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Hassan Jaber

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HARDY-SOBOLEV EQUATIONS ON COMPACT RIEMANNIAN MANIFOLDS

HASSAN JABER

ABSTRACT. Let (M, g) be a compact Riemannian Manifold of dimension $n \geq 3$, $x_0 \in M$, and $s \in (0, 2)$. We let $2^*(s) := \frac{2(n-s)}{n-2}$ be the critical Hardy-Sobolev exponent. We investigate the existence of positive distributional solutions $u \in C^0(M)$ to the critical equation

$$\Delta_g u + a(x)u = \frac{u^{2^*(s)-1}}{d_g(x, x_0)^s} \quad \text{in } M$$

where $\Delta_g := -\operatorname{div}_g(\nabla)$ is the Laplace-Beltrami operator, and d_g is the Riemannian distance on (M, g) . Via a minimization method in the spirit of Aubin, we prove existence in dimension $n \geq 4$ when the potential a is sufficiently below the scalar curvature at x_0 . In dimension $n = 3$, we use a global argument and we prove existence when the mass of the linear operator $\Delta_g + a$ is positive at x_0 . As a byproduct of our analysis, we compute the best first constant for the related Riemannian Hardy-Sobolev inequality.

Let (M, g) be a compact Riemannian Manifold of dimension $n \geq 3$ without boundary. Given $s \in (0, 2)$, $x_0 \in M$, and $a \in C^0(M)$, we consider distributional solutions $u \in C^0(M)$ to the equation

$$(1) \quad \Delta_g u + a(x)u = \frac{u^{2^*(s)-1}}{d_g(x, x_0)^s} \quad \text{in } M$$

where $2^*(s) := \frac{2(n-s)}{n-2}$ is the Hardy-Sobolev exponent. More precisely, let $H_1^2(M)$ be the completion of $C^\infty(M)$ for the norm $u \mapsto \|u\|_2 + \|\nabla u\|_2$. The exponent $2^*(s)$ is critical in the following sense: the Sobolev space $H_1^2(M)$ is continuously embedded in the weighted Lebesgue space $L^p(M, d_g(\cdot, x_0)^{-s})$ if and only if $1 \leq p \leq 2^*(s)$, and this embedding is compact if and only if $1 \leq p < 2^*(s)$.

There is an important literature on Hardy-Sobolev equations in the Euclidean setting of a domain of \mathbb{R}^n , in particular to show existence or non-existence of solutions, see for instance Ghoussoub-Kang [5], Ghoussoub-Yuan [7], Li-Ruf-Guo-Niu [15], Musina [16], Pucci-Servadei [17], Kang-Peng [12], and the references therein. In particular, in the spirit of Brezis-Nirenberg, Ghoussoub-Yuan [7] proved the existence of solutions for equations like (1) when $n \geq 4$ and the potential a achieves negative values at the interior singular point x_0 . In the present manuscript, our objective is both to study the influence of the curvature when dealing with a Riemannian Manifold, and to tackle dimension $n = 3$.

We consider the functional

$$J(u) := \frac{\int_M (|\nabla u|_g^2 + au^2) dv_g}{\left(\int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}}}; \quad u \in H_1^2(M) \setminus \{0\},$$

which is well-defined due to the above-mentioned embeddings. Here dv_g denotes the Riemannian element of volume. When the operator $\Delta_g + a$ is coercive, then, up to multiplication by a positive constant, critical points of the functional J (if they exist) are solutions to equation (1). In the sequel, we assume that $\Delta_g + a$ is coercive. In the spirit of Aubin [1], we investigate the existence of solutions to (1) by minimizing the functional J : it is classical for this type of problem that the difficulty is the lack of compactness for the critical embedding. Since the resolution of the Yamabe problem (see [1], [19] and [24]), it is also well known that there exists a dichotomy between high dimension (see Aubin [1]) where the arguments are local, and small dimension (see Schoen [19]) where the arguments are global.

In the sequel, we let $\text{Scal}_g(x)$ be the scalar curvature at $x \in M$. We let $G_{x_0} : M \setminus \{x_0\} \rightarrow \mathbb{R}$ be the Green's function at x_0 for the operator $\Delta_g + a$ (this is defined since the operator is coercive). In dimension $n = 3$, there exists $m(x_0) \in \mathbb{R}$ such that for all $\alpha \in (0, 1)$

$$G_{x_0}(x) = \frac{1}{\omega_2 d_g(x, x_0)} + m(x_0) + O(d_g(x, x_0)^\alpha) \text{ when } x \rightarrow x_0.$$

Here and in the sequel, ω_k denote the volume of the canonical k -dimensional unit sphere \mathbb{S}^k , $k \geq 1$. The quantity $m(x_0)$ is referred to as the mass of the point $x_0 \in M$. Our main result states as follows:

Theorem 1. *Let $x_0 \in M$, $s \in (0, 2)$, and $a \in C^0(M)$ be such that the operator $\Delta_g + a$ is coercive. We assume that*

$$(2) \quad \left\{ \begin{array}{ll} a(x_0) < c_{n,s} \text{Scal}_g(x_0) & \text{if } n \geq 4 \\ m(x_0) > 0 & \text{if } n = 3. \end{array} \right\}$$

with $c_{n,s} := \frac{(n-2)(6-s)}{12(2n-2-s)}$. Then there exists a positive solution $u \in C^0(M) \cap H_1^2(M)$ to the Hardy-Sobolev equation (1). Moreover, $u \in C^{0,\theta}(M)$ for all $\theta \in (0, \min\{1, 2-s\})$ and we can choose u as a minimizer of J .

As a consequence of the Positive Mass Theorem (see [20], [21]), we get (see Druet [3] and Proposition 2 in Section 4 below) that $m(x_0) > 0$ for $n = 3$ when $a \leq \text{Scal}_g/8$, with the additional assumption that (M, g) is not conformally equivalent to the canonical 3-sphere if $a \equiv \text{Scal}_g/8$.

Theorem 1 suggests some remarks. For equations of scalar curvature type, that is when $s = 0$, a similar result was obtained by Aubin [1] (for $n \geq 4$) and by Schoen [19] (see also Druet [3]) (for $n = 3$): however, when $s \in (0, 2)$, the problem is subcritical outside the singular point x_0 , and therefore it is natural to get a condition at this point. Another remark is that, when $s = 0$, Aubin (see [1]) obtained the constant $c_{n,0}$ when $n \geq 4$, the potential $c_{n,0} \text{Scal}_g$ being such that the Yamabe equation is conformally invariant. When $s \in (0, 2)$, the critical equation enjoys no suitable conformal invariance due to the singular term $d_g(\cdot, x_0)^{-s}$, and, despite our existence result involves the scalar curvature, one gets another constant $c_{n,s}$.

It is also to notice that, unlike the case $s = 0$, the solutions to equations like (1) are not C^2 . This lead us to handle with care issues related to the maximum principle, for which we develop a suitable approach. As in Aubin, the minimization approach leads to computing some test-function estimates. However, unlike the case $s = 0$,

the terms involved in the expansion of the functional are not explicit and we need to collect them suitably to obtain the explicit value of $c_{n,s}$ above.

The proof of Theorem 1 uses the best constant in the Hardy-Sobolev inequality. It follows from the Hardy-Sobolev embedding that there exist $A, B > 0$ such that

$$(3) \quad \left(\int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq A \int_M |\nabla u|_g^2 dv_g + B \int_M u^2 dv_g$$

for all $u \in H_1^2(M)$. We let $A_0(M, g, s, x_0)$ be the best first constant of the Riemannian Hardy-Sobolev inequality, that is

$$(4) \quad A_0(M, g, s, x_0) := \inf\{A > 0; (3) \text{ holds for all } u \in H_1^2(M)\}.$$

We prove the following:

Theorem 2. *Let (M, g) be a compact Riemannian Manifold of dimension $n \geq 3$, $x_0 \in M$, $s \in (0, 2)$ and $2^*(s) = \frac{2(n-s)}{n-2}$. Then*

$$A_0(M, g, s, x_0) = K(n, s),$$

where $K(n, s)$ is the optimal constant of the Euclidean Hardy-Sobolev inequality, that is

$$(5) \quad K(n, s)^{-1} := \inf_{\varphi \in C_c^\infty(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla \varphi|^2 dX}{\left(\int_{\mathbb{R}^n} \frac{|\varphi|^{2^*(s)}}{|X|^s} dX \right)^{\frac{2}{2^*(s)}}}$$

Theorem 2 was proved by Aubin [2] for the case $s = 0$. The value of $K(n, s)$ is

$$K(n, s) = [(n-2)(n-s)]^{-1} \left(\frac{1}{2-s} \omega_{n-1} \frac{\Gamma^2(n-s/2-s)}{\Gamma(2(n-s)/2-s)} \right)^{-\frac{2-s}{n-s}}.$$

It was computed independently by Aubin [2], Rodemich [18] and Talenti [22] for the case $s = 0$, and the value for $s \in (0, 2)$ has been computed by Lieb (see [14], Theorem 4.3).

A natural question is to know whether the infimum $A_0(M, g, s, x_0)$ is achieved or not, that is if there exists $B > 0$ such that equality (3) holds for all $u \in H_1^2(M)$ with $A = K(n, s)$. The answer is positive: this is the object of the work [11].

A very last remark is that Theorem 1 holds when M is a compact manifold with boundary provided x_0 lies in the interior. In particular, we extend Ghossoub-Yuan's [7] result to dimension $n = 3$:

Theorem 3. *Let Ω be a smooth bounded domain of \mathbb{R}^3 and let $x_0 \in \Omega$ be an interior point. For $a \in C^0(\overline{\Omega})$ such that $\Delta + a$ is coercive, we define the Robin function as $R(x, y) := \omega_2^{-1} |x - y|^{-1} - G_x(y)$ where G is the Green's function for $\Delta + a$ with Dirichlet boundary condition. We assume that $R(x_0, x_0) < 0$. Then there exists a function $u \in C^{0,\theta}(\overline{\Omega})$ for all $\theta \in (0, \min\{1, 2-s\})$ to the Hardy-Sobolev equation*

$$\Delta u + a(x)u = \frac{u^{2^*(s)-1}}{|x - x_0|^s}, u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega.$$

This paper is organized as follows. In Section 1, we prove Theorem 2. In Section 2, we prove a general existence theorem for solutions to equation (1). In Section 3, we compute the full expansion of the functional J taken at the relevant test-functions

for dimension $n \geq 4$. In Section 4, we perform the test-functions estimate for the specific dimension $n = 3$ and prove Theorems 1 and 3.

After this work was completed, we learned that Thiam [23] has independently studied similar issues.

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1. THE BEST CONSTANT IN THE HARDY-SOBOLEV INEQUALITY

In this section, we will prove Theorem 2. For that, we begin by the following proposition :

Proposition 1. *Let (M, g) be a compact Riemannian Manifold of dimension $n \geq 3$, $x_0 \in M$, $s \in (0, 2)$. For any $\epsilon > 0$, there exists $B_\epsilon > 0$ such that*

$$(6) \quad \left(\int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq (K(n, s) + \epsilon) \int_M |\nabla u|_g^2 dv_g + B_\epsilon \int_M u^2 dv_g.$$

for all $u \in H_1^2(M)$.

Thiam [23] proved a result in the same spirit with addition of an extra remainder term. The case $s = 0$ has been proved by Aubin [2] (see also [9], [10] for an exposition in book form). We adapt this proof to our case.

Proof. Step 1: Covering of M by geodesic balls. For any $x \in M$, we denote as \exp_x the exponential map at x with respect to the metric g . In the sequel, for any $r > 0$ and $z \in M$, $\mathbb{B}_r(z) \subset M$ denotes the ball of center 0 and of radius r for the Riemannian distance d_g . For any $x \in M$ and any $\rho > 0$, there exist $r = r(x, \rho) \in (0, i_g(M)/2)$, $\lim_{\rho \rightarrow 0} r(x, \rho) = 0$ (here, $i_g(M)$ denotes the injectivity radius of (M, g)) such that the exponential chart $(\mathbb{B}_{2r}(x), \exp_x^{-1})$ satisfies the following properties: on $\mathbb{B}_{2r}(x)$, we have that

$$\begin{aligned} (1 - \rho)\delta &\leq g \leq (1 + \rho)\delta, \\ (1 - \rho)^{\frac{n}{2}} dx &\leq dv_g \leq (1 + \rho)^{\frac{n}{2}} dx, \\ D_\rho^{-1}|T|_\delta &\leq |T|_g \leq D_\rho|T|_\delta, \text{ for all } T \in \chi(T^*M) \end{aligned}$$

where $\lim_{\rho \rightarrow +\infty} D_\rho = 1$, $\chi(T^*M)$ denotes the space of 1-covariant tensor fields on M , δ is the Euclidean metric on \mathbb{R}^n , that is the standard scalar product on \mathbb{R}^n , and we have assimilated g to the local metric $(\exp_x)^*g$ on \mathbb{R}^n via the exponential map.

It follows from the compactness of M that there exists $N \in \mathbb{N}$ (depending on ρ) and $x_1, \dots, x_{N-1} \in M \setminus \mathbb{B}_{\frac{r_0}{2}}(x_0)$ (depending on ρ) such that

$$M \setminus \mathbb{B}_{r_0}(x_0) \subset \cup_{m=1}^{N-1} \mathbb{B}_{r_m}(x_m),$$

where $r_0 = r(x_0, \rho)$ and $r_m = r(x_m, \rho)$, $r_m < r_0^2$, for $m = 0, \dots, N-1$.

Step 2: We claim that for all $\epsilon > 0$ there exists $\rho_0 = \rho_0(\epsilon) > 0$ such that $\lim_{\epsilon \rightarrow 0} \rho_0(\epsilon) = 0$ and for all $\rho \in (0, \rho_0)$, all $m \in \{0, \dots, N-1\}$ and all $u \in C_c^\infty(\mathbb{B}_{r_m}(x_m))$, we have that :

$$(7) \quad \left(\int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq \left(K(n, s) + \frac{\epsilon}{2} \right) \int_M |\nabla u|_g^2 dv_g.$$

Indeed, it follows from (5) that for all $\varphi \in C_c^\infty(\mathbb{R}^n)$:

$$(8) \quad \left(\int_{\mathbb{R}^n} \frac{|\varphi|^{2^*(s)}}{|X|_\delta^s} dX \right)^{\frac{2}{2^*(s)}} \leq K(n, s) \int_{\mathbb{R}^n} |\nabla \varphi|_\delta^2 dX.$$

We consider $\rho > 0$, $m \in \{0, \dots, N\}$ and $u \in C_c^\infty(\mathbb{B}_{r_m}(x_m))$ such that $(\mathbb{B}_{r_m}(x_m), \exp_{x_m}^{-1})$ is an exponential card as in Step 1. We distinguish two cases :

Case 2.1 : If $m = 0$ then using the properties of the exponential card $(\mathbb{B}_{r_0}(x_0), \exp_{x_0}^{-1})$, developed in Step 1, and the Euclidean Hardy-Sobolev inequality (8), we write

$$\begin{aligned} \left(\int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} &\leq (1 + \rho)^{\frac{n}{2^*(s)}} K(n, s) \int_{\mathbb{R}^n} |\nabla(u \circ \exp_{x_m})|_\delta^2 dX \\ &\leq D_\rho^2 (1 + \rho)^{\frac{n}{2^*(s)}} (1 - \rho)^{\frac{-n}{2}} K(n, s) \int_M |\nabla u|_g^2 dv_g. \end{aligned}$$

Letting $\rho \rightarrow 0$, we get (7), for all $u \in C_c^\infty(\mathbb{B}_{r_0}(x_0))$, when $m = 0$. This proves (7) in the Case 2.1.

Case 2.2 : If $m \in \{1, \dots, N-1\}$ then for all $x \in \mathbb{B}_{r_m}(x_m)$, we have :

$$d_g(x, x_0) \geq \lambda_0 > 0,$$

with $\lambda_0 = \frac{r_0}{2} - r_m$. Thanks to Hölder's inequalities and inequalities of Gagliardo-Nirenberg-Sobolev, we can write that :

$$\begin{aligned} \left(\int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} &\leq \frac{\text{vol}(\mathbb{B}_{r_m}(x_m))^{2\left(\frac{1}{2^*(s)} - \frac{1}{2^*}\right)}}{\lambda_0^{\frac{2s}{2^*(s)}}} \left(\int_{\mathbb{B}_{r_m}(x_m)} |u|^{2^*} dv_g \right)^{\frac{2}{2^*}} \\ &\leq Q'_\rho \int_M |\nabla u|_g^2 dv_g, \end{aligned}$$

where $\lim_{\rho \rightarrow 0} Q'_\rho = 0$ and $2^* := 2n/(n-2)$ is the Sobolev exponent. Letting $\rho \rightarrow 0$, we get (7), for all $u \in C_c^\infty(\mathbb{B}_{r_m}(x_m))$, when $m \geq 1$. This ends Step 2.

Step 3 : We fix $\epsilon > 0$, $\rho \in (0, \rho_0(\epsilon))$ and x_1, \dots, x_N as in Step 1 and Step 2. We consider now $(\alpha_m)_{m=0, \dots, N-1}$ a C^∞ -partition of unity subordinate to the covering $(\mathbb{B}_{r_m}(x_m))_{m=0, \dots, N-1}$ of M and define, for all $m = 0, \dots, N-1$, a function η_m on M by

$$\eta_m = \frac{\alpha_m^3}{\sum_{i=0}^{N-1} \alpha_i^3}.$$

We can see easily that $(\eta_m)_{m=0, \dots, N-1}$ is a C^∞ -partition of unity subordinate to the covering $(\mathbb{B}_{r_m}(x_m))_{m=0, \dots, N-1}$ of M s.t. $\eta_m^{\frac{1}{2}} \in C^1(M)$, for every $m = 0, \dots, N-1$. We let $H > 0$ satisfying for each $m = 0, \dots, N-1$:

$$(9) \quad |\nabla \eta_m^{\frac{1}{2}}|_g \leq H.$$

Step 4 : In this step, we will prove the Hardy-Sobolev inequality on $C^\infty(M)$. Indeed, we let $\epsilon > 0$ and $(\eta_m)_{m=0, \dots, N-1}$ be a C^∞ -partition of unity as in Step 3

and consider $u \in C^\infty(M)$. Since $\frac{2^*(s)}{2} > 1$, we get that :

$$\begin{aligned} & \left(\int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq \left(\int_M \frac{|\sum_{m=0}^{N-1} \eta_m u^2|^{\frac{2^*(s)}{2}}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \\ & \leq \left\| \sum_{m=0}^{N-1} \eta_m u^2 \right\|_{L^{\frac{2^*(s)}{2}}(M, d_g(x, x_0)^{-s})} \leq \sum_{m=0}^{N-1} \|\eta_m u^2\|_{L^{\frac{2^*(s)}{2}}(M, d_g(x, x_0)^{-s})} \\ & \leq \sum_{m=0}^{N-1} \left(\int_M \frac{|\eta_m^{\frac{1}{2}} u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}}. \end{aligned}$$

Using inequality (7) in Step 2 and by density ($\eta_m^{\frac{1}{2}} u \in C^1(M)$), we get that

$$\left(\int_M \frac{|\eta_m^{\frac{1}{2}} u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq (K(n, s) + \frac{\epsilon}{2}) \int_M |\nabla(\eta_m^{\frac{1}{2}} u)|_g^2 dv_g.$$

Hence

$$\begin{aligned} \left(\int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} & \leq (K(n, s) + \frac{\epsilon}{2}) \sum_{m=0}^{N-1} \int_M \left(\eta_m |\nabla u|_g^2 + 2\eta_m^{\frac{1}{2}} |\nabla u|_g |u| |\nabla \eta_m^{\frac{1}{2}}|_g \right. \\ & \quad \left. + |u|^2 |\nabla \eta_m^{\frac{1}{2}}|_g^2 \right) dv_g. \end{aligned}$$

Using the Cauchy-Schwarz inequality and (9) from Step 3, we get that:

$$(10) \quad \left(\int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq (K(n, s) + \frac{\epsilon}{2}) (\|\nabla u\|_2^2 + 2NH\|\nabla u\|_2 \|u\|_2 + NH^2 \|u\|_2^2).$$

We choose now $\epsilon_0 > 0$ s.t.

$$(11) \quad (K(n, s) + \frac{\epsilon}{2})(1 + \epsilon_0) \leq K(n, s) + \epsilon.$$

Since

$$(12) \quad 2NH\|\nabla u\|_2 \|u\|_2 \leq \epsilon_0 \|\nabla u\|_2^2 + \frac{(NH)^2}{\epsilon_0} \|u\|_2^2,$$

then by combining (10) with (11) and (12), we get that :

$$\left(\int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq (K(n, s) + \epsilon) \int_M |\nabla u|_g^2 dv_g + B_\epsilon \int_M |u|^2 dv_g,$$

where $B_\epsilon = \frac{(NH)^2}{\epsilon_0} + NH^2(K(n, s) + \frac{\epsilon}{2})$. This proves inequality (6) for functions $u \in C^\infty(M)$. The inequality for $H_1^2(M)$ follows by density. This ends the proof of Proposition 1. \square

Proof of Theorem 2: We let $A \in \mathbb{R}$ be such that there exists $B > 0$ such that inequality (3) holds for all $u \in H_1^2(M)$. Therefore, we have that

$$(13) \quad \left(\int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq A \int_M |\nabla u|_g^2 dv_g + B \int_M |u|^2 dv_g.$$

We consider $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\text{Supp } \phi \subset \mathbb{B}_R(0)$, $R > 0$ and $(\mathbb{B}_{\rho_0}(x_0), \exp_{x_0}^{-1})$ an exponential chart centered at x_0 with $\rho_0 \in (0, i_g(M))$. For all $\mu > 0$ sufficiently small ($\mu \leq \frac{\rho_0}{R}$), we let $\phi_\mu \in C^\infty(\mathbb{B}_{\rho_0}(x_0))$ be such that

$$\phi_\mu(x) = \phi(\mu^{-1} \exp_{x_0}^{-1}(x))$$

for all $x \in \mathbb{B}_{\rho_0}(x_0)$. Applying, by density, (13) to ϕ_μ , we write :

$$(14) \quad \left(\int_{\mathbb{B}_{\mu R}(x_0)} \frac{|\phi_\mu|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq A \int_{\mathbb{B}_{\mu R}(x_0)} |\nabla \phi_\mu|_g^2 dv_g + B \int_{\mathbb{B}_{\mu R}(x_0)} \phi_\mu^2 dv_g.$$

For all $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$(1 - \epsilon)\delta \leq g \leq (1 + \epsilon)\delta$$

in $\mathbb{B}_{R_\epsilon}(x_0)$, where g is assimilated to the local metric $(\exp_{x_0})^*g$ on \mathbb{R}^n . Then, for all $\mu > 0$ sufficiently small such that $R\mu < R_\epsilon$, we get successively that :

$$(15) \quad \int_{\mathbb{B}_{\mu R}(x_0)} \frac{|\phi_\mu|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \geq (1 - \epsilon)^{\frac{n}{2}} \mu^{n-s} \int_{\mathbb{B}_R(0)} \frac{\phi^{2^*(s)}(X)}{|X|^s} dX,$$

$$(16) \quad \int_{\mathbb{B}_{\mu R}(x_0)} |\nabla \phi_\mu|_g^2 dv_g \leq (1 + \epsilon)^{\frac{n}{2}+1} \mu^{n-2} \int_{\mathbb{B}_R(0)} |\nabla \phi|_\delta^2 dX$$

and

$$(17) \quad \int_{\mathbb{B}_{\mu R}(x_0)} \phi_\mu^2 dv_g \leq (1 + \epsilon)^{\frac{n}{2}} \mu^n \int_{\mathbb{B}_R(0)} \phi^2 dX.$$

Plugging the estimates (15), (16) and (17) into (14), letting $\mu \rightarrow 0$ and then $\epsilon \rightarrow 0$, we get that

$$\left(\int_{\mathbb{R}^n} \frac{\phi^{2^*(s)}(X)}{|X|^s} dX \right)^{\frac{2}{2^*(s)}} \leq A \int_{\mathbb{R}^n} |\nabla \phi|_\delta^2 dX, \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n).$$

It then follows from the definition of $K(n, s)$ that $A \geq K(n, s)$. Therefore, it follows from the definition of $A_0(M, g, s, x_0)$ that $A_0(M, g, s, x_0) \geq K(n, s)$. By Proposition 1, we have that $A_0(M, g, s, x_0) \leq K(n, s)$. Therefore, $A_0(M, g, s, x_0) = K(n, s)$. This proves Theorem 2.

Remark: Proposition 1 does not allow to conclude whether $A_0(M, g, s, x_0)$ is achieved or not, that is of one can take $\epsilon = 0$ in (6). Indeed, in our construction, when $\epsilon \rightarrow 0$, $r_m \rightarrow 0$ and then $H \geq |\nabla \eta_m^{\frac{1}{2}}|_g \rightarrow +\infty$ (see the proof of Proposition 1). This implies that $\lim_{\epsilon \rightarrow 0} B_\epsilon = +\infty$. Proving that $A_0(M, g, s, x_0)$ is achieved required different techniques and blow-up analysis: this is the object of the article [11].

2. A GENERAL EXISTENCE THEOREM

This section is devoted to the proof of the following Theorem:

Theorem 4. *Let (M, g) be a compact Riemannian Manifold of dimension $n \geq 3$ without boundary. We fix $s \in (0, 2)$, $x_0 \in M$, and $a \in C^0(M)$ such that $\Delta_g + a$ is coercive. We assume that*

$$(18) \quad \inf_{u \in H_1^2(M) \setminus \{0\}} J(u) < \frac{1}{K(n, s)}$$

Then the infimum of J on $H_1^2(M) \setminus \{0\}$ is achieved by a positive function $u \in H_1^2(M) \cap C^0(M)$. Moreover, up to homothety, u is a solution to (1) and $u \in C^{0,\theta}(M) \cap C_{loc}^{1,\alpha}(M \setminus \{x_0\})$ for all $\theta \in (0, \min\{1, 2-s\})$ and $\alpha \in (0, 1)$.

The existence of a minimizer of J in $H_1^2(M) \setminus \{0\}$ has been proved independently by Thiam [23].

We prove Theorem 4 via the classical subcritical approach. For any $q \in (2, 2^*(s))$, we define

$$J_q(u) := \frac{\int_M (|\nabla u|_g^2 + au^2) dv_g}{\left(\int_M \frac{|u|^q}{d_g(x, x_0)^s} dv_g\right)^{\frac{2}{q}}}; \quad u \in H_1^2(M),$$

and

$$\mathcal{H}_q = \left\{ u \in H_1^2(M); \int_M \frac{|u|^q}{d_g(x, x_0)^s} dv_g = 1 \right\}.$$

Finally, we define:

$$\lambda_q = \inf_{u \in H_1^2(M) \setminus \{0\}} J_q(u).$$

We fix $q \in (2, 2^*(s))$. Since the embedding $H_1^2(M) \hookrightarrow L^q(M, d_g(\cdot, x_0)^{-s})$ is compact, there exists a minimizer for $\lambda_q^{(s)}$. More precisely, there exists $u_q \in H_1^2(M) \setminus \{0\} \cap \mathcal{H}_q$, $u_q \geq 0$ a.e. such that u_q verifies weakly the subcritical Hardy-Sobolev equation :

$$\Delta_g u_q + a u_q = \lambda_q \frac{u_q^{q-1}}{d_g(x, x_0)^s} \text{ in } M.$$

In particular, we have that $\lambda_q = J_q(u_q)$.

Now we proceed in several steps.

Step 1: We claim that the sequence $(\lambda_q)_q$ converge to $\lambda_{2^*(s)}$ when $q \rightarrow 2^*(s)$.

The proof follows the standard method described in [25] and [1] for instance. We omit the proof.

Step 2: As one checks, the sequence $(u_q)_q$ is bounded in $H_1^2(M)$ independently of q . Therefore, there exists $u \in H_1^2(M)$, $u \geq 0$ a.e. such that, up to a subsequence, $(u_q)_q$ converge to u weakly in $H_1^2(M)$ and strongly in $L^2(M)$, moreover, the convergence holds a.e. in M . It is classical (see [25] and [1]) that $u \in H_1^2(M)$ is a weak solution to

$$\Delta_g u + a u = \lambda_{2^*(s)} \frac{u^{2^*(s)-1}}{d_g(x, x_0)^s} \text{ in } M; \quad u \geq 0 \text{ a.e. in } M.$$

Step 3: We claim that $u \not\equiv 0$ is a minimizer of $J^{(s)}$ and that $(u_q)_q \rightarrow u$ strongly in $H_1^2(M)$.

Indeed, it follows from the hypothesis (18) that there exists $\epsilon_0 > 0$ such that

$$(19) \quad \lambda_{2^*(s)}(K(n, s) + \epsilon_0) < 1.$$

Now from Proposition 1, we know that there exists $B_{\epsilon_0} > 0$ such that for all $q \in (2, 2^*(s))$:

$$(20) \quad \left(\int_M \frac{|u_q|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq (K(n, s) + \epsilon_0) \int_M |\nabla u_q|^2 dv_g + B_{\epsilon_0} \int_M u_q^2 dv_g.$$

Hölder inequality and $u_q \in H_q$ yield:

$$(21) \quad \left(\int_M \frac{|u_q|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \geq 1 + o(1)$$

where $o(1) \rightarrow 0$ when $q \rightarrow 2^*(s)$. Combining (20) and (21), we get :

$$\left[(K(n, s) + \epsilon_0)\lambda_q + B_{\epsilon_0} \int_M u_q^2 dv_g \right] \geq 1 + o(1),$$

where $o(1) \rightarrow 0$ when $q \rightarrow 2^*(s)$. Letting $q \rightarrow 2^*(s)$ in the last relation, we write :

$$(K(n, s) + \epsilon_0)\lambda_{2^*(s)} + B(\epsilon_0) \int_M u^2 dv_g \geq 1.$$

It then follows from (19) that $B_{\epsilon_0} \int_M u^2 dv_g > 0$, and then $u \not\equiv 0$. It is then classical that $u \in H_1^2(M)$ is a minimizer and that $u_q \rightarrow u$ strongly in $H_1^2(M)$ when $q \rightarrow 2^*(s)$.

Step 4: We claim that $u \in C^{0, \theta}(M)$, for all $\theta \in (0, \min\{1, 2 - s\})$. Following the method used in [6] (see Proposition 8.1) inspired from the strategy developed by Trudinger [24] for the Yamabe problem, we get that $u \in L^p(M)$, for all $p \geq 1$. Defining $f_u(x) := \frac{u(x)^{2^*(s)-1}}{d_g(x, x_0)^s}$, we then get from Hölder inequality that $f_u \in L^p(M)$, for all $p \in [1, \frac{n}{s})$. Since $\Delta_g u + au = f_u$ and $u \in H_1^2(M)$ and $s \in (0, 2)$, it follows from standard elliptic theory (see [8]) that $u \in C^{0, \theta}(M)$, for all $\theta \in (0, \min\{1, 2 - s\})$.

Step 5: We claim that $u \in C_{loc}^{1, \alpha}(M \setminus \{x_0\})$, for all $\alpha \in (0, 1)$. Indeed, since $u \in L^p(M)$ for all $p > 1$ (see Step 4), we get that $f_u \in L_{loc}^p(M \setminus \{x_0\})$ for all $p > 1$. Since $\Delta_g u + au = f_u$ and $u \in H_1^2(M)$, then, up to taking $p > n$ sufficiently large, it follows from standard elliptic theory (see [8]) that $u \in C_{loc}^{1, \alpha}(M \setminus \{x_0\})$ for all $\alpha \in (0, 1)$.

Remark: If $a \in C^{0, \gamma}(M)$ for some $\gamma \in (0, 1)$ then, using the same argument as above, we get that $u \in C_{loc}^{2, \gamma}(M \setminus \{x_0\})$.

Step 6: We claim that $u > 0$ on M . Indeed, we consider $x_1 \neq x_0$ such that $\mathbb{B}_{2r}(x_1) \subset\subset M \setminus \{x_0\}$, with $r > 0$ sufficiently small and a function h defined on $\mathbb{B}_{2r}(x_1)$ by $h(x) := a(x) - \lambda_{2^*(s)} \frac{|u(x)|^{2^*(s)-2}}{d_g(x, x_0)^s}$. Clearly, we have that $h \in C^0(\overline{\mathbb{B}_{2r}(x_1)})$. Since $u \in H_1^2(\mathbb{B}_{2r}(x_1))$, $u \geq 0$ and $(\Delta_g + h)u = 0$ on $\mathbb{B}_{2r}(x_1)$. It then follows from standard elliptic theory (see [8], Theorem 8.20) that there exists $C = C(M, g, x_1, r) > 0$ such that $\sup_{\mathbb{B}_r(x_1)} u \leq C \inf_{\mathbb{B}_r(x_1)} u$. This implies that $u|_{\mathbb{B}_r(x_1)} > 0$. Therefore, $u(x) > 0$ for all $x \in M \setminus \{x_0\}$.

We are left with proving that $u(x_0) > 0$. We argue by contradiction and we assume that $u(x_0) = 0$.

Step 6.1.: We claim that u is differentiable at x_0 . Here again, we follow the method used in [6] (see Proposition 8.1). Since $u \in C^{0, \alpha}(M)$, for all $\alpha \in (0, \min\{1, 2 - s\})$ (from Step 4) and $u(x_0) = 0$ then for any $\alpha \in (0, \min\{1, 2 - s\})$, there exists a constant $C_1(\alpha) = C(M, g, \alpha) > 0$ such that

$$(22) \quad |u(x)| \leq C_1(\alpha) d_g(x, x_0)^\alpha$$

for all $x \in M$. Therefore, we have that

$$(23) \quad \Delta_g u + au = f_u,$$

where with (22), we have that

$$(24) \quad |f_u(x)| \leq \frac{C_2(\alpha)}{d_g(x, x_0)^{s-\alpha(2^*(s)-1)}}$$

for all $x \in M \setminus \{x_0\}$.

We claim that $u \in C^{0,\alpha}(M)$, for all $\alpha \in (0, 1)$.

Indeed, we define $\alpha_1 := \sup\{\alpha \in (0, 1) ; u \in C^{0,\alpha}(M)\}$ and $N'_s = s - \alpha_1(2^*(s) - 1)$ and distinguish the following cases :

- *Case 6.1.1* $N'_s \leq 0$. In this case, up to taking α close enough to α_1 , we get that $f_u \in L^p(M)$, for all $p \geq 1$. It follows from (23) and standard elliptic theory that there exists $\theta \in (0, 1)$ such that $u \in C^{1,\theta}(M)$. This proves that $\alpha_1 = 1$ in Case 6.1.1.
- *Case 6.1.2* $0 < N'_s < 1$. In this case, up to taking α close enough to α_1 , we get that $f_u \in L^p(M)$, for all $p < \frac{n}{N'_s}$. Since $1 > N'_s$ then there exists $p \in (n, \frac{n}{N'_s})$ such that $f_u \in L^p(M)$. Therefore, (23) and standard elliptic theory yield the existence of $\theta \in (0, 1)$ such that $u \in C^{1,\theta}(M)$. This proves that $\alpha_1 = 1$ in Case 6.1.2.
- *Case 6.1.3* $N'_s = 1$. In this case, up to taking α close enough to α_1 , we get that $f_u \in L^p(M)$, for all $p < n$. This implies that for any $p \in (\frac{n}{2}, n)$, we have that $f_u \in L^p(M)$. Equation (23) and standard elliptic theory then yields $u \in C^{0,\theta}(M)$ for all $\theta \in (0, 1)$. This proves that $\alpha_1 = 1$ in Case 6.1.3.
- *Case 6.1.4* $N'_s > 1$. In this case, up to taking α close enough to α_1 , we get that $f_u \in L^p(M)$, for all $p < \frac{n}{N'_s}$. Therefore, (23), $N'_s \in (1, 2)$ (because $N'_s > 0$ and $s < 2$), and standard elliptic theory yield $u \in C^{0,\theta}(M)$ for all $\theta < 2 - N'_s$. It then follows from the definition of α_1 that $\alpha_1 \geq 2 - N'_s$. This leads to a contradiction with the definition of N'_s . Then Case 6.1.4 does not occur.

These four cases imply that $u \in C^{0,\alpha}(M)$, for all $\alpha \in (0, 1)$. This proves the claim.

In order to end Step 6.1, we proceed as the above, let $N''_s = s - 2^*(s) + 1$ and distinguish two cases :

- *Case 6.1.5* $N''_s \leq 0$. In this case, up to taking α close enough to 1, we have that $f_u \in L^p(M)$, for all $p \geq 1$. Therefore, (23) and elliptic theory yield $u \in C^1(M)$. This proves Step 6.1 in Case 6.1.5.
- *Case 6.1.6* $N''_s > 0$. In this case, up to taking α close enough to 1, we have that $f_u \in L^p(M)$ for all $p < \frac{n}{N''_s}$. Since $1 > N''_s$, there exists $p \in (n, \frac{n}{N''_s})$ such that $f_u \in L^p(M)$. Therefore, it follows from (23) and elliptic theory that $u \in C^1(M)$. This proves the claim of Step 6.1 in Case 6.1.6.

This ends Step 6.1.

Step 6.2: We prove the contradiction here. Since $u \in C^1(M)$, we are able to follow the strategy of [8] (see Lemma 3.4) to adapt Hopf's strong maximum principle. We let $\Omega \subset M \setminus \{x_0\}$ be an open set such that $x_0 \in \partial\Omega$ and $\partial\Omega$ satisfies an interior sphere condition at x_0 , then there exists an exponential chart $(\mathbb{B}_{2r_y}(y), \exp_y^{-1})$, $y \in \Omega$, $r_y > 0$ small enough such that $\mathbb{B}_{r_y}(y) \cap \partial\Omega = \{x_0\}$. We consider $C > 0$ such that

$$L_{g,C}(-u) := -(\Delta_g + C)(-u) \geq (\Delta_g + a)(u) \geq 0$$

on Ω . We fix $\rho \in (0, r_y)$ and introduce the function v_ρ defined on the annulus $\mathbb{B}_{r_y}(y) \setminus \mathbb{B}_\rho(y)$ by $v_\rho(x) = e^{-kr^2} - e^{-kr_y^2}$ where $r := d_g(x, y)$ and $k > 0$ to be

determined. Now, if $\lambda(x)$ is the smaller eigenvalue of g^{-1} then that for any $x \in \mathbb{B}_{r_y}(y) \setminus \mathbb{B}_\rho(y)$ we have that:

$$L_{g,C}v_\rho(x) \geq e^{-kr^2} \left[4k^2\lambda(x)r^2 - 2k \left(\sum_{i=1}^n g^{ii} + \Gamma_0 r \right) - C \right]$$

where $\Gamma_0 = \Gamma_0(g)$. Hence we choose k large enough so that $L_{g,C}v_\rho \geq 0$ on $\mathbb{B}_{r_y}(y) \setminus \mathbb{B}_\rho(y)$. Since $-u < 0$ on $\partial\mathbb{B}_\rho(y)$ then there exists a constant $\epsilon > 0$ such that $-u + \epsilon v_\rho \leq 0$ on $\partial\mathbb{B}_\rho(y)$. Thus we have $-u + \epsilon v_\rho \in H_1^2(\mathbb{B}_{r_y}(y) \setminus \mathbb{B}_\rho(y))$, $-u + \epsilon v_\rho \leq 0$ on $\partial\mathbb{B}_\rho(y)$ and $L_{g,C}(-u + \epsilon v_\rho) \geq 0$ on $\mathbb{B}_{r_y}(y) \setminus \mathbb{B}_\rho(y)$. It follows from the weak maximum principle (see Theorem 8.1 in [8]) that

$$(25) \quad -u + \epsilon v_\rho \leq 0, \quad \text{on } \mathbb{B}_{r_y}(y) \setminus \mathbb{B}_\rho(y)$$

In the sequel, $\mathbb{B}_r(0)$ denotes a ball in (\mathbb{R}^n, δ) centered at the origin and of radius r . Now we define $\tilde{u} = u \circ \exp_y$ and $\tilde{v}_\rho = v \circ \exp_y$ on $\mathbb{B}_{r_y}(0)$. By (25), we get :

$$(26) \quad \epsilon \tilde{v}_\rho \leq \tilde{u}, \quad \text{on } \mathbb{B}_{r_y}(0) \setminus \mathbb{B}_\rho(0)$$

We define $X_0 := \exp_y^{-1}(x_0)$. Since $\hat{u}(X_0) = \hat{v}_\rho(X_0) = 0$, then, by (26), we can write that

$$\frac{\partial \hat{u}}{\partial \nu}(X_0) := \liminf_{t \xrightarrow{<0} 0} \frac{\tilde{u}(X_0 + t\nu) - \tilde{u}(X_0)}{t} \leq \epsilon \liminf_{t \xrightarrow{<0} 0} \frac{\tilde{v}_\rho(X_0 + t\nu) - \tilde{v}_\rho(X_0)}{t} := \epsilon \frac{\partial \hat{v}_\rho}{\partial \nu}(X_0),$$

where ν is the outer normal vector field on $\mathbb{B}_{r_y}(y)$. Therefore $\frac{\partial \hat{u}}{\partial \nu}(X_0) \leq \epsilon \frac{\partial \hat{v}_\rho}{\partial \nu}(X_0)$, but $\frac{\partial \hat{v}_\rho}{\partial \nu}(x_0) = v'_\rho(R)$, it follows that

$$\frac{\partial \hat{u}}{\partial \nu}(X_0) \leq \epsilon v'_\rho(r_y) < 0.$$

This is a contradiction since $\min_M u = u(x_0)$ and therefore $\nabla \tilde{u}(X_0) = \nabla u(x_0) = 0$. This ends the proof of Step 6.

3. TEST-FUNCTIONS FOR $n \geq 4$

We consider the test-function sequence $(u_\epsilon)_{\epsilon > 0}$ defined, for any $\epsilon > 0, x \in M$, by

$$(27) \quad u_\epsilon(x) = \left(\frac{\epsilon^{1-\frac{s}{2}}}{\epsilon^{2-s} + d_g(x, x_0)^{2-s}} \right)^{\frac{n-2}{2-s}},$$

the function Φ defined on \mathbb{R}^n by

$$(28) \quad \Phi(X) = (1 + |X|^{2-s})^{-\frac{n-2}{2-s}}.$$

Since u_ϵ is a Lipschitz function, we have that $u_\epsilon \in H_1^2(M)$, for any $\epsilon > 0$. Given $\rho \in (0, i_g(M))$, where $i_g(M)$ is the injectivity radius on M , we recall that $\mathbb{B}_\rho(x_0)$ be the geodesic ball of center x_0 and radius ρ . Cartan's expansion of the metric g (see [13]) in the exponential chart $(\mathbb{B}_\rho(x_0), \exp_{x_0}^{-1})$ yields

$$(29) \quad \det(g)(x) = 1 - \frac{R_{\alpha\beta}}{3}(x_0)x^\alpha x^\beta + O(r^3),$$

where the x^α 's are the coordinates of x , $r^2 = \sum_\alpha (x^\alpha)^2$ and $(R_{\alpha\beta})$ is the Ricci curvature. Integrating on the unit sphere \mathbb{S}^{n-1} yields

$$\int_{\mathbb{S}^{n-1}} \sqrt{\det(g)}(r\theta) d\theta = \omega_{n-1} \left[1 - \frac{\text{Scal}_g(x_0)}{6n} r^2 + O(r^3) \right].$$

3.1. Estimate of the gradient term. At first, we estimate $\int_M |\nabla u_\epsilon|_g^2 dv_g$. For that, we write for all $x \in M$:

$$|\nabla u_\epsilon|_g^2(x) = (n-2)^2 \epsilon^{n-2} \frac{r^{2(1-s)}}{(\epsilon^{2-s} + r^{2-s})^{\frac{2(n-s)}{2-s}}}$$

where $r = d_g(x, x_0)$. Therefore, using (27) and the change of variable $t = r\epsilon^{-1}$, we get that

$$\begin{aligned} \int_{\mathbb{B}_\rho(x_0)} |\nabla u_\epsilon|_g^2 dv_g &= (n-2)^2 \epsilon^{n-2} \omega_{n-1} \times \int_0^\rho \frac{r^{n+1} \left(1 - \frac{\text{Scal}_g(x_0)}{6n} r^2 + O(r^3)\right) dr}{r^{2s} (\epsilon^{2-s} + r^{2-s})^{\frac{2(n-s)}{2-s}}} \\ (30) \qquad \qquad \qquad &= (n-2)^2 \omega_{n-1} \int_0^{\frac{\rho}{\epsilon}} \frac{t^{n+1} \left(1 - \frac{\text{Scal}_g(x_0)}{6n} (\epsilon t)^2 + O((\epsilon t)^3)\right) dt}{t^{2s} (1 + t^{2-s})^{\frac{2(n-s)}{2-s}}}. \end{aligned}$$

Straightforward computations yield

$$(31) \qquad \int_0^{+\infty} \frac{t^{n+1} dt}{t^{2s} (1 + t^{2-s})^{\frac{2(n-s)}{2-s}}} = (n-2)^{-2} \omega_{n-1}^{-1} \int_{\mathbb{R}^n} |\nabla \Phi|^2 dX,$$

and

$$(32) \qquad \epsilon^2 \int_0^{+\infty} \frac{t^{n+3} dt}{t^{2s} (1 + t^{2-s})^{\frac{2(n-s)}{2-s}}} = \begin{cases} \epsilon^2 (n-2)^{-2} \omega_{n-1}^{-1} \int_{\mathbb{R}^n} |X|^2 |\nabla \Phi|^2 dX & \text{if } n \geq 5, \\ \epsilon^2 \ln \frac{1}{\epsilon} & \text{if } n = 4, \\ O(\epsilon) & \text{if } n = 3. \end{cases}$$

Since

$$\int_{M \setminus \mathbb{B}_\rho(x_0)} |\nabla u_\epsilon|_g^2 dv_g = O(\epsilon^{n-2}),$$

when $\epsilon \rightarrow 0$, putting together (30) with (31) and (32) yield

$$(33) \qquad \int_M |\nabla u_\epsilon|^2 dv_g = \begin{cases} \int_{\mathbb{R}^n} |\nabla \Phi|^2 dX - \frac{\int_{\mathbb{R}^n} |X|^2 |\nabla \Phi|^2 dX}{6n} \text{Scal}_g(x_0) \epsilon^2 + o(\epsilon^2) & \text{if } n \geq 5, \\ \int_{\mathbb{R}^n} |\nabla \Phi|^2 dX - \frac{\omega_3}{6} \text{Scal}_g(x_0) \epsilon^2 \ln \frac{1}{\epsilon} + O(\epsilon^2) & \text{if } n = 4, \\ \int_{\mathbb{R}^n} |\nabla \Phi|^2 dX + O(\epsilon) & \text{if } n = 3, \end{cases}$$

Arguing as the above and using that $a \in C^0(M)$, we get that :

$$(34) \qquad \int_M a u_\epsilon^2 dv_g = \begin{cases} \epsilon^2 a(x_0) \int_{\mathbb{R}^n} \Phi^2 dX + o(\epsilon^2) & \text{if } n \geq 5, \\ a(x_0) \omega_3 \epsilon^2 \ln \frac{1}{\epsilon} + O(\epsilon^2) & \text{if } n = 4, \\ O(\epsilon) & \text{si } n = 3, \end{cases}$$

and

$$(35) \qquad \int_M \frac{|u_\epsilon|^{2^*(s)}}{d_g(x, x_0)^s} dv_g = \begin{cases} \int_{\mathbb{R}^n} \frac{|\Phi|^{2^*(s)}}{|X|^s} dX - \epsilon^2 \frac{\text{Scal}_g(x_0)}{6n} \int_{\mathbb{R}^n} |X|^{2-s} |\Phi|^{2^*(s)} dX + o(\epsilon^2) & \text{if } n \geq 4, \\ \int_{\mathbb{R}^3} \frac{|\Phi|^{2^*(s)}}{|X|^s} dX + O(\epsilon) & \text{if } n = 3. \end{cases}$$

From Lieb [14], we know that Φ is an extremal for (5), that is

$$(36) \quad \frac{\int_{\mathbb{R}^n} |\nabla \Phi|^2 dX}{\left(\int_{\mathbb{R}^n} \frac{|\Phi|^{2^*(s)}}{|X|^s} dX \right)^{\frac{2}{2^*(s)}}} = K(n, s)^{-1}$$

Combining (33), (34) and (35) and this last equation, we obtain, for any $\epsilon > 0$, the following results :

$$(37) \quad J(u_\epsilon) = K(n, s)^{-1} \left(1 + \left\{ \begin{array}{ll} (C_1(n, s)a(x_0) - C_2(n, s)\text{Scal}_g(x_0)) \epsilon^2 + o(\epsilon^2) & \text{if } n \geq 5 \\ \omega_3 \left(\int_{\mathbb{R}^4} |\nabla \Phi|^2 dX \right)^{-1} \left(a(x_0) - \frac{1}{6} \text{Scal}_g(x_0) \right) \epsilon^2 \ln\left(\frac{1}{\epsilon}\right) + O(\epsilon^2) & \text{if } n = 4 \\ O(\epsilon) & \text{if } n = 3 \end{array} \right\} \right)$$

where

$$C_1(n, s) := \frac{\int_{\mathbb{R}^n} |\Phi|^2 dX}{\int_{\mathbb{R}^n} |\nabla \Phi|^2 dX}$$

$$C_2(n, s) := \frac{1}{6n} \frac{\int_{\mathbb{R}^n} |X|^2 |\nabla \Phi|^2 dX}{\int_{\mathbb{R}^n} |\nabla \Phi|^2 dX} - \frac{2}{2^*(s)6n} \frac{\int_{\mathbb{R}^n} |X|^{2-s} |\Phi|^{2^*(s)} dX}{\int_{\mathbb{R}^n} \frac{|\Phi|^{2^*(s)}}{|X|^s} dX}$$

Unlike the case $s = 0$, it is not possible to compute explicitly the constants $C_1(n, s)$ and $C_2(n, s)$. However, we are able to explicit their quotient, which is enough to prove our theorem. We need the following lemma taken from Aubin [1] :

Lemma 1. *Let $p, q \in \mathbb{R}_+^*$ such that $p - q > 1$ and assume that $I_p^q = \int_0^{+\infty} \frac{t^q dt}{(1+t)^p}$, then*

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q \text{ and } I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q.$$

Indeed, an integration by parts shows that $I_p^q = \frac{p}{q+1} I_{p+1}^{q+1}$. On the other hand, we can easily see that $I_p^q = I_{p+1}^q + I_{p+1}^{q+1}$. Together, the above relations yield the lemma.

We apply Lemma 1 to the computation of $C_2(n, s)/C_1(n, s)$ when $n \geq 5$. We have that

$$(38) \quad \frac{C_2(n, s)}{C_1(n, s)} = \frac{1}{6n} \frac{\int_{\mathbb{R}^n} |X|^2 |\nabla \Phi|^2 dX}{\int_{\mathbb{R}^n} \Phi^2 dX} - \frac{2}{2^*(s)6n} \frac{\int_{\mathbb{R}^n} |X|^{2-s} |\Phi|^{2^*(s)} dX}{\int_{\mathbb{R}^n} \frac{|\Phi|^{2^*(s)}}{|X|^s} dX} \cdot \frac{\int_{\mathbb{R}^n} |\nabla \Phi|^2 dX}{\int_{\mathbb{R}^n} |\Phi|^2 dX}$$

Independently

$$\frac{\int_{\mathbb{R}^n} |X|^2 |\nabla \Phi|^2 dX}{\int_{\mathbb{R}^n} |\Phi|^2 dX} = \frac{(n-2)^2 \int_0^{+\infty} \frac{r^{n+3-2s} dr}{(1+r^{2-s})^{\frac{2(n-s)}{2-s}}}}{\int_0^{+\infty} \frac{r^{n-1} dr}{(1+r^{2-s})^{\frac{2(n-2)}{2-s}}}},$$

up to taking $t = r^{2-s}$ and using the Lemma 1, we get that :

$$(39) \quad \frac{\int_{\mathbb{R}^n} |X|^2 |\nabla \Phi|^2 dX}{\int_{\mathbb{R}^n} |\Phi|^2 dX} = \frac{\frac{(n-2)^2}{2-s} \int_0^{+\infty} \frac{t^{\frac{n}{2-s}+1} dt}{(1+t)^{\frac{2(n-2)}{2-s}}}}{\frac{1}{2-s} \int_0^{+\infty} \frac{t^{\frac{n}{2-s}+2} dt}{(1+t)^{\frac{2(n-2)}{2-s}}}} = \frac{n(n-2)(n+2-s)}{2(2n-2-s)},$$

$$(40) \quad \frac{\int_{\mathbb{R}^n} |X|^{2-s} \cdot |\Phi|^{2^*(s)} dX}{\int_{\mathbb{R}^n} |\Phi|^2 dX} = \frac{n(n-4)}{2(n-2)(2n-2-s)}$$

and

$$(41) \quad \frac{\int_{\mathbb{R}^n} |\nabla \Phi|^2 dX}{\int_{\mathbb{R}^n} |X|^{-s} \cdot |\Phi|^{2^*(s)} dX} = (n-2)(n-s).$$

Therefore, plugging (39), (40) and (41) into (38) yields

$$\frac{C_2(n, s)}{C_1(n, s)} = \frac{(n-2)(6-s)}{12(2n-2-s)}$$

when $n \geq 5$. As a conclusion, the expansion (37) rewrites

$$(42) \quad J(u_\epsilon) = K(n, s)^{-1} \left(1 + \begin{cases} \frac{\int_{\mathbb{R}^n} |\Phi|^2 dX}{\int_{\mathbb{R}^n} |\nabla \Phi|^2 dX} (a(x_0) - c_{n,s} \text{Scal}_g(x_0)) \epsilon^2 + o(\epsilon^2) & \text{if } n \geq 5 \\ \frac{\omega_3}{\int_{\mathbb{R}^4} |\nabla \Phi|^2 dX} (a(x_0) - c_{n,s} \text{Scal}_g(x_0)) \epsilon^2 \ln\left(\frac{1}{\epsilon}\right) + O(\epsilon^2) & \text{if } n = 4 \\ O(\epsilon) & \text{if } n = 3 \end{cases} \right)$$

where

$$(43) \quad c_{n,s} := \frac{(n-2)(6-s)}{12(2n-2-s)}.$$

As a consequence, we then get the following theorem:

Theorem 5. *Let (M, g) be a compact Riemannian Manifold of dimension $n \geq 3$. Let $a \in C^0(M)$ such that $\Delta_g + a$ is coercive, $x_0 \in M$ and $s \in (0, 2)$. Then for all $n \geq 3$, we have that*

$$(44) \quad \inf_{v \in H_1^2(M) \setminus \{0\}} J(v) \leq K(n, s)^{-1}.$$

Moreover, if $n \geq 4$ and $a(x_0) < c_{n,s} \text{Scal}_g(x_0)$, where $c_{n,s}$ is as (43), then inequality (44) is strict.

4. TEST-FUNCTIONS: THE CASE $n = 3$

The argument used for $n \geq 4$ is local in the sense that the expansion (42) only involves the values of a and Scal_g at the singular point x_0 . When $n = 3$, the first-order in (42) of Section 3 has an undetermined sign. It is well-known since Schoen [19] that the relevant quantity to use in small dimension is the mass, which is a global quantity.

We follow the technique developed by Druet [3] for test-function in dimension 3. The case of a manifold with boundary is discussed at the end of this section. We define the Green-function G_{x_0} of the elliptic operator $\Delta_g + a$ on x_0 as the unique function strictly positive and symmetric verifying, in the sense of distribution,

$$(45) \quad \Delta_g G_{x_0} + a G_{x_0} = D_{x_0},$$

where D_{x_0} is the Dirac mass at x_0 . We fix $\rho \in (0, i_g(M)/2)$ and we consider a cut-off function $\eta \in C_c^\infty(\mathbb{B}_{2\rho}(x_0))$ such that $\eta \equiv 1$ on $\mathbb{B}_\rho(x_0)$. Then there exists

$\beta_{x_0} \in H_1^2(M)$ such that we can write G_{x_0} as follow :

$$(46) \quad \omega_2 G_{x_0}(x) = \frac{\eta(x)}{d_g(x, x_0)} + \beta_{x_0}(x)$$

for all $x \in M$. According to (45) and (46), we have that

$$(47) \quad \Delta_g \beta_{x_0} + a \beta_{x_0} = f_{x_0}$$

where

$$(48) \quad f_{x_0}(x) := -\Delta_g \left(\frac{\eta(x)}{d_g(x, x_0)} \right) - \frac{(a\eta)(x)}{d_g(x, x_0)} \text{ for all } x \in M \setminus \{x_0\}.$$

In particular, for all $p \in (1, 3)$, we have $f_{x_0} \in L^p(M)$. Therefore, it follows from standard elliptic theory that $\beta_{x_0} \in C^0(M) \cap C_{loc}^1(M \setminus \{x_0\}) \cap H_2^p(M)$ for all $p \in (1, 3)$. In particular, the mass satisfies $m(x_0) = \beta_{x_0}(x_0)$. For any $\epsilon > 0$, we define, on M , the function

$$v_\epsilon = \eta u_\epsilon + \sqrt{\epsilon} \beta_{x_0},$$

where u_ϵ is the general test-function defined as (27). This section is devoted to computing the expansion of $J(v_\epsilon)$. We compute the different terms separately.

4.1. The leading term $\int_M (|\nabla v_\epsilon|_g^2 + a v_\epsilon^2) dv_g$. Integration by parts and using the definition of v_ϵ , we write, for any $\epsilon > 0$, that :

$$(49) \quad \int_M (|\nabla v_\epsilon|_g^2 + a v_\epsilon^2) dv_g = \int_M \eta^2 u_\epsilon \Delta_g u_\epsilon dv_g + \int_M u_\epsilon^2 \eta \Delta_g \eta dv_g \\ - \int_M \eta (\nabla \eta, \nabla u_\epsilon^2)_g dv_g + \int_M a \eta^2 u_\epsilon^2 dv_g + \int_M (\Delta_g \beta_{x_0} + a \beta_{x_0}) (\epsilon \beta + 2\sqrt{\epsilon} \eta u_\epsilon) dv_g.$$

Writing u_ϵ^2 in the form :

$$(50) \quad u_\epsilon^2(x) = \frac{\epsilon}{d_g(x, x_0)^2} + O(\epsilon^{5-2s}),$$

with $O(1) \in C^2(M \setminus \mathbb{B}_\rho(x_0))$ uniformly bounded with respect to ϵ , we obtain that

$$(51) \quad \int_M u_\epsilon^2 \eta \Delta_g \eta dv_g = \epsilon \int_{M \setminus \mathbb{B}_\rho(x_0)} \frac{\eta \Delta_g \eta}{d_g(x, x_0)^2} dv_g + o(\epsilon),$$

and

$$(52) \quad \int_M \eta (\nabla \eta, \nabla u_\epsilon^2)_g dv_g = \epsilon \int_{M \setminus \mathbb{B}_\rho(x_0)} \eta (\nabla \eta, \nabla \frac{1}{d_g(x, x_0)^2})_g dv_g + o(\epsilon).$$

By integrating by parts, using (50) and since $\partial_\nu \eta = 0$ then we write

$$(53) \quad \int_M u_\epsilon^2 \eta \Delta_g \eta dv_g - \int_M \eta (\nabla \eta, \nabla u_\epsilon^2)_g dv_g = \epsilon \int_{M \setminus \mathbb{B}_\rho(x_0)} \frac{|\nabla \eta|_g^2}{d_g(x, x_0)^2} dv_g + o(\epsilon)$$

We have also that

$$(54) \quad \int_M a \eta^2 u_\epsilon^2 dv_g = \epsilon \int_M \frac{a \eta^2}{d_g(x, x_0)^2} dv_g + R_1(\epsilon) + o(\epsilon),$$

where, as in (30),

$$R_1(\epsilon) = O \left(\epsilon^{3-s} \int_{\mathbb{B}_\rho(x_0)} \frac{a \eta^2 dv_g}{d_g(x, x_0)^2 (\epsilon^{2-s} + d_g(x, x_0)^{2-s})} \right) = O \left(\epsilon^2 \int_0^{\frac{\rho}{\epsilon}} \frac{dt}{1+t^{2-s}} \right) = o(\epsilon).$$

This latest relation and (54) give that

$$(55) \quad \int_M a\eta^2 u_\epsilon^2 dv_g = \epsilon \int_M \frac{a\eta^2}{d_g(x, x_0)^2} dv_g + o(\epsilon).$$

Writing now u_ϵ in the form

$$(56) \quad u_\epsilon(x) = \frac{\sqrt{\epsilon}}{d_g(x, x_0)} + O(\epsilon^{\frac{5}{2}-s}),$$

with $O(1) \in C^2(M \setminus \mathbb{B}_\rho(x_0))$ we get that

$$(57) \quad \int_{M \setminus \mathbb{B}_\rho(x_0)} \eta^2 u_\epsilon \Delta_g u_\epsilon dv_g = \epsilon \int_{M \setminus \mathbb{B}_\rho(x_0)} \frac{\eta^2}{d_g(x, x_0)} \Delta_g \left(\frac{1}{d_g(x, x_0)} \right) dv_g + o(\epsilon).$$

Since u_ϵ is radially symmetrical, denoting Δ_δ as the Laplacian in the Euclidean metric δ , we get with a change of variable and Cartan's expansion of the metric (29) that

$$\int_{\mathbb{B}_\rho(x_0)} u_\epsilon \Delta_\delta u_\epsilon dv_g = \int_{\mathbb{R}^n} \Phi \Delta_\delta \Phi dX + o(\epsilon),$$

where Φ is defined in (28). Since

$$\Delta_g u_\epsilon = \Delta_\delta u_\epsilon - \partial_r(\ln \det(g)) \partial_r u_\epsilon$$

in g -normal coordinates, we have that

$$(58) \quad \begin{aligned} \int_{\mathbb{B}_\rho(x_0)} u_\epsilon \Delta_g u_\epsilon dv_g &= \int_{\mathbb{R}^n} \Phi \Delta_\delta \Phi dX + o(\epsilon) \\ &+ \epsilon \int_{\mathbb{B}_\rho(x_0)} \frac{d_g(x, x_0)^{1-s} \partial_r(\ln \det(g))}{2(\epsilon^{2-s} + d_g(x, x_0)^{2-s})^{\frac{4-s}{2-s}}} dv_g + o(\epsilon) \end{aligned}$$

when $\epsilon \rightarrow 0$. Similar computations to the ones we just developed give that

$$\begin{aligned} \epsilon \int_{\mathbb{B}_\rho(x_0)} \frac{d_g(x, x_0)^{1-s} \partial_r(\ln \det(g))}{2(\epsilon^{2-s} + d_g(x, x_0)^{2-s})^{\frac{4-s}{2-s}}} dv_g &= \epsilon \int_{\mathbb{B}_\rho(x_0)} \frac{\partial_r(\ln \det(g))}{2d_g(x, x_0)^3} dv_g \\ &+ O\left(\epsilon^{3-s} \int_{\mathbb{B}_\rho(x_0)} \frac{\partial_r(\ln \det(g))}{d_g(x, x_0)^3 (\epsilon^{2-s} + d_g(x, x_0)^{2-s})} dv_g \right), \end{aligned}$$

Cartan's expansion of the metric g , (29) and to this latest relation yield

$$(59) \quad \epsilon \int_{\mathbb{B}_\rho(x_0)} \frac{d_g(x, x_0)^{1-s} \partial_r(\ln \det(g))}{2(\epsilon^{2-s} + d_g(x, x_0)^{2-s})^{\frac{4-s}{2-s}}} dv_g = \epsilon \int_{\mathbb{B}_\rho(x_0)} \frac{\partial_r(\ln \det(g))}{2d_g(x, x_0)^3} dv_g + o(\epsilon)$$

Relations (57), (58) and (59) yield

$$(60) \quad \begin{aligned} \int_M \eta^2 u_\epsilon \Delta_g u_\epsilon dv_g &= \int_{\mathbb{R}^n} \Phi \Delta_\delta \Phi dX + \epsilon \int_{\mathbb{B}_\rho(x_0)} \frac{\partial_r(\ln \det(g))}{2d_g(x, x_0)^3} dv_g \\ &+ \epsilon \int_{M \setminus \mathbb{B}_\rho(x_0)} \frac{\eta^2}{d_g(x, x_0)} \Delta_g \left(\frac{1}{d_g(x, x_0)} \right) dv_g + o(\epsilon) \end{aligned}$$

when $\epsilon \rightarrow 0$. At last, using again the expansion (60) of u_ϵ , we obtain that :

$$\int_M (\Delta_g \beta_{x_0} + a \beta_{x_0})(\epsilon \beta_{x_0} + 2\sqrt{\epsilon} \eta u_\epsilon) dv_g = \epsilon \int_M (\Delta_g \beta_{x_0} + a \beta_{x_0}) \left(\beta_{x_0} + 2 \frac{\eta}{d_g(x, x_0)} \right) dv_g + O \left(\epsilon^{3-s} \int_M \frac{(\Delta_g \beta_{x_0} + a \beta_{x_0}) \eta}{d_g(x, x_0) (\epsilon^{2-s} + d_g(x, x_0)^{2-s})} dv_g \right).$$

The latest relation and (47) allow to write :

$$\int_M (\Delta_g \beta_{x_0} + a \beta_{x_0})(\epsilon \beta_{x_0} + 2\sqrt{\epsilon} \eta u_\epsilon) dv_g = \epsilon \int_M (\Delta_g \beta_{x_0} + a \beta_{x_0}) \left(\beta_{x_0} + 2 \frac{\eta}{d_g(x, x_0)} \right) dv_g + o(\epsilon).$$

Since $\beta_{x_0} \in C^0(M) \cap H_2^p(M)$ for all $p \in (\frac{3}{2}, 3)$, it follows from (45) and (46) that

$$\int_M (\Delta_g \beta_{x_0} + a \beta_{x_0}) \left(\beta_{x_0} + \frac{\eta}{d_g(x, x_0)} \right) dv_g = \omega_2 \beta_{x_0}(x_0).$$

Then the last couple of relations give that

$$\begin{aligned} \int_M (\Delta_g \beta_{x_0} + a \beta_{x_0})(\epsilon \beta_{x_0} + 2\sqrt{\epsilon} \eta u_\epsilon) dv_g &= \epsilon \omega_2 \beta_{x_0}(x_0) + o(\epsilon) \\ (61) \qquad \qquad \qquad &+ \epsilon \int_M (\Delta_g \beta_{x_0} + a \beta_{x_0}) \left(\beta_{x_0} + \frac{\eta}{d_g(x, x_0)} \right) dv_g \end{aligned}$$

when $\epsilon \rightarrow 0$. Knowing, from (47) and (48), that

$$\begin{aligned} \int_M (\Delta_g \beta_{x_0} + a \beta_{x_0}) \frac{\eta}{d_g(x, x_0)} dv_g &= - \int_{\mathbb{B}_\rho(x_0)} \frac{\partial_r(\ln \det(g))}{2d_g(x, x_0)^3} dv_g - \int_M \frac{a\eta^2}{d_g(x, x_0)^2} dv_g + \\ &- \int_{M \setminus \mathbb{B}_\rho(x_0)} \frac{\eta}{d_g(x, x_0)} \Delta_g \left(\frac{\eta}{d_g(x, x_0)} \right) dv_g, \end{aligned}$$

and combining (53), (55), (60) and (61) with (49), we get that

$$(62) \quad \int_M (|\nabla v_\epsilon|^2 + a v_\epsilon^2) dv_g = \int_{\mathbb{R}^n} \Phi \Delta_\delta \Phi dX + \epsilon \omega_2 \beta_{x_0}(x_0) + o(\epsilon)$$

when $\epsilon \rightarrow 0$.

4.2. Estimate of $\int_M \frac{v_\epsilon^{2^*(s)}}{d_g(x, x_0)^s} dv_g$. Since $s \in (0, 2)$ then $6 - 2s > 2$. Therefore there exists $C(s) > 0$ such that for all $X, Y \in \mathbb{R}$, we have :

$$||X + Y|^{6-2s} - |X|^{6-2s} - (6 - 2s)XY|X|^{4-2s}| \leq C(s) (|X|^{4-2s}Y^2 + |Y|^{6-2s})$$

This allows to write

$$\begin{aligned} \int_M \frac{v_\epsilon^{2^*(s)}}{d_g(x, x_0)^s} dv_g &= \int_M \frac{(\eta u_\epsilon + \sqrt{\epsilon} \beta_{x_0})^{6-2s}}{d_g(x, x_0)^s} dv_g \\ &= \int_{\mathbb{B}_\rho(x_0)} \frac{(u_\epsilon + \sqrt{\epsilon} \beta_{x_0})^{6-2s}}{d_g(x, x_0)^s} dv_g + O(\epsilon^{3-s}) \\ &= \int_{\mathbb{B}_\rho(x_0)} \frac{u_\epsilon^{6-2s} + (6 - 2s)|u_\epsilon|^{6-2s-2} u_\epsilon \sqrt{\epsilon} \beta_{x_0}}{d_g(x, x_0)^s} dv_g + R_2(\epsilon) + o(\epsilon) \end{aligned} \tag{63}$$

where

$$(64) \quad R_2(\epsilon) = O \left(\int_{\mathbb{B}_\rho(x_0)} \frac{u_\epsilon^{4-2s} \epsilon \beta_{x_0}^2 + \epsilon^{3-s} \beta_{x_0}^{6-2s}}{d_g(x, x_0)^s} dv_g \right) = o(\epsilon)$$

Direct computations yield

$$(65) \quad \int_{\mathbb{B}_\rho(x_0)} \frac{u_\epsilon^{6-2s}}{d_g(x, x_0)^s} dv_g = \int_{\mathbb{R}^n} \frac{\Phi^{2^*(s)}(X)}{|X|^s} dX + o(\epsilon),$$

when $\epsilon \rightarrow 0$. Using that $\beta_{x_0} \in C^{0,\theta}(M)$ for all $\theta \in (0, 1)$, we get that

$$(66) \quad \begin{aligned} \int_{\mathbb{B}_\rho(x_0)} \frac{(6-2s)|u_\epsilon|^{6-2s-2} u_\epsilon \sqrt{\epsilon} \beta_{x_0}}{d_g(x, x_0)^s} dv_g &= \epsilon^{3-s} (6-2s) \beta_{x_0}(x_0) \omega_2 \int_0^\rho \frac{r^{2-s} dr}{(\epsilon^{2-s} + r^{2-s})^{\frac{5-2s}{2-s}}} + o(\epsilon) \\ &= \epsilon (6-2s) \beta_{x_0}(x_0) \omega_2 \int_0^{\frac{\rho}{\epsilon}} \frac{t^{2-s} dt}{(1+t^{2-s})^{\frac{5-2s}{2-s}}} + o(\epsilon) \\ &= \epsilon (6-2s) \beta_{x_0}(x_0) \omega_2 \int_0^{+\infty} \frac{t^{2-s} dt}{(1+t^{2-s})^{\frac{5-2s}{2-s}}} + o(\epsilon), \end{aligned}$$

when $\epsilon \rightarrow 0$. Since $\Delta_\delta \Phi = (3-s) \frac{\Phi^{2^*(s)-1}}{|X|^s}$ in \mathbb{R}^n , a changes of variable and an integration by parts yields

$$(67) \quad \begin{aligned} \omega_2 \int_0^{+\infty} \frac{t^{2-s} dt}{(1+t^{2-s})^{\frac{5-2s}{2-s}}} &= \int_{\mathbb{R}^n} \frac{\Phi^{2^*(s)-1}(X)}{|X|^s} dX \\ &= (3-s)^{-1} \lim_{R \rightarrow +\infty} \int_{\partial B_R(0)} -\partial_\nu \Phi dX, \end{aligned}$$

where ν is the normal vector field on the Euclidean ball $B_R(0)$. Since $\partial_\nu \Phi = -|X|^{1-s} (1+|X|^{2-s})^{-\frac{3+s}{2-s}}$ for all $X \in \mathbb{R}^n$, passing to the limit in (67) yields

$$\omega_2 \int_0^{+\infty} \frac{t^{2-s} dt}{(1+t^{2-s})^{\frac{5-2s}{2-s}}} = (3-s)^{-1} \omega_2.$$

Hence, the latest relation and (66) give that

$$(68) \quad \int_{\mathbb{B}_\rho(x_0)} \frac{(6-2s)|u_\epsilon|^{6-2s-2} u_\epsilon \sqrt{\epsilon} \beta_{x_0}}{d_g(x, x_0)^s} dv_g = \epsilon (2\beta_{x_0}(x_0) \omega_2) + o(\epsilon),$$

when $\epsilon \rightarrow 0$. Combining (64), (65) and (68) with (63), we get that

$$(69) \quad \int_M \frac{v_\epsilon^{2^*(s)}}{d_g(x, x_0)^s} dv_g = \int_{\mathbb{R}^n} \frac{\Phi^{2^*(s)}(X)}{|X|^s} dX + \epsilon (2\beta_{x_0}(x_0) \omega_2) + o(\epsilon),$$

when $\epsilon \rightarrow 0$.

4.3. Expansion of $J(v_\epsilon)$ and proof of Theorem 1. Equality (69), (62) and (67) yield

$$(70) \quad \begin{aligned} J(v_\epsilon) &= \frac{\int_M (|\nabla v_\epsilon|_g^2 + a v_\epsilon^2) dv_g}{\left(\int_M \frac{v_\epsilon^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}}} \\ &= K(3, s)^{-1} \left(1 - \epsilon \frac{2\beta_{x_0}(x_0) \omega_2}{\int_{\mathbb{R}^n} |x|^{-s} \Phi^{2^*(s)} dx} + o(\epsilon) \right) \end{aligned}$$

when $\epsilon \rightarrow 0$. Noting that $m(x_0) = \beta_{x_0}(x_0)$, we then get the following as a consequence of (70):

Theorem 6. *Let (M, g) be a compact Riemannian Manifold of dimension $n = 3$. Let $a \in C^0(M)$ such that $\Delta_g + a$ is coercive, $x_0 \in M$ and $s \in (0, 2)$. Assume that the mass at x_0 is positive, that is $\beta_{x_0}(x_0) > 0$. Then we have that*

$$\inf_{v \in H_1^2(M) \setminus \{0\}} J(v) < K(n, s)^{-1}.$$

Proof of Theorem 1. Theorem 1 follows from the existence result (Theorem 4) and the upper-bounds (Theorem 5 and Theorem 6).

Proof of Theorem 3. As one checks, the estimates (42) and (70) hold when M is a smooth compact manifold with boundary provided x_0 lies in the interior. Then Theorem 1 extends to such a case, and Theorem 3 is a corollary.

4.4. Examples with positive mass.

Proposition 2. *Let (M, g) be a compact Riemannian Manifold of dimension $n = 3$. Let $a \in C^0(M)$ such that $\Delta_g + a$ is coercive, $x_0 \in M$ and $s \in (0, 2)$. If $\{a \not\leq c_{3,0} \text{Scal}_g\}$ or $\{a \equiv c_{3,0} \text{Scal}_g$ and (M, g) is not conformally equivalent to the canonical n -sphere $\}$ then we have that :*

$$\inf_{v \in H_1^2(M) \setminus \{0\}} J(v) < K(3, s)^{-1}.$$

Indeed, the positivity of the mass in this case was proved by Druet [4]. We incorporate the proof for the sake of self-completeness.

Lemma 2. *Let (M, g) be a compact Riemannian Manifold of dimension $n = 3$. We consider $a, a' \in C^0(M)$ such that operators $\Delta_g + a$ and $\Delta_g + a'$ are coercive. We denote as G_x, G'_x their respective Green's function at any point $x \in M$. We assume that $a \not\leq a'$. Then $\beta_x > \beta'_x$ for all $x \in M$, where $\beta_x, \beta'_x \in C^{0,\theta}(M)$, $\theta \in (0, 1)$ are such that*

$$(71) \quad \omega_2 G_x = \frac{\eta_x}{d_g(x, \cdot)} + \beta_x \text{ and } \omega_2 G'_x = \frac{\eta_x}{d_g(x, \cdot)} + \beta'_x.$$

Proof. We fix $x \in M$ and we define $h_x = \beta'_x - \beta_x$, where β'_x and β_x are as in (71). Noting $L := \Delta_g + a$ and $L' := \Delta_g + a'$, we have that $L'(h_x) = -(a' - a)G_x \leq 0$. Since $h_x \in H_2^p(M)$ for all $p \in (1, 3)$, then for all $y \in M$, Green's formula yields

$$h_x(y) = - \int_M G'_y(z) (a' - a)(z) G_x(z) dv_g(z).$$

Therefore $h_x \leq 0$ since $a \leq a'$. Moreover, since $a \not\leq a'$, we have that $h_x < 0$. This ends the proof. \square

Proof of Proposition 2 : We consider the operator $L^0 := \Delta_g + c_{3,0} \text{Scal}_g$, β^0 the mass of (M, g) corresponding to L^0 . The Positive Mass Theorem (see [20], [21]) gives that $\beta_x^0(x) \geq 0$, the equality being achieved only in the conformal class of the canonical sphere. It then follows from Lemma 2 that $\beta_{x_0}(x_0) > 0$ when $\{a \not\leq c_{3,0} \text{Scal}_g\}$ or $\{a \equiv c_{3,0} \text{Scal}_g$ and (M, g) is not conformally equivalent to the unit n -sphere $\}$. It then follows from Theorem 6 that $\inf_{v \in H_1^2(M) \setminus \{0\}} J(v) < K(3, s)^{-1}$.

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UNIVERSITÉ DE LORRAINE, INSTITUT ELIE CARTAN DE LORRAINE, UMR 7502, VANDŒUVRE-LÈS-NANCY, F-54506, FRANCE.

E-mail address: Hassan.Jaber@univ-lorraine.fr