h\|^2 + 2 \max\left(\frac{1}{h_1^3 h_2}, \frac{1}{h_1 h_2^3}\right) \left( \left\| \frac{\partial K}{\partial u} \right\|^2 + \left\| \frac{\partial K}{\partial v} \right\|^2 \right) \frac{C}{T} + \frac{\|K\|^2}{N h_1 h_2} \quad (17)$$

with  $C$  the constant which appears in Proposition 1.

This bound comes from the bias-variance decomposition:

$$\mathbb{E}[\|\widehat{f}_h - f\|^2] \leq 2\|f - f_h\|^2 + 2\|\mathbb{E}[\widehat{f}_h] - f_h\|^2 + \mathbb{E}[\|\widehat{f}_h - \mathbb{E}[\widehat{f}_h]\|^2].$$

The first term is a bias term due to the approximation of  $f$  by  $f_h$ , it decreases when  $h$  decreases. The third term is a variance term which increases when  $h$  decreases. Finally, the middle term is an error term due to the approximation of the  $\phi_j$ 's by the  $\widehat{A}_j(T)$  also increasing when  $h$  decreases. Note that the orders are consistent with the result of Comte *et al.* (2013), Proposition 1 for the multiplicative model. In fact in the case of a single random effect, multiplicative in the drift, the second term has the order  $1/(Th^3)$  and the third term:  $1/(Nh)$ .

To choose the best  $h$ , a compromise must be done between terms which are increasing in  $h$  and the one decreasing with  $h$ .

**4.2. Rates of convergence.** We consider anisotropic Nikolskii classes of functions which are well fitted to evaluate the order of the bias term (see Goldenshluger and Lepski (2011) and Comte and Lacour (2013) for example).

**Definition 3.** A function  $f$  is in the Nikolskii class  $\mathcal{N}(\beta, R)$  if  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  admits partial derivatives of order  $\lfloor \beta_i \rfloor$ ,  $i \in \{1, 2\}$  and with  $y_1 = (y, 0), y_2 = (0, y)$  for all  $y \in \mathbb{R}$ , they satisfy for  $i \in \{1, 2\}$

$$\left( \int \left| \frac{\partial^{\lfloor \beta_i \rfloor} f}{(\partial x_i)^{\lfloor \beta_i \rfloor}}((x_1, x_2) + y_i) - \frac{\partial^{\lfloor \beta_i \rfloor} f}{(\partial x_i)^{\lfloor \beta_i \rfloor}}(x_1, x_2) \right|^2 dx \right)^{1/2} \leq R |y|^{\beta_i - \lfloor \beta_i \rfloor},$$

$$\left\| \frac{\partial^{\lfloor \beta_i \rfloor} f}{(\partial x_i)^{\lfloor \beta_i \rfloor}} \right\| \leq R$$

(with  $\lfloor \beta \rfloor$  denotes the largest integer strictly less than  $\beta$ ).

Recall that kernel  $K : \mathbb{R} \rightarrow \mathbb{R}$  is of order  $l \in \mathbb{N}^*$  if for  $j = 0, \dots, l$ ,  $\int |x|^j |K(x)| dx < +\infty$ , and  $\int x^j K(x) dx = 0$ . In this context we are now able to bound the bias term  $\|f - f_h\|^2$ .

**Proposition 4.** Under (A1)-(A5), if  $f \in \mathcal{N}(\beta, R)$  and  $K(x) = K(x_1, x_2) = K_1(x_1)K_2(x_2)$  where  $K_1$  is a kernel of order  $l_1 := \lfloor \beta_1 \rfloor$  and  $K_2$  a kernel of order  $l_2 := \lfloor \beta_2 \rfloor$  and if  $\int |K_1(u)u^{\beta_1}| du < \infty$  and  $\int |K_2(u)u^{\beta_2}| du < \infty$ . Then there are two constants  $c_1, c_2 > 0$  such that

$$\|f - f_h\|^2 \leq c_1 h_1^{2\beta_1} + c_2 h_2^{2\beta_2}. \quad (18)$$

In order to minimize the MISE, if we only consider the first and the last term of the right hand side of (17), we have to minimize the following function of two variables:

$$\phi(h_1, h_2) = \frac{1}{N h_1 h_2} + h_1^{2\beta_1} + h_2^{2\beta_2}.$$

With  $1/\bar{\beta} = 1/(2\beta_1) + 1/(2\beta_2)$ , we get the minimizing values

$$h_1^* = O\left(N^{-\frac{1}{2\beta_1} \frac{\bar{\beta}}{\beta+1}}\right), \quad h_2^* = O\left(N^{-\frac{1}{2\beta_2} \frac{\bar{\beta}}{\beta+1}}\right) \quad (19)$$

and this implies

$$\phi(h_1^*, h_2^*) = O\left(N^{-\frac{4\beta_1\beta_2}{(2\beta_1+1)(2\beta_2+1)-1}}\right) = O\left(N^{-\frac{2\bar{\beta}}{2+2\bar{\beta}}}\right).$$

Therefore, we require (see (17))

$$\frac{1}{T} \max\left(\frac{1}{h_1^{*3} h_2^*}, \frac{1}{h_1^* h_2^{*3}}\right) \leq \frac{1}{T h_1^* h_2^*} \left(\frac{1}{h_1^{*2}} + \frac{1}{h_2^{*2}}\right) \leq N^{-\frac{4\beta_1\beta_2}{(2\beta_1+1)(2\beta_2+1)-1}}.$$

This holds under the following condition

$$T \geq N^{1+\frac{4\beta_1}{(2\beta_1+1)(2\beta_2+1)-1}} + N^{1+\frac{4\beta_2}{(2\beta_1+1)(2\beta_2+1)-1}}. \quad (20)$$

This condition is implied by  $T \geq N^3$ , and if  $\beta_i > 1/2$  by  $T \geq N^2$ . We are now able to state the order of the risk, for large  $N$  and  $T$ .

**Corollary 5.** Under the assumptions of Proposition 4 and under conditions (19) and (20), we obtain

$$\mathbb{E}[\|\widehat{f}_{h^*} - f\|^2] = O\left(N^{-\frac{2\bar{\beta}}{2\bar{\beta}+2}}\right) \text{ with } 1/\bar{\beta} = 1/(2\beta_1) + 1/(2\beta_2).$$

In practice the regularity parameters  $\beta$  and  $R$  are unknown. Thus we propose a data-driven way of choosing  $h$  based on the (Goldenshluger and Lepski, 2011) method.

**4.3. Data-driven bandwidth and adaptation.** The idea is to replace the unknown term  $\|f_h - f\|^2$  by an estimator. For this we introduce the iterated auxiliary kernel estimators

$$\widehat{f}_{h,h'}(x) := K_{h'} \star \widehat{f}_h(x) = \frac{1}{N} \sum_{j=1}^N K_{h'} \star K_h(x - \widehat{A}_j(T)).$$

Let  $\mathcal{H}_{N,T}$  a finite set of bandwidths  $h = (h_1, h_2)$ . Following Goldenshluger and Lepski's approach, we define the estimator of the bias term by

$$B(h) = \sup_{h' \in \mathcal{H}_{N,T}} \left( \|\widehat{f}_{h,h'} - \widehat{f}_{h'}\|^2 - v(h') \right)_+ \quad (21)$$

where  $\kappa_1, \kappa_2$  are numerical constants and

$$v(h) := \kappa_1 \frac{\|K\|^2 \|K\|_1^2}{N h_1 h_2} + \kappa_2 \|K\|_1^2 \left( \frac{1}{h_1^3 h_2} + \frac{1}{h_1 h_2^3} \right) \left( \left\| \frac{\partial K}{\partial u} \right\|^2 + \left\| \frac{\partial K}{\partial v} \right\|^2 \right) \frac{C}{T}, \quad (22)$$

a term which has the same order as the variance terms in (17) with  $C$  the constant from Proposition 1. The bandwidth is finally selected as follows

$$\widehat{h} = \operatorname{argmin}_{h \in \mathcal{H}_{N,T}} (B(h) + v(h)). \quad (23)$$

Then we obtain the following Theorem.

**Theorem 6.** *We assume that  $\mathcal{H}_{N,T}$  satisfies  $0 < h_i < 1, i = 1, 2, 1/(N h_1 h_2) \leq 1, 1/(h_1^3 h_2) \leq T, 1/(h_1 h_2^3) \leq T, \forall c > 0, \exists \Sigma(c) < \infty, \sum_{h \in \mathcal{H}_{N,T}} (h_1 h_2)^{-1/2} e^{-c/\sqrt{h_1 h_2}} \leq \Sigma(c)$ . Under (A1)-(A5), and under the assumptions of Proposition 1, there exist numerical constants  $\kappa_1, \kappa_2$  such that*

$$\mathbb{E}[\|\widehat{f}_{\widehat{h}} - f\|^2] \leq C_1 \inf_{h \in \mathcal{H}_{N,T}} \{\|f - f_h\|^2 + v(h)\} + \frac{C_2}{N} \quad (24)$$

with  $C_1$  a positive constant depending on  $\|K\|_1, C_2$  positive constant depending on  $\|f\|, \left\| \frac{\partial K}{\partial u} \right\|^2 + \left\| \frac{\partial K}{\partial v} \right\|^2, \|K\|_1, \|K\|_{4/3}$  and  $C$ .

This is a non-asymptotic result proving that the bias variance compromise is automatically realized by the final estimator  $\widehat{f}_{\widehat{h}}$  which is therefore adaptive. Theorem 6 is proved in Dion (2014) for a single random effect. As the proof is analogous we do not repeat it here. The key is to prove that  $\mathbb{E}[B(h)] \leq \|f - f_h\|^2 + c/N$ . The theoretical study gives  $\kappa_1 \geq \max(40/\|K\|_1^2, 40)$  and  $\kappa_2 \geq \max(10/\|K\|_1^2, 10)$ . Nevertheless, the constants  $\kappa_1, \kappa_2$  obtained in the proof are not optimal and thus it is standard to calibrate them from a simulation study.

The conditions of Theorem 6 are fulfilled for example with the set  $\mathcal{H}_{N,T} = \{(1/k_1^2, 1/k_2^4), k_1 = 1 \dots, N^{1/4}, k_2 = 1 \dots, N^{1/8}, N^2 \leq T\}$ .

In practice, note that in  $v(h)$ , the constant  $C$  is unknown, and must be replaced by a rough estimator. From the proofs of Propositions 1 and 7 it appears that  $C = \mathbb{E}[C(\phi_j)]$  for an explicit function  $C(\phi_j)$ , thus  $C$  can be estimated by  $(1/N) \sum_{j=1}^N C(\widehat{A}_j(T))$ .

## 5. SIMULATION STUDY

In the following we investigate our nonparametric estimation procedure on the two examples of Sections 2.2 and 3.3. For simplicity we denote  $f := f_{\widehat{h}}$  and

$$\widehat{f}_1(x_1) = \int \widehat{f}(x_1, x_2) dx_2, \quad \widehat{f}_2(x_2) = \int \widehat{f}(x_1, x_2) dx_1. \quad (25)$$

For the O-U model we compute the exact solution of the stochastic differential equation. First we simulate a realisation of  $\phi$ :  $\varphi = {}^t(\varphi_1 \ \varphi_2)$  according to the distributions given below, then  $X^\varphi(0) \sim \mathcal{N}\left(\frac{\varphi_1}{\varphi_2}, \frac{\sigma^2}{2\varphi_2}\right)$  and  $G_1, \dots, G_K$  *i.i.d.*  $\mathcal{N}(0, 1)$  independent of  $X^\varphi(0)$ ,

$$X^\varphi((k+1)\Delta) = X^\varphi(k\Delta)e^{-\Delta\varphi_2} + \frac{\varphi_1}{\varphi_2}(1 - e^{-\Delta\varphi_2}) + \sqrt{\frac{\sigma^2}{2\varphi_2}(1 - \exp(-2\Delta\varphi_2))}G_{k+1}. \quad (26)$$

We simulate  $N$  independent discretized realisations and obtain discrete observations:  $X_j(k\Delta), k \in \{1, \dots, K\}$ .

For the C-I-R model, we use a discretization scheme given in Alfonsi (2005): for  $\delta$  small enough and  $\varphi$  fixed, the scheme is given by  $X_{(k+1)\delta}^\varphi \cong X_{k\delta}^\varphi + (\varphi_1 - \sigma^2/2 - \varphi_2 X_{(k+1)\delta}^\varphi)\delta + \sigma \sqrt{X_{(k+1)\delta}^\varphi}(W_{(k+1)\delta} - W_{k\delta})$ . It produces an explicit formula for the process:

$$X^\varphi((k+1)\delta) = \left( \frac{\sqrt{\delta}\sigma G_{k+1} + \sqrt{\sigma^2\delta G_{k+1}^2 + 4(\delta(\varphi_1 - \sigma^2/2) + X^\varphi(k\delta))(1 + \varphi_2\delta)}}{2 + 2\varphi_2\delta} \right)^2 \quad (27)$$

We simulate  $\varphi = {}^t(\varphi_1 \ \varphi_2)$  and  $X^\varphi(0) \sim \Gamma(2\frac{\varphi_1}{\sigma^2}, \frac{\sigma^2}{2\varphi_2})$ . The values  $\delta = T/20000$ ,  $\Delta = 10 \times \delta$  and  $K = T/\Delta = 2000$  are chosen.

We choose independent components  $\phi_1, \phi_2$ , thus the density of  $\phi$  is  $f(x_1, x_2) = f_1(x_1) \times f_2(x_2)$ . For the O-U model we choose:

- $\phi_1 \sim \mathcal{N}(1, 0.25)$ ,  $\phi_2 \sim \Gamma(10.1, 0.25)$ ,
- $\phi_1 \sim \mathcal{N}(1, 0.25)$ ,  $\phi_2 \sim 1/\Gamma(3, 0.25)$
- $\phi_1 \sim \Gamma(1.8, 0.8)$ ,  $\phi_2 \sim \Gamma(2, 1)$
- $\phi_1 \sim \Gamma(1.8, 0.8)$ ,  $\phi_2 \sim 1/\Gamma(3, 0.25)$

For the C-I-R model as the two random effects must be positive random variables and satisfy  $\phi_1 > 2\sigma^2$ , we choose:

- $\phi_1 \sim 2\sigma^2 + \Gamma(5, 0.5)$ ,  $\phi_2 \sim 1 + \Gamma(1.8, 0.8)$
- $\phi_1 \sim 2\sigma^2 + \Gamma(5, 0.5)$ ,  $\phi_2 \sim 1/\Gamma(3, 0.25)$ .

Note that the third case for the O-U model does not satisfy all the assumptions because  $\mathbb{E}[\phi_2^{-10}]$  does not exist. In the same way the first case for the C-I-R model does not satisfy  $\mathbb{E}[\phi_2^{-4}] < \infty$ . However the following study shows that the estimation procedure is quite robust and thus the theoretical assumptions may be relaxed. Beside, note that the chosen distribution for  $\phi_1$  for the C-I-R model satisfies the assumption  $\mathbb{E}[\psi^2(2\phi_1/\sigma^2 - 1)] < \infty$ . Indeed, when  $\phi_1 \sim 2\sigma^2 + \Gamma(k, \theta)$ ,  $\mathbb{E}[\psi^2(2\phi_1/\sigma^2 - 1)] = \int_{2\sigma^2}^{+\infty} \psi^2(2x/\sigma^2 - 1)e^{-(x-2\sigma^2)/\theta}(x-2\sigma^2)^{k-1}dx$ . One can use the asymptotic equivalent  $\psi(x) \underset{+\infty}{\sim} \log(x)$  to see that the integral is convergent.

Then, we compute the estimators  $\hat{A}_j(T)$ 's given by (13) for different values of the cut-off  $\kappa$  including  $\kappa = 0$  (no cut-off). In Dion (2014) the kernel estimator with data-driven bandwidth selected by the Goldenshluger and Lepski criterion is implemented for one random effect. It is also implemented using the *R*-function *density* choosing the bandwidth by cross-validation or with the rule of thumb. The Goldenshluger and Lepski method performs better for mixture densities but is computationally costly. Both methods are comparable for unimodal densities. This is why here we have chosen to use the *R*-function *kde2d* for the choice of the bandwidths with a standard bivariate Gaussian kernel.

Finally to choose the cut-off constant  $\kappa$ , different functions  $f$  have been investigated with different values of  $\sigma, \Delta$  and a large number of repetitions. Comparing the MISEs obtained as functions of the constant  $\kappa$  yields to select values making a good compromise over all experiences. Finally we choose  $\kappa = 0.125$  and we also implement  $\kappa = 0$  (no cut-off) for comparison. For  $\kappa = 0$  the estimator is denoted  $\tilde{f}$  and its marginals  $\tilde{f}_1$  and  $\tilde{f}_2$ .

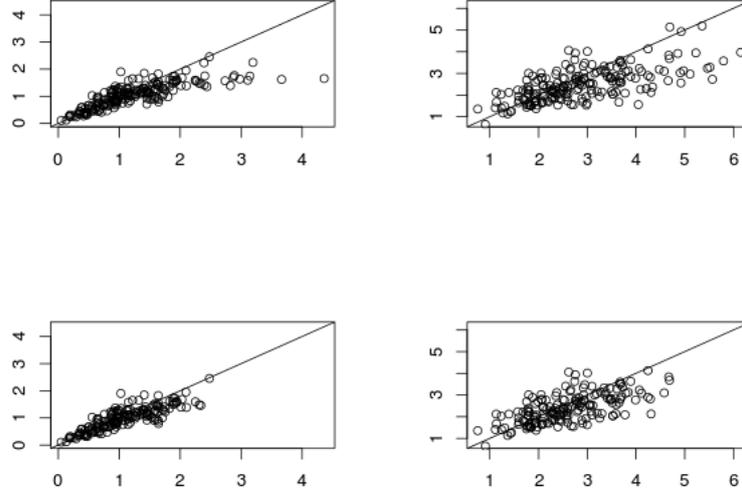


FIGURE 1. O-U example. First line:  $\phi_{j,1}$ 's as a function of  $A_{j,1}(T)$  on the left and  $\phi_{j,2}$ 's as a function of  $A_{j,2}(T)$  on the right. Second line:  $\phi_{j,1}$ 's as a function of  $\hat{A}_{j,1}(T)$  on the left and  $\phi_{j,2}$ 's as a function of  $\hat{A}_{j,2}(T)$  on the right. With  $\phi_1 \sim \mathcal{N}(1, 0.25)$ ,  $\phi_2 \sim \Gamma(10.1, 0.25)$ ,  $T = 10$ ,  $N = 200$ ,  $\sigma = 0.1$

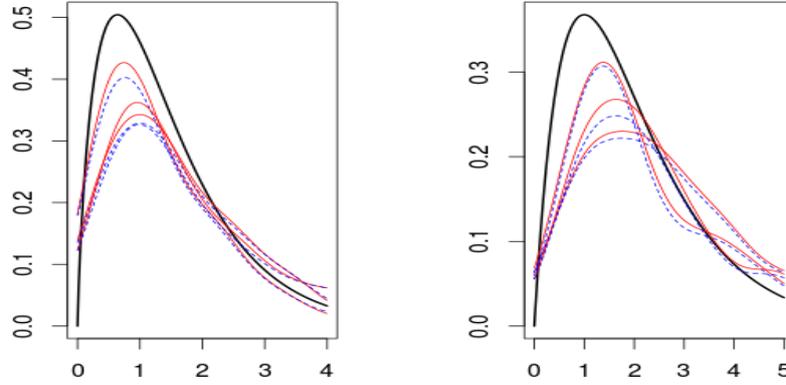


FIGURE 2. O-U example. Simulation with  $\phi_1 \sim \Gamma(1.8, 0.8)$ ,  $\phi_2 \sim \Gamma(2, 1)$ ,  $T = 10$ ,  $N = 200$ ,  $\sigma = 0.1$ . Comparison: the black bold line  $f(x_1, x_2) = f_1(x_1) \times f_2(x_2)$ , on the left 3 estimators  $\tilde{f}_1(x_1)$  the dotted blue curves and  $\hat{f}_1(x_1)$  in red, on the right 3 estimators  $\tilde{f}_2(x_2)$  the dotted blue curves and  $\hat{f}_2(x_2)$  in red

On Figures 1 and 4, we represent the  $\phi_j$ 's as a function of the estimation  $A_j(T)$  on the first line ( $\kappa = 0$ ), on the left for the first coordinate on the right for the second one. On the second line we represent the  $\phi_j$ 's as a function of the estimation  $\hat{A}_j(T)$  ( $\kappa = 0.125$ ). Between the first line and the second line, we clearly see the influence of the truncation. The cut-off seems to have the adequate

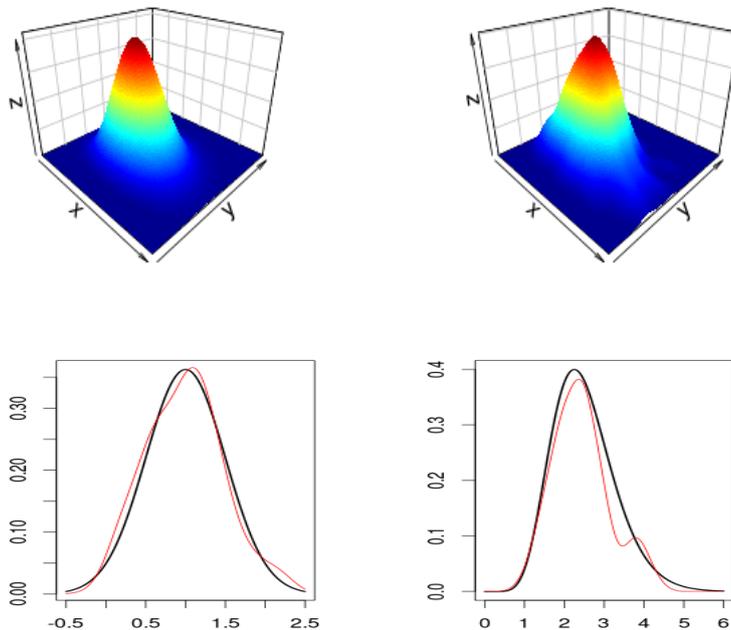


FIGURE 3. O-U example. Simulation with  $\phi_1 \sim \mathcal{N}(1, 0.25)$ ,  $\phi_2 \sim \Gamma(10.1, 0.25)$ ,  $T = 100$ ,  $N = 200$ ,  $\sigma = 0.1$ . First line:  $f$  on the left and  $\hat{f}$  on the right. Second line:  $x_1 \rightarrow f(x_1, x_2)$  in bold black,  $x_1 \rightarrow \hat{f}(x_1, x_2)$  in red for a fixed  $x_2$  on the left. On the right  $x_2 \rightarrow f(x_1, x_2)$  in bold black,  $x_2 \rightarrow \hat{f}(x_1, x_2)$  in red for a fixed  $x_1$ .

effect on the estimation of the random effect: it deletes the values which are too far away from the diagonal axe.

On Figures 2 and 5, we represent on the left 3 estimators  $\tilde{f}_1$  (dotted curve) and  $\hat{f}_1$  (red curve), on the right 3 estimators  $\tilde{f}_2$  (dotted curve) and  $\hat{f}_2$  (red curve), and the true densities  $f_1$  and  $f_2$  in bold black. Again the action of the cut-off is clear especially for  $\hat{f}_1$ .

On Figures 3 and 6, we represent on the first line the true density  $f$  on the left and on the line the estimator  $\hat{f}$ . On the second line we represent  $x_1 \rightarrow f(x_1, x_2)$  on the left, and  $x_1 \rightarrow \hat{f}(x_1, x_2)$  for a fixed  $x_2$ . On the right  $x_2 \rightarrow f(x_1, x_2)$  and  $x_2 \rightarrow \hat{f}(x_1, x_2)$  in red for a fixed  $x_1$ . The estimation seems close to the true density.

In order to evaluate the performances of our estimator on these two examples we compare their empirical MISE computed from 100 simulated data sets, for different designs. Results are presented in Tables 1 and 2.

From Table 1 the influence of the cut-off is often visible when  $N = 20$ . It almost disappears when  $N = 200$ . We can see for a given value of  $N$ , the influence of  $T$  is weak. Nevertheless the best results are obtained when  $T = 100$  and  $N = 200$ . When  $\phi_1$  is Gaussian the estimation seems much harder than when it is Gamma. Note that the influence of  $\sigma$  is important: the factor  $\sigma^2/T$  cannot be too large for good estimation.

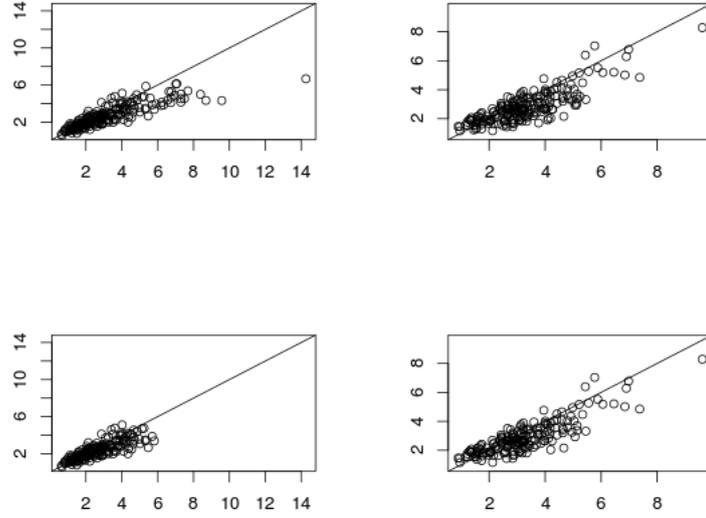


FIGURE 4. C-I-R example. First line:  $\phi_{j,1}$ 's as a function of  $A_{j,1}(T)$  on the left and  $\phi_{j,2}$ 's as a function of  $A_{j,2}(T)$  on the right. Second line:  $\phi_{j,1}$ 's as a function of  $\hat{A}_{j,1}(T)$  on the left and  $\phi_{j,2}$ 's as a function of  $\hat{A}_{j,2}(T)$  on the right. With  $\phi_1 \sim 2\sigma^2 + \Gamma(5, 0.5)$ ,  $\phi_2 \sim 1/\Gamma(8, 0.05)$ ,  $T = 10$ ,  $N = 200$ ,  $\sigma = 0.1$ .

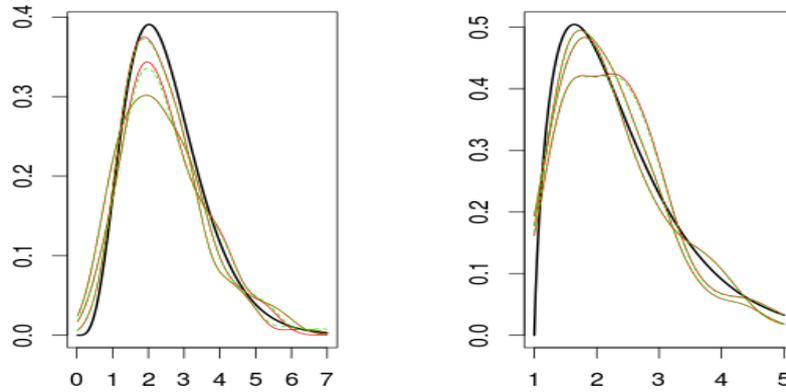


FIGURE 5. C-I-R example. Simulation with  $\phi_1 \sim 2\sigma^2 + \Gamma(5, 0.5)$ ,  $\phi_2 \sim \Gamma(1.8, 0.8)$ ,  $T = 100$ ,  $N = 200$ ,  $\sigma = 0.1$ . The black bold line is  $f(x_1, x_2) = f_1(x_1) \times f_2(x_2)$ , on the left 3 estimators  $\tilde{f}_1(x_1)$  the dotted green curves and  $\hat{f}_1(x_1)$  in red, on the right 3 estimators  $\tilde{f}_2(x_2)$  the dotted green curves and  $\hat{f}_2(x_2)$  in red

From Table 2 when  $N = 200$  the influence of the cut-off is clear: the MISEs of  $\tilde{f}$  are larger than the MISEs of  $\hat{f}$ . When  $N = 200$  the increase of  $T$  makes estimation better.

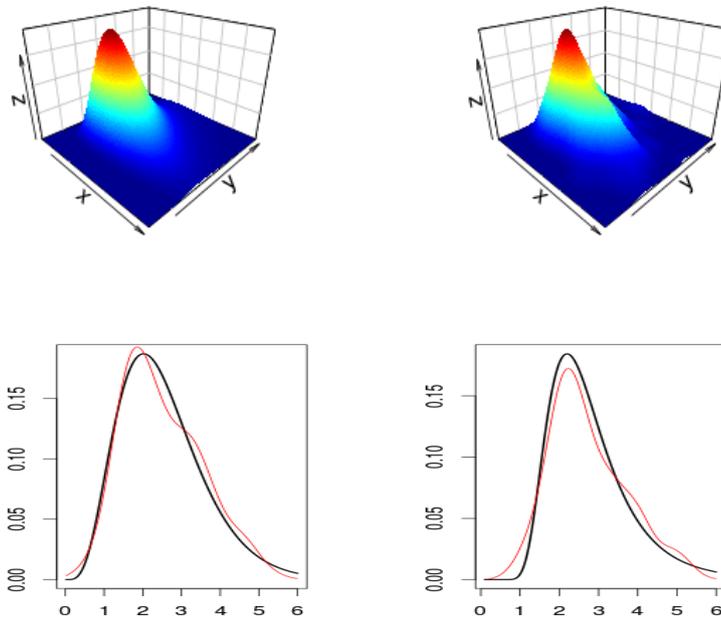


FIGURE 6. C-I-R example. Simulation with  $\phi_1 \sim 2\sigma^2 + \Gamma(5, 0.5)$ ,  $\phi_2 \sim 1/\Gamma(8, 0.05)$ ,  $T = 100$ ,  $N = 200$ ,  $\sigma = 0.1$ . First line:  $f$  on the left and  $\hat{f}$  on the right. Second line:  $x_1 \rightarrow f(x_1, x_2)$  in bold black,  $x_1 \rightarrow \hat{f}(x_1, x_2)$  in red for a fixed  $x_2$  one the left. On the right  $x_2 \rightarrow f(x_1, x_2)$  in bold black,  $x_2 \rightarrow \hat{f}(x_1, x_2)$  in red for a fixed  $x_1$ .

Tables 3 highlights the different roles of  $T$  and  $\Delta$ . In fact, we notice that when  $T$  increases (from 10 to 100) with the same  $\Delta$  the MISEs are divided by 2. And if  $T$  increases and also  $\Delta$  (from  $0.005 \times 2$  to  $0.05 \times 2$ ) we can still notice a slight improvement but less clearly. However,  $T/\Delta$  has clearly no influence and this is why in the previous tables we fixed the number of observations  $T/\Delta = 2000$  ( $\Delta = 10 \times \delta$ ). Table 3 right highlights again the role of  $N$ .

## 6. CONCLUDING REMARKS

This work provides an estimator of a bivariate random effect in the drift of a stochastic differential equation based on the observation of one trajectory given by (1). The definition of the estimator using a cut-off parameter allows us to study and bound its  $L^2$ -risk. The main contribution of this paper is to produce an estimator of the common bivariate density of the random effects, from  $N$  *i.i.d.* trajectories. The Orsntein-Uhlenbeck model and the Cox-Ingersoll-Ross model both with two random effects can be studied through our estimation procedure. Furthermore simulation results illustrate the good properties of our estimator.

The method developed here for two random effects can be extended for  $d$  random effects. Extensions of the present study to more general models are interesting but require further work and other tools to accommodate the estimation procedure. First an estimation procedure should be set up for other

TABLE 1. O-U example. MISE computed from 100 simulations

Case	$\sigma$	$T$	$N$	$\tilde{f}$	$\hat{f}$	$\hat{f}_1$	$\hat{f}_2$
$\phi_1 \sim \mathcal{N}, \phi_2 \sim \Gamma$	0.1	10	20	0.064	0.064	0.067	0.073
			200	0.039	0.038	0.038	0.047
	100	20	20	0.046	0.046	0.047	0.030
			200	0.010	0.010	0.009	0.006
1	100	20	20	0.046	0.046	0.044	0.033
			200	0.012	0.012	0.011	0.007
$\phi_1 \sim \mathcal{N}, \phi_2 \sim 1/\Gamma$	0.1	10	20	0.077	0.074	0.113	0.086
			200	0.050	0.031	0.015	0.030
	100	20	20	0.068	0.068	0.083	0.041
			200	0.024	0.032	0.039	0.011
1	100	20	20	0.061	0.061	0.061	0.044
			200	0.021	0.021	0.035	0.011
$\phi_1 \sim \Gamma, \phi_2 \sim \Gamma$	0.1	10	20	0.028	0.026	0.056	0.047
			200	0.015	0.011	0.019	0.028
	100	20	20	0.024	0.024	0.029	0.029
			200	0.009	0.010	0.007	0.012
1	100	20	20	0.022	0.023	0.025	0.025
			200	0.008	0.009	0.007	0.011
$\phi_1 \sim \Gamma, \phi_2 \sim 1/\Gamma$	0.1	10	20	0.043	0.040	0.058	0.060
			200	0.030	0.024	0.029	0.036
	100	20	20	0.029	0.030	0.027	0.035
			200	0.008	0.008	0.006	0.009
1	100	20	20	0.029	0.029	0.027	0.034
			200	0.008	0.008	0.007	0.008

TABLE 2. C-I-R example. MISE computed from 100 simulations

Case	$\sigma$	$T$	$N$	$\tilde{f}$	$\hat{f}$	$\hat{f}_1$	$\hat{f}_2$
$\phi_1 \sim \Gamma, \phi_2 \sim \Gamma$	0.1	10	20	0.031	0.028	0.036	0.074
			200	0.019	0.015	0.017	0.047
	100	20	20	0.022	0.021	0.027	0.031
			200	0.005	0.004	0.006	0.006
1	100	200	200	0.006	0.006	0.012	0.010
			200	0.006	0.006	0.012	0.010
$\phi_1 \sim \Gamma, \phi_2 \sim 1/\Gamma$	0.1	10	20	0.028	0.027	0.033	0.063
			200	0.016	0.014	0.012	0.039
	100	20	20	0.020	0.019	0.027	0.025
			200	0.005	0.005	0.006	0.006
1	100	200	200	0.007	0.007	0.013	0.008
			200	0.007	0.007	0.013	0.008

TABLE 3. CIR. MISE computed from 100 simulations, with  $N = 20$  on the left and  $N = 200$  on the right,  $\sigma = 0.1$  in the case of the gamma distributions

$T$	$\delta$	$\Delta$	$T/\Delta$	$\tilde{f}$	$\hat{f}$	$T$	$\delta$	$\Delta$	$T/\Delta$	$\tilde{f}$	$\hat{f}$
10	0.005	$0.005 \times 2$	1000	0.038	0.037	10	0.005	$0.005 \times 2$	1000	0.029	0.025
100	0.05	$0.05 \times 2$	1000	0.021	0.025	100	0.05	$0.05 \times 2$	1000	0.006	0.007
100	0.005	$0.005 \times 2$	10000	0.019	0.020	100	0.005	$0.005 \times 2$	10000	0.006	0.007

drift forms. Secondly, models including a linear random effect in the diffusion coefficient could be investigated, for instance using the ideas developed in Delattre *et al.* (2014).

The nonparametric estimation of the drift function in SDEs with no random effect has been investigated in the literature (see *e.g.* Kutoyants, 2004; Comte *et al.*, 2007; Schmisser, 2013). In the case of SDEs with random effects and a general drift  $b(x, \varphi_1, \varphi_2)$ , the nonparametric estimation of the function  $b$  is open and of interest.

## 7. PROOFS

**7.1. An additional result and its proof.** We denote  $\mathcal{L}_\varphi$  the generator of the process given by (1) when  $\phi_j = \varphi$ , given for all  $F \in \mathcal{C}^2((l, r))$ , by  $\mathcal{L}_\varphi F(x) := (\sigma^2(x)/2)F''(x) + ({}^t b(x)\varphi)F'(x)$ . Then when  $F$  is a matrix, the notation  $\mathcal{L}_\varphi F$  only indicated that we apply the operator on each coefficient of the matrix. We also define in this sense  $F'$  as the derivative matrix coefficients by coefficients.

**Proposition 7.** *Consider the processes  $(X_j(t))$  given by (1) under **(A1)**-**(A4)**. Let  $g = (g_{i,k})_{1 \leq i, k \leq 2}$  a matrix of  $S_2(\mathbb{R})$ , such that  $\pi_\varphi g_{i,k}^2 < +\infty$  for all  $\varphi \in \Phi$ . Assume that*

$$F_\varphi^g = \begin{pmatrix} F_\varphi^{g_{1,1}} & F_\varphi^{g_{1,2}} \\ F_\varphi^{g_{2,1}} & F_\varphi^{g_{2,2}} \end{pmatrix}$$

*satisfies  $-\mathcal{L}_\varphi F_\varphi^g = g - \pi_\varphi g$  for all  $\varphi \in \Phi$ . Let  $J(\phi_j)$  be a non negative measurable function of  $\phi_j$  such that for all  $p \geq 1$*

$$\mathbb{E} \left[ J(\phi_j) \left( \frac{1}{T^p} \pi_{\phi_j} \left( \|F_{\phi_j}^g\|_F^{2p} \right) + \pi_{\phi_j} \left( \|(F_{\phi_j}^g)'\|_F^{2p} \sigma^{2p} \right) \right) \right] < \infty.$$

*Then for all  $p \geq 1$ , there exists a constant  $C_p > 0$  depending on  $p$  such that*

$$\begin{aligned} \mathbb{E} \left[ J(\phi_j) \left\| \sqrt{T} \left( \frac{1}{T} \int_0^T g(X_j(s)) ds - \pi_{\phi_j}(g) \right) \right\|_F^{2p} \right] &\leq C_p \left( \frac{1}{T^p} \mathbb{E} \left[ J(\phi_j) \pi_{\phi_j} \left( \|F_{\phi_j}^g\|_F^{2p} \right) \right] \right. \\ &\quad \left. + \mathbb{E} \left[ J(\phi_j) \pi_{\phi_j} \left( \|(F_{\phi_j}^g)'\|_F^{2p} \sigma^{2p} \right) \right] \right). \end{aligned}$$

We demonstrate the result without the function  $J$  but it can be added all along. The subscript  $j$  is omitted for simplicity. Denote  $g_\varphi := g - \pi_\varphi g$ . Ito's formula applied to  $F_\varphi^g$  term by term, and the equality  $\mathcal{L}_\varphi F_\varphi^g = -g_\varphi$ , lead to

$$\begin{aligned} F_\varphi^g(X^\varphi(T)) &= F_\varphi^g(X^\varphi(0)) + \int_0^T \mathcal{L}_\varphi F_\varphi^g(X^\varphi(s)) ds + \int_0^T (F_\varphi^g)'(X^\varphi) \sigma(X^\varphi(s)) dW(s) \\ &= F_\varphi^g(X^\varphi(0)) - \int_0^T g_\varphi(X^\varphi(s)) ds + \int_0^T (F_\varphi^g)'(X^\varphi(s)) \sigma(X^\varphi(s)) dW(s). \end{aligned}$$

Thus,

$$\int_0^T g_\varphi(X^\varphi(s)) ds = \int_0^T (F_\varphi^g)'(X^\varphi(s)) \sigma(X^\varphi(s)) dW(s) + (F_\varphi^g(X^\varphi(0)) - F_\varphi^g(X^\varphi(T))).$$

Then with Hölder's inequality it yields

$$\mathbb{E} \left[ \left\| \int_0^T g_\varphi(X^\varphi(s)) ds \right\|_F^{2p} \right] \leq 2^{2p-1} (\mathbb{T}_a + \mathbb{T}_b)$$

with

$$\mathbb{T}_a := \mathbb{E} \left[ \left\| \int_0^T (F_\varphi^g)'(X^\varphi(s)) \sigma(X^\varphi(s)) dW(s) \right\|_F^{2p} \right], \quad \mathbb{T}_b := \mathbb{E} \left[ \left\| F_\varphi^g(X^\varphi(0)) - F_\varphi^g(X^\varphi(T)) \right\|_F^{2p} \right]. \quad (28)$$

We study first the term  $\mathbb{T}_a$  given by (28). Again, Hölder's inequality gives

$$\begin{aligned} \mathbb{T}_a &= \mathbb{E} \left[ \left( \sum_{1 \leq i, k \leq 2} \left( \int_0^T (F_\varphi^{g_{i,k}})'(X^\varphi(s)) \sigma(X^\varphi(s)) dW(s) \right)^2 \right)^p \right] \\ &\leq 4^{p-1} \mathbb{E} \left[ \sum_{1 \leq i, k \leq 2} \left( \int_0^T (F_\varphi^{g_{i,k}})'(X^\varphi(s)) \sigma(X^\varphi(s)) dW(s) \right)^{2p} \right]. \end{aligned}$$

The first term of the sum can be studied with the Burkholder-Davis-Gundy (B.D.G.) inequality (see *e.g.* Revuz and Yor, 1999; Le Gall, 2010). Thus there exists a constant  $c_p > 0$  such that

$$\begin{aligned} \mathbb{E} \left[ \left( \int_0^T (F_\varphi^{g_{1,1}})'(X^\varphi(s)) \sigma(X^\varphi(s)) dW(s) \right)^{2p} \right] &\leq c_p \mathbb{E} \left[ \left( \int_0^T ((F_\varphi^{g_{1,1}})'(X^\varphi(s)) \sigma(X^\varphi(s)))^2 ds \right)^p \right] \\ &\leq c_p T^{p-1} \mathbb{E} \left[ \int_0^T ((F_\varphi^{g_{1,1}})'(X^\varphi(s)) \sigma(X^\varphi(s)))^{2p} ds \right] \\ &= c_p T^{p-1} T \pi_\varphi(((F_\varphi^{g_{1,1}})' \sigma)^{2p}). \end{aligned}$$

Finally there is a constant  $C_p := 4^{p-1} c_p$  verifying

$$\mathbb{T}_a \leq C_p T^p \pi_\varphi \left( \left( \sum_{1 \leq i, k \leq 2} ((F_\varphi^{g_{i,k}})' \sigma)^{2p} \right) \sigma^{2p} \right) \leq C_p T^p \pi_\varphi \left( \| (F_\varphi^g)' \|_F^{2p} \sigma^{2p} \right).$$

Furthermore, we study term  $\mathbb{T}_b$  (28) and Cauchy-Schwarz's inequality leads to

$$\begin{aligned} \mathbb{T}_b &\leq \mathbb{E} \left[ \left( (\|F_\varphi^g(X^\varphi(0))\|_F^{2p} + \|F_\varphi^g(X^\varphi(T))\|_F^{2p})^{1/(2p)} 2^{1-1/(2p)} \right)^{2p} \right] \\ &\leq 2^{2p-1} \mathbb{E} \left[ \|F_\varphi^g(X^\varphi(0))\|_F^{2p} + \|F_\varphi^g(X^\varphi(T))\|_F^{2p} \right] = 2^{2p} \pi_\varphi(\|F_\varphi^g\|_F^{2p}). \end{aligned}$$

Finally, with  $C'_p := \max(2^{2p}, C_p)$  it yields

$$\mathbb{E} \left[ \frac{1}{T^p} \left\| \int_0^T g_\varphi(X^\varphi(s)) ds \right\|_F^{2p} \right] \leq C'_p \left( \pi_\varphi \left( \| (F_\varphi^g)' \|_F^{2p} \sigma^{2p} \right) + \frac{1}{T^p} \pi_\varphi(\|F_\varphi^g\|_F^{2p}) \right). \quad \square$$

**7.2. Proof of Proposition 1.** The subscript  $j$  is omitted for simplicity.

*Proof.* First note that  $\widehat{A} - \phi = (A - \phi) \mathbf{1}_{B(T)} - \phi \mathbf{1}_{B(T)^c}$ . Thus:

$$T \|\widehat{A} - \phi\|_2^2 = T \|(A - \phi) \mathbf{1}_{B(T)} - \phi \mathbf{1}_{B(T)^c}\|_2^2 \leq 2(\mathbb{T}_1 + \mathbb{T}_2)$$

with

$$\mathbb{T}_1 := T \|(A - \phi)\|_2^2 \mathbf{1}_{B(T)}, \quad \mathbb{T}_2 := T \|\phi\|_2^2 \mathbf{1}_{B(T)^c}. \quad (29)$$

• Study of  $\mathbb{T}_1$ . We have

$$\begin{aligned} \mathbb{T}_1 &= \mathbf{1}_{B(T)} T \|V(T)^{-1} M(T)\|_2^2 = \mathbf{1}_{B(T)} \left\| TV(T)^{-1} \frac{M(T)}{\sqrt{T}} \right\|_2^2 \\ &= \mathbf{1}_{B(T)} \left\| (TV(T)^{-1} - L^{-1} + L^{-1}) \frac{M(T)}{\sqrt{T}} \right\|_2^2 \leq 2\mathbb{T}'_1 + 2\mathbb{T}_1'' \end{aligned}$$

with

$$\mathbb{T}'_1 := \mathbf{1}_{B(T)} \left\| (TV(T)^{-1} - L^{-1}) \frac{M(T)}{\sqrt{T}} \right\|_2^2, \quad \mathbb{T}_1'' := \mathbf{1}_{B(T)} \left\| L^{-1} \frac{M(T)}{\sqrt{T}} \right\|_2^2 \quad (30)$$

Let us study this two terms separately. Note that

$$\begin{aligned} \mathbb{E} \left( \frac{V(T)}{T} \middle| \phi \right) &= \frac{1}{T} \int_0^T \mathbb{E} \left( \frac{b^t b}{\sigma^2} (X(s)) \middle| \phi \right) ds, \\ \mathbb{E} \left( \frac{b^t b}{\sigma^2} (X(s)) \middle| \phi = \varphi \right) &= \mathbb{E} \left( \frac{b^t b}{\sigma^2} (X^\varphi(s)) \right) = \mathbb{E} \left( \frac{b^t b}{\sigma^2} (X^\varphi(0)) \right) = \pi_\varphi \left( \frac{b^t b}{\sigma^2} \right). \end{aligned}$$

Thus

$$L := \pi_\phi \left( \frac{b^t b}{\sigma^2} \right) = \mathbb{E} \left( \frac{V(T)}{T} \middle| \phi \right).$$

- Study of  $\mathbb{T}_1''$ . As  $L$  is  $\mathcal{F}_0$  measurable,

$$\begin{aligned} \mathbb{E}[\mathbb{T}_1''] &\leq \mathbb{E} \left[ \|L^{-1}\|_F^2 \left\| \frac{M(T)}{\sqrt{T}} \right\|_2^2 \right] = \mathbb{E} \left[ \|L^{-1}\|_F^2 \mathbb{E} \left[ \left( \frac{M_1^2(T)}{T} + \frac{M_2^2(T)}{T} \right) \middle| \mathcal{F}_0 \right] \right] \\ &= \mathbb{E} \left[ \|L^{-1}\|_F^2 \left( \frac{\langle M_1 \rangle_T}{T} + \frac{\langle M_2 \rangle_T}{T} \right) \right] = \mathbb{E} \left[ \|L^{-1}\|_F^2 \text{Tr} \left( \frac{V(T)}{T} \right) \right] \end{aligned}$$

Thus we obtain:

$$\mathbb{E}[\mathbb{T}_1''] \leq \mathbb{E} \left[ \|L^{-1}\|_F^2 \mathbb{E} \left[ \text{Tr} \left( \frac{V(T)}{T} \right) \middle| \phi \right] \right] = \mathbb{E} \left[ \|L^{-1}\|_F^2 \text{Tr}(L) \right].$$

Using Property 9 (see Appendix), we must assume :

$$\mathbb{E} \left[ \frac{\|L\|_F^2}{\det(L)^2} \text{Tr}(L) \right] < \infty. \quad (31)$$

- Study of  $\mathbb{T}'_1$  given by (30). This term will use the cut-off  $B(T)$ . We have:

$$\begin{aligned} TV(T)^{-1} - L^{-1} &= (TV(T)^{-1}L - I)L^{-1} \\ &= V(T)^{-1}(TL - V(T))L^{-1}. \end{aligned} \quad (32)$$

Then we obtain:

$$\begin{aligned} (TV(T)^{-1} - L^{-1}) \frac{M(T)}{\sqrt{T}} &= V(T)^{-1}(TL - V(T))L^{-1} \frac{M(T)}{\sqrt{T}} \\ &= \underbrace{TV(T)^{-1} \left( L - \frac{V(T)}{T} \right)}_{\text{matrix}} \underbrace{L^{-1} \frac{M(T)}{\sqrt{T}}}_{\text{vector}} \end{aligned}$$

and

$$\mathbb{E}[\mathbb{T}'_1] \leq \mathbb{E} \left[ \|TV^{-1}(T)\|_F^2 \mathbf{1}_{B(T)} \left\| L - \frac{V(T)}{T} \right\|_F^2 \left\| L^{-1} \frac{M(T)}{\sqrt{T}} \right\|_2^2 \right].$$

But  $\|V^{-1}(T)\|_F^2 = \det(V(T)^{-2}) \|V(T)\|_F^2 = \lambda_1(T)^{-2} + \lambda_2(T)^{-2}$ . And the definition of the set  $B(T)$  implies:

$$\|TV^{-1}(T)\|_F^2 \leq \frac{2T}{\kappa^2}. \quad (33)$$

Thus the Hölder inequality yields

$$\mathbb{E}[\mathbb{T}'_1] \leq \mathbb{E} \left[ \frac{2T}{\kappa^2} \left\| L - \frac{V(T)}{T} \right\|_F^2 \left\| L^{-1} \frac{M(T)}{\sqrt{T}} \right\|_2^2 \right] \leq \frac{2}{\kappa^2} (\mathbb{E}[\mathbb{T}_c] \mathbb{E}[\mathbb{T}_d])^{1/2}$$

with

$$\mathbb{T}_c := \left\| \sqrt{T} \left( L - \frac{V(T)}{T} \right) \right\|_F^4, \quad \mathbb{T}_d := \left\| L^{-1} \frac{M(T)}{\sqrt{T}} \right\|_2^4. \quad (34)$$

- Study of term  $\mathbb{T}_c$ . We apply here Proposition 7 with  $p = 2$  and  $g = b^{tb}/\sigma^2$  and get

$$\mathbb{E}[\mathbb{T}_c] \leq C \left( \frac{1}{T^2} \mathbb{E}[\|H_\phi\|_F^4] + \mathbb{E}[\|H'_\phi\|_F^4 \sigma^4] \right) < \infty. \quad (35)$$

- Study of term  $\mathbb{T}_d$ . We notice that:

$$\mathbb{E}[\mathbb{T}_d] \leq \mathbb{E} \left[ \|L^{-1}\|_F^4 \left\| \frac{M(T)}{\sqrt{T}} \right\|_2^4 \right].$$

Proceeding as for  $\mathbb{T}_1$  and using the B.D.G. inequality implies that for some constant  $c > 0$ ,

$$\begin{aligned} \mathbb{E} \left[ \|L^{-1}\|_F^4 \left\| \frac{M(T)}{\sqrt{T}} \right\|_2^4 \right] &= \mathbb{E} \left[ \|L^{-1}\|_F^4 \left( \frac{M_1^2(T) + M_2^2(T)}{T} \right)^2 \right] \\ &\leq c \mathbb{E} \left[ \|L^{-1}\|_F^4 \mathbb{E} \left[ \left( \text{Tr} \left( \frac{V(T)}{T} \right) \right)^2 \middle| \phi \right] \right]. \end{aligned}$$

And by the Hölder inequality,

$$\begin{aligned} \mathbb{E} \left[ \left( \text{Tr} \left( \frac{V(T)}{T} \right) \right)^2 \middle| \phi = \varphi \right] &= \mathbb{E} \left[ \left( \frac{1}{T} \int_0^T \left( \frac{b_1^2}{\sigma^2} (X^\varphi(s)) + \frac{b_2^2}{\sigma^2} (X^\varphi(s)) \right) ds \right)^2 \right] \\ &\leq \mathbb{E} \left[ \frac{1}{T} \int_0^T \left( \frac{b_1^2}{\sigma^2} (X^\varphi(s)) + \frac{b_2^2}{\sigma^2} (X^\varphi(s)) \right)^2 ds \right] = \text{Tr} \left( \pi_\varphi \left( \left( \frac{b^{tb}}{\sigma^2} \right)^2 \right) \right). \end{aligned}$$

Thus,

$$\mathbb{E}[\mathbb{T}_d] \leq \mathbb{E} \left[ \|L^{-1}\|_F^4 \text{Tr} \left( \pi_\phi \left( \left( \frac{b^{tb}}{\sigma^2} \right)^2 \right) \right) \right].$$

Finally we must assume that

$$\mathbb{E} \left[ \frac{\|L\|_F^4}{\det(L)^4} \text{Tr} \left( \pi_\phi \left( \left( \frac{b^{tb}}{\sigma^2} \right)^2 \right) \right) \right] < \infty. \quad (36)$$

- Study of term  $\mathbb{T}_2$ . Term  $\mathbb{T}_2$  is given by (29). Note that

$$\mathbb{E}[\mathbb{T}_2] = T \mathbb{E}[\mathbb{E}[\|\phi\|_2^2 \mathbf{1}_{\{B(T)^c\}} \middle| \phi]] = T \mathbb{E}[\|\phi\|_2^2 \mathbb{E}[\mathbf{1}_{\{B(T)^c\}} \middle| \phi]].$$

We need to compute the quantity  $\mathbb{P}(B(T)^c | \phi = \varphi)$  for all  $\varphi$ . We have:

$$B(T)^c = \left\{ \frac{V(T)}{T} \leq \frac{\kappa}{\sqrt{T}} I_2 \right\} = \left\{ L - \frac{V(T)}{T} \geq L - \frac{\kappa}{\sqrt{T}} I_2 \right\}.$$

If we denote  $l := \pi_\varphi \left( \frac{b^{tb}}{\sigma^2} \right)$ ,

$$l - \frac{\kappa}{\sqrt{T}} I_2 \geq \frac{l}{2} \Leftrightarrow l - \frac{l}{2} \geq \frac{\kappa}{\sqrt{T}} I_2 \Leftrightarrow l \geq \frac{2\kappa}{\sqrt{T}} I_2.$$

Furthermore,

$$\begin{aligned} \mathbb{P}\left(\frac{V(T)}{T} \leq \frac{\kappa}{\sqrt{T}} I_2 \mid \phi = \varphi\right) &\leq \mathbf{1}_{l \geq \frac{2\kappa}{\sqrt{T}} I_2} \mathbb{P}\left(\frac{V(T)}{T} \leq \frac{\kappa}{\sqrt{T}} I_2 \mid \phi = \varphi\right) + \mathbf{1}_{l \leq \frac{2\kappa}{\sqrt{T}} I_2} \\ &\leq \mathbb{T}_\alpha + \mathbb{T}_\beta \end{aligned}$$

with

$$\mathbb{T}_\alpha := \mathbb{P}\left(l - \frac{V(T)}{T} \geq \frac{l}{2} \mid \phi = \varphi\right), \quad \mathbb{T}_\beta := \mathbf{1}_{l \leq \frac{2\kappa}{\sqrt{T}} I_2}. \quad (37)$$

We study the two terms separately. First:

$$\mathbb{T}_\beta = \mathbf{1}_{l \leq \frac{2\kappa}{\sqrt{T}} I_2} = \mathbf{1}_{l^{-1} \geq \frac{\sqrt{T}}{2\kappa} I_2} = \mathbf{1}_{\min\{\frac{1}{\lambda_{\varphi,1}}, \frac{1}{\lambda_{\varphi,2}}\} \geq \frac{\sqrt{T}}{2\kappa}} \leq \left(\frac{4\kappa^2}{T}\right) \min\left\{\frac{1}{\lambda_{\varphi,1}}, \frac{1}{\lambda_{\varphi,2}}\right\}$$

Secondly, for the term  $\mathbb{T}_\alpha$  given by formula (37), we use Property 11 (Appendix)

$$l - \frac{V(T)}{T} \geq \frac{l}{2} \Rightarrow l_{i,i} - \frac{V_{i,i}(T)}{T} \geq \frac{l_{i,i}}{2}, \quad i = 1, 2$$

then

$$\mathbb{T}_\alpha \leq \mathbb{P}\left(l_{1,1} - \frac{V_{1,1}(T)}{T} \geq \frac{l_{1,1}}{2} \mid \phi = \varphi\right) + \mathbb{P}\left(l_{2,2} - \frac{V_{2,2}(T)}{T} \geq \frac{l_{2,2}}{2} \mid \phi = \varphi\right).$$

For the two terms of the right hand side we are able now to use Markov inequality:

$$\mathbb{T}_\alpha \leq \left(\frac{2}{l_{1,1}}\right)^2 \mathbb{E}\left[\left(l_{1,1} - \frac{V_{1,1}(T)}{T}\right)^2 \mid \phi = \varphi\right] + \left(\frac{2}{l_{2,2}}\right)^2 \mathbb{E}\left[\left(l_{2,2} - \frac{V_{2,2}(T)}{T}\right)^2 \mid \phi = \varphi\right].$$

Finally we obtain

$$\begin{aligned} \mathbb{E}[T_2] &\leq T \mathbb{E}\left[\|\phi\|_2^2 \left(\left(\frac{2}{L_{1,1}}\right)^2 \left(L_{1,1} - \frac{V_{1,1}(T)}{T}\right)^2 + \left(\frac{2}{L_{2,2}}\right)^2 \left(L_{2,2} - \frac{V_{2,2}(T)}{T}\right)^2\right.\right. \\ &\quad \left.\left.+ \left(\frac{4\kappa^2}{T}\right) \min\left\{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}\right\}\right)\right] \end{aligned} \quad (38)$$

$$\leq \mathbb{E}\left[4\|\phi\|_2^2 \left(\frac{1}{L_{1,1}^2} + \frac{1}{L_{2,2}^2}\right) \left\|\sqrt{T}\left(L - \frac{V(T)}{T}\right)\right\|^2\right] + \mathbb{E}\left[4\kappa^2 \|\phi\|_2^2 \min\left\{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}\right\}\right] \quad (39)$$

We are able to conclude this proof using the Proposition 7 with  $J(\phi) = \|\phi\|_2^2 \left(\frac{1}{L_{1,1}^2} + \frac{1}{L_{2,2}^2}\right)$ . Indeed, as  $T\mathbb{E}[\|\hat{A} - \phi\|^2] \leq 4\mathbb{E}[T_1]4\mathbb{E}[T_1] + 2\mathbb{E}[T_2]$ , gathering the bounds (31), (35), (36), and (38) we obtain that there is a constant  $C > 0$  such that  $\mathbb{E}[\|\hat{A}_j(T) - \phi_j\|_2^2] \leq C/T$ .  $\square$

### 7.3. Proof of proposition 2.

*Proof.* With the Cauchy-Schwarz inequality, we obtain the following decomposition

$$\begin{aligned} \mathbb{E}[\|\hat{f}_h - f\|^2] &= \|f - \mathbb{E}[\hat{f}_h]\|^2 + \mathbb{E}[\|\mathbb{E}[\hat{f}_h] - \hat{f}_h\|^2] \\ &\leq 2\|f - \hat{f}_h\|^2 + 2\|f_h - \mathbb{E}[\hat{f}_h]\|^2 + \mathbb{E}[\|\hat{f}_h - \mathbb{E}[\hat{f}_h]\|^2]. \end{aligned}$$

Let us study the second term of the bound. For all  $x = (x_1, x_2) \in \mathbb{R}^2$ ,

$$\mathbb{E}[\hat{f}_h(x)] = \frac{1}{N} \sum_{j=1}^N \mathbb{E}[K_h(x - \hat{A}_j(T))] = \mathbb{E}[K_h(x - \hat{A}_1(T))],$$

thus  $f_h \neq \mathbb{E}[\widehat{f}_h]$ . The subscript  $j$  (or 1) is omitted in the following for simplicity. Note that

$$f_h(x) = f_h(x_1, x_2) = f \star K_h(x_1, x_2) = \frac{1}{h_1 h_2} \int_{\mathbb{R}^2} f(u, v) K \left( \frac{x_1 - u}{h_1}, \frac{x_2 - v}{h_2} \right) dudv = \mathbb{E}[K_h(x - \phi)].$$

Then to study  $\|f_h - \mathbb{E}[\widehat{f}_h]\|^2 = \int_{\mathbb{R}^2} \mathbb{E}[K_h(x - \widehat{A}(T)) - K_h(x - \phi)]^2 dx$  we denote  $DK_h = {}^t(\frac{\partial K_h}{\partial u}, \frac{\partial K_h}{\partial v})$ , with

$$\frac{\partial K_h}{\partial u}(u, v) = \frac{1}{h_1^2 h_2} \frac{\partial K}{\partial u} \left( \frac{u}{h_1}, \frac{v}{h_2} \right), \quad \frac{\partial K_h}{\partial v}(u, v) = \frac{1}{h_1 h_2^2} \frac{\partial K}{\partial v} \left( \frac{u}{h_1}, \frac{v}{h_2} \right)$$

and applying Taylor's formula with  $U(x, t) := (U_1(x, t), U_2(x, t)) := x - \phi + t(\phi - \widehat{A}(T))$ , it yields

$$K_h(x - \widehat{A}(T)) - K_h(x - \phi) = \int_0^1 DK_h(U(x, t)) \cdot (\phi - \widehat{A}(T)) dt.$$

Thus

$$\begin{aligned} \|f_h - \mathbb{E}[\widehat{f}_h]\|^2 &\leq \int \mathbb{E} \left[ (K_h(x - \phi) - K_h(x - \widehat{A}(T)))^2 \right] dx = \mathbb{E} \left[ \int (K_h(x - \phi) - K_h(x - \widehat{A}(T)))^2 dx \right] \\ &\leq \mathbb{E} \left[ 2 \int_{\mathbb{R}^2} \left( \int_0^1 \frac{2}{h_1^2 h_2} \frac{\partial K}{\partial u} \left( \frac{U_1(x, t)}{h_1}, \frac{U_2(x, t)}{h_2} \right) dt \right)^2 dx_1 dx_2 (\phi_1 - \widehat{A}_1(T))^2 \right] \\ &\quad + \mathbb{E} \left[ 2 \int_{\mathbb{R}^2} \left( \int_0^1 \frac{2}{h_1 h_2^2} \frac{\partial K}{\partial v} \left( \frac{U_1(x, t)}{h_1}, \frac{U_2(x, t)}{h_2} \right) dt \right)^2 dx_1 dx_2 (\phi_2 - \widehat{A}_2(T))^2 \right]. \end{aligned}$$

The first term of the previous sum is bounded according to Proposition 1:

$$\begin{aligned} &\mathbb{E} \left[ \int_{\mathbb{R}^2} \left( \int_0^1 \frac{1}{h_1^2 h_2} \frac{\partial K}{\partial u} \left( \frac{U_{1,1}}{h_1}, \frac{U_{1,2}}{h_2} \right) dt \right)^2 dx_1 dx_2 (\phi_{1,1} - \widehat{A}_{1,1}(T))^2 \right] \\ &\leq \mathbb{E} \left( \frac{1}{h_1^4 h_2^2} \int_0^1 dt \int_{\mathbb{R}^2} \left( \frac{\partial K}{\partial u}(y_1, y_2) \right)^2 dy_1 dy_2 h_1 h_2 (\phi_{1,1} - \widehat{A}_{1,1}(T))^2 \right) \\ &\leq \frac{1}{h_1^3 h_2} \int_{\mathbb{R}^2} \left( \frac{\partial K}{\partial u}(y_1, y_2) \right)^2 dy_1 dy_2 \mathbb{E}[(\phi_{1,1} - \widehat{A}_{1,1}(T))^2] \leq \frac{C}{h_1^3 h_2 T} \left\| \frac{\partial K}{\partial u} \right\|^2 \end{aligned}$$

with  $C$  the constant from Equation (15). The same arguments works for the second term and finally it yields

$$\|f_h - \mathbb{E}[\widehat{f}_h]\|^2 \leq C \max \left( \frac{1}{h_1^3 h_2}, \frac{1}{h_1 h_2^3} \right) \frac{\left\| \frac{\partial K}{\partial u} \right\|^2 + \left\| \frac{\partial K}{\partial v} \right\|^2}{T}.$$

Finally, the last term is bounded by

$$\mathbb{E}[\|\widehat{f}_h - \mathbb{E}[\widehat{f}_h]\|^2] \leq \frac{\|K\|^2}{N h_1 h_2}.$$

□

**7.4. Proof of Proposition 4.** For all  $x = (x_1, x_2) \in \mathbb{R}^2$ ,

$$f_h(x) - f(x) = f \star K_h(x) - f(x) = \int_{\mathbb{R}^2} K(v) [f(x - vh) - f(x)] dv.$$

Then,

$$f(x_1 - v_1 h_1, x_2 - v_2 h_2) - f(x_1, x_2) = f(x_1 - v_1 h_1, x_2 - v_2 h_2) - f(x_1 - v_1 h_1, x_2) + f(x_1 - v_1 h_1, x_2) - f(x_1, x_2).$$

As in Tsybakov (2009) we apply Taylor's formula to the two partial functions:  $t \mapsto f(x_1 - v_1 h_1, t)$  at the order  $l_2$  and  $t \mapsto f(t, x_2)$  at the order  $l_1$ . Using the orders of the two kernels yields for the first term,

$$\begin{aligned}
 B_1(x) &:= \int_{\mathbb{R}^2} K_1(v_1)K_2(v_2)[f(x_1 - v_1h_1, x_2 - v_2h_2) - f(x_1 - v_1h_1, x_2)]dv_1dv_2 \\
 &= - \int_{\mathbb{R}^2} \frac{(h_2v_2)^{l_2}}{(l_2 - 1)!} \int_0^1 (1-t)^{l_2-1} \frac{\partial^{l_2}}{\partial x_2^{l_2}} f(x_1 - v_1h_1, x_2 - v_2h_2 + th_2v_2) dt K_1(v_1)K_2(v_2)dv_1dv_2 \\
 &= - \int_{\mathbb{R}^2} \frac{(h_2v_2)^{l_2}}{(l_2 - 1)!} \int_0^1 (1-t)^{l_2-1} \left[ \frac{\partial^{l_2}}{\partial x_2^{l_2}} f(x_1 - v_1h_1, x_2 - v_2h_2 + th_2v_2) \right. \\
 &\quad \left. - \frac{\partial^{l_2}}{\partial x_2^{l_2}} f(x_1 - v_1h_1, x_2 - v_2h_2) \right] dt K_1(v_1)K_2(v_2)dv_1dv_2
 \end{aligned} \tag{40}$$

and the analogue term  $B_2(x)$ . To evaluate  $\|f - f_h\|^2 = \int (f_h(x) - f(x))^2 dx$  we remind the generalized Minkowski's inequality (see Tsybakov, 2009): for any measurable function  $f$  on  $\mathbb{R}^2$ , we have

$$\int \left( \int f(u, x) du \right)^2 dx \leq \left[ \int \left( \int f^2(u, x) dx \right)^{1/2} du \right]^2.$$

Looking at the first term (40), and applying twice the above inequality implies for  $f \in \mathcal{N}(\beta, R)$ :

$$\begin{aligned}
 \int_{\mathbb{R}^2} B_1^2(x) dx &\leq \left( \int_{\mathbb{R}^2} |K_1(v_1)K_2(v_2)| \frac{h_2|v_2|^{l_2}}{(l_2 - 1)!} \int_0^1 (1-t)^{l_2-1} \left[ \int_{\mathbb{R}^2} \left( \frac{\partial^{l_2}}{\partial x_2^{l_2}} f(x_1 - v_1h_1, x_2 - v_2h_2 + t h_2v_2) \right. \right. \right. \\
 &\quad \left. \left. \left. - \frac{\partial^{l_2}}{\partial x_2^{l_2}} f(x_1 - v_1h_1, x_2 - v_2h_2) \right)^2 dx_1 dx_2 \right]^{1/2} dt dv_1 dv_2 \right)^2 \\
 &\leq \left[ \int_{\mathbb{R}^2} |K_1(v_1)K_2(v_2)| h_2^{\beta_2} \frac{|v_2|^{\beta_2}}{l_2!} R dv_1 dv_2 \right]^2 \leq C_2^2 h_2^{2\beta_2}
 \end{aligned}$$

with  $C_2 := (R/l_2!) \int |K_2(v_2)v_2^{\beta_2}| dv_2$ . Finally,

$$\begin{aligned}
 \int (f_h(x) - f(x))^2 dx &\leq 2 \int_{\mathbb{R}^2} B_1^2(x) dx + 2 \int_{\mathbb{R}^2} B_2^2(x) dx \\
 &\leq 2C_1^2 h_1^{2\beta_1} + 2C_2^2 h_2^{2\beta_2}. \quad \square
 \end{aligned}$$

## APPENDIX

### 7.5. Useful results of algebra.

**Property 8.** For all  $A, B \in M_2(\mathbb{R})$ ,  $\|AB\|_F \leq \|A\|_F \|B\|_F$  and for all  $x \in \mathbb{R}^2$ ,  $\|Ax\|_2 \leq \|A\|_F \|x\|_2$  where  $\|\cdot\|_2$  is the euclidean norm on  $\mathbb{R}^2$ .

**Property 9.** If  $A \in S_2(\mathbb{R})$  and  $(\lambda_1, \lambda_2) \in \mathbb{R}^2$  are its eigenvalues,  $\|A\|_F^2 = \lambda_1^2 + \lambda_2^2$ . Furthermore if  $A$  is invertible,  $(\lambda_1, \lambda_2) \in \mathbb{R}^{*2}$  and  $\|A^{-1}\|_F^2 = \left( \frac{1}{\det(A)} \right)^2 \|A\|_F^2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}$ .

**Property 10.** If  $A \in S_2(\mathbb{R})$  with eigenvalues  $\lambda_1, \lambda_2$ , then  $A \geq I_2 \Leftrightarrow \min(\lambda_1, \lambda_2) \geq 1$ .

**Property 11.** For all  $A \in S_2^+(\mathbb{R})$ ,  $a_{i,i} \geq 0$  for  $i = 1, 2$ . Then if  $A, B \in S_2(\mathbb{R})$ ,

$$A \geq B \Rightarrow a_{i,i} \geq b_{i,i} \quad i = 1, 2.$$

**7.6. Details on examples.** We refer to Genon-Catalot *et al.* (2000) and Genon-Catalot and Larédo (2014) for details and properties on the generator infinitesimal. Nevertheless we recall one of them. For  $f \in \mathcal{C}^2((l, r))$ ,

$$\mathcal{L}_\varphi f(x) = \frac{1}{2}\sigma^2(x)f''(x) + ({}^t b(x)\varphi)f'(x) = \frac{1}{2m_\varphi(x)} \left( \frac{f'}{s_\varphi} \right)'(x). \quad (41)$$

Thus for  $g \in L_{\pi_\varphi}^2$  when we are looking for the associated  $F_\varphi^g$  such that  $\mathcal{L}_\varphi F_\varphi^g = -(g - \pi_\varphi(g))$  we can use the relation  $(F_\varphi^g)'(x) = -2s_\varphi(x) \int_l^x (g(u) - \pi_\varphi(g))m_\varphi(u)du$ .

**For the O-U model.**

We want to explicit  $H_\phi = F_\phi^{b{}^t b/\sigma^2}$ . Let us denote:  $g = b{}^t b/\sigma^2 = (g_{i,k})_{1 \leq i, k \leq 2}$ . First:  $g_{1,1}(x) - \pi_\varphi g_{1,1} = 0 = -\mathcal{L}_\varphi F_\varphi^{g_{1,1}}(x)$ . For example  $F_\varphi^{g_{1,1}}(x) = 1$  is suitable. Then,  $g_{1,2}(x) - \pi_\varphi g_{1,2} = -\frac{1}{\sigma^2} \left( x - \frac{\varphi_1}{\varphi_2} \right)$ . We look for  $F_\varphi^{g_{1,2}}(x) = -\frac{a}{\sigma^2} \left( x - \frac{\varphi_1}{\varphi_2} \right)$  and we obtain

$$F_\varphi^{g_{1,2}}(x) = -\frac{1}{\varphi_2 \sigma^2} \left( x - \frac{\varphi_1}{\varphi_2} \right).$$

Finally

$$g_{2,2}(x) - \pi_\varphi g_{2,2} = \frac{x^2}{\sigma^2} - \frac{1}{2\varphi_2} + \frac{\varphi_1^2}{\sigma^2 \varphi_2^2} = \frac{1}{\sigma^2} \left[ \left( x - \frac{\varphi_1}{\varphi_2} \right)^2 + 2\frac{\varphi_1}{\varphi_2} \left( x - \frac{\varphi_1}{\varphi_2} \right) \right] - \frac{1}{2\varphi_2}.$$

We look for  $F_\varphi^{g_{2,2}}$  with the same functional form and we obtain

$$F_\varphi^{g_{2,2}}(x) = \frac{2\varphi_1}{\sigma^2 \varphi_2^2} \left( x - \frac{\varphi_1}{\varphi_2} \right) + \frac{1}{2\sigma^2 \varphi_2} \left[ \left( x - \frac{\varphi_1}{\varphi_2} \right)^2 - \frac{\sigma^2}{2\varphi_2} \right].$$

**For the C-I-R model.**

We want to explicit  $H_\phi = F_\phi^{b{}^t b/\sigma^2}$ . First:  $g_{1,1}(x) - \pi_\varphi g_{1,1} = 1/(\sigma^2 x) - 2\varphi_2/((2\varphi_1 - \sigma^2)\sigma^2)$ . Here we use formula (41). We have

$$(F_\varphi^{g_{1,1}})'(x) = -2s_\varphi(x) \int_0^x \left( \frac{1}{\sigma^2 u} - \frac{2\varphi_2}{\sigma^2(2\varphi_1 - \sigma^2)} \right) m_\varphi(u) du$$

with  $s_\varphi(x) = e^{2\varphi_2 x/\sigma^2} x^{-2\varphi_1/\sigma^2}$ . This is equivalent to

$$(F_\varphi^{g_{1,1}})'(x) = -\frac{2}{\sigma^4} e^{2\varphi_2 x/\sigma^2} x^{-2\varphi_1/\sigma^2} \left[ \int_0^x e^{-2\varphi_2 u/\sigma^2} u^{2\varphi_1/\sigma^2 - 2} du - \int_0^x \frac{2\varphi_2}{2\varphi_1 - \sigma^2} e^{-2\varphi_2 u/\sigma^2} u^{2\varphi_1/\sigma^2 - 1} du \right]$$

and an integration by part in the first integral gives

$$(F_\varphi^{g_{1,1}})'(x) = -\frac{2}{\sigma^4} e^{2\varphi_2 x/\sigma^2} x^{-2\varphi_1/\sigma^2} \left[ -\frac{e^{-2\varphi_2 x/\sigma^2} x^{2\varphi_1/\sigma^2 - 1}}{2\varphi_1/\sigma^2 - 1} \right] = \frac{2}{\sigma^2(2\varphi_1 - \sigma^2)x}.$$

We finally set

$$F_\varphi^{g_{1,1}}(x) = \frac{2}{\sigma^2(2\varphi_1 - \sigma^2)} \log(x).$$

Furthermore  $g_{1,2}(x) - \pi_\varphi g_{1,2} = 0 = -\mathcal{L}_\varphi F_\varphi^{g_{1,2}}(x)$ , and then for example  $F_\varphi^{g_{1,2}}(x) = 1$ . At last  $g_{2,2}(x) - \pi_\varphi g_{2,2} = \frac{1}{\sigma^2} \left( x - \frac{\varphi_1}{\varphi_2} \right)$  and we obtain

$$F_\varphi^{g_{2,2}}(x) = \frac{1}{\varphi_2 \sigma^2} \left( x - \frac{\varphi_1}{\varphi_2} \right).$$

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