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► **To cite this version:**

Nathanaël Fijalkow. Profinite Techniques for Probabilistic Automata and the Optimality of the Markov Monoid Algorithm. 2015. hal-01102610v2

HAL Id: hal-01102610

<https://hal.science/hal-01102610v2>

Preprint submitted on 29 Jan 2015 (v2), last revised 12 Feb 2016 (v3)

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Profinite Techniques for Probabilistic Automata and the Optimality of the Markov Monoid Algorithm

Nathanaël Fijalkow

LIAFA, Paris 7, France
University of Warsaw, Poland

Abstract. We consider the value 1 problem for probabilistic automata over finite words. This problem is known to be undecidable. However, different algorithms have been proposed to partially solve it. The aim of this paper is to prove that one such algorithm, called the Markov Monoid algorithm, is optimal.

To this end, we develop a profinite theory for probabilistic automata. This new framework gives a topological account by constructing the free prostochastic monoid. We use it in two ways. First, to characterize the computations realized by the Markov Monoid algorithm, and second to prove its optimality.

1 Introduction

In 1963 Rabin [Rab63] introduced the notion of probabilistic automata, which are finite automata with randomized transitions. This powerful model has been widely studied since then and has applications, for instance in image processing [CK97], computational biology [DEKM99] and speech processing [Moh97]. This paper follows a long line of work that studies the algorithmic properties of probabilistic automata. For instance, Schützenberger [Sch61] proved in 1961 that *language equivalence* is decidable in polynomial time, and even faster with randomized algorithms, which led to applications in software verification [KMO⁺11].

However, many natural decision problems are *undecidable*; for example the *emptiness*, the *isolation* and the *value 1* problems are undecidable, as shown in [Paz71,BMT77,GO10]. To overcome such undecidability results, a lot of effort went into finding subclasses of probabilistic automata for which natural decision problems become decidable. For instance, Chadha et al. and Korthikanti et al. look at restrictions implying that containment in ω -regular specifications is decidable [KVAK10,CKV⁺11], and investigate whether assuming isolated cut-points leads to decidability for the emptiness problem [CSV13].

In this paper, we consider the value 1 problem: it asks, given a probabilistic automaton, whether there exist words accepted with probability arbitrarily close to 1. This problem has been shown undecidable [GO10], but attracted a lot of attention recently (see, for instance, [BBG12,CT12,FGO12,FGHO14,FGKO14]).

It has been shown in [FGKO14] that the so-called Markov Monoid algorithm introduced in [FGO12] is the most correct algorithm of all the algorithms proposed in these papers. Indeed, all proposed subclasses of probabilistic automata turned out to be included in the subclass of leaktight automata, for which the Markov Monoid algorithm correctly solves the value 1 problem.

The aim of this paper is to prove that the Markov Monoid algorithm is optimal.

What is an optimality argument? It consists in constructing a maximal subclass of probabilistic automata for which the problem is decidable. We can reverse the point of view, and equivalently construct an optimal algorithm, *i.e.* an algorithm that correctly solves a subset of the instances, such that no algorithm correctly solves a superset of these instances. However, it is clear that no such strong statement holds, as one can always from any algorithm obtain a better algorithm by precomputing finitely many instances. Hence our optimality argument has to be weaker.

We show that the Markov Monoid algorithm is *in some sense* optimal, by showing that no algorithm can correctly solve *substantially* more instances than the Markov Monoid algorithm. To this end, we first characterize the computations of the Markov Monoid algorithm: roughly speaking, it captures exactly all *polynomial behaviours*. We then show that no algorithm can capture both polynomial and *super-polynomial behaviours*, supporting the claim that the Markov Monoid algorithm is optimal.

To make sense of the notion of *convergence speeds*, we rely on topological techniques. We develop a profinite theory for probabilistic automata, called prostochastic theory. This is inspired by the profinite approach for (classical) automata [Pin09,GGP10], and for distance automata as developed in Szymon Toruńczyk’s PhD thesis [Tor11].

Section 3 is devoted to constructing the free prostochastic monoid and showing some of its properties. In particular, we define the acceptance of a prostochastic word by a probabilistic automaton, and show that the value 1 problem reformulates as the emptiness problem for probabilistic automata over prostochastic words. The free prostochastic monoid is represented in Figure 1, as a diamond. It contains the set of finite words, represented on the bottom of the picture, and the sets of polynomial and super-polynomial prostochastic words.

Our main result is the following (the missing definitions are given in Section 2 and 4):

Theorem 1. [*Optimality of the Markov Monoid algorithm*]

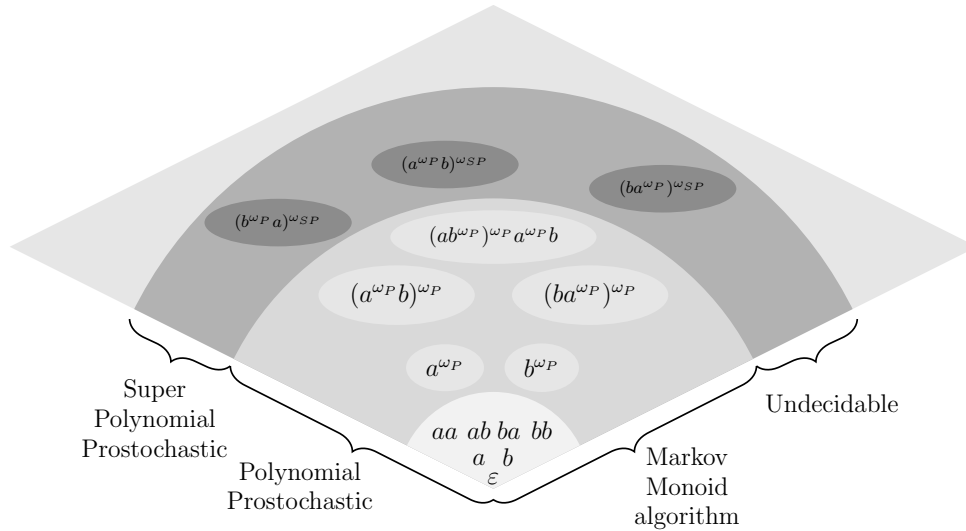


Fig. 1. The Free Prostochastic Monoid.

1. (Characterization) The Markov Monoid algorithm answers “YES” on input A if, and only if, there exists a polynomial prostochastic word accepted by A ,
2. (Undecidability) The following problem is undecidable: given a probabilistic automaton A as input, determine whether there exists a super-polynomial prostochastic word accepted by A .

To construct prostochastic words, we define two limit operators: an operator ω_P , where P stands for “polynomial”, and an operator ω_{SP} , where SP stands for “super-polynomial”.

The polynomial prostochastic words are built using concatenation and the operator ω_P . On an intuitive level, this does not allow for different convergence speeds to compete. Indeed, part of the proof consists in showing that the polynomial prostochastic words are fast, a notion made precise in Section 4.3. On the other hand, the super-polynomial prostochastic words are built using both operators ω_P and ω_{SP} , which allows for two convergence speeds to interfere, leading to undecidability.

We prove the first half of the theorem above in Section 4.4, and the second half in Section 4.5.

Acknowledgments

This paper and its author owe a lot to Szymon Toruńczyk's PhD thesis and its author, to Sam van Gool for his expertise on Profinite Theory, to Mikołaj Bogańczyk for his insightful remarks and to Jean-Éric Pin for his numerous questions and comments. The opportunity to present partial results on this topic in several scientific meetings has been a fruitful experience, and I thank everyone that took part in it.

2 Probabilistic Automata and the Value 1 Problem

We work with finite words over a finite alphabet A . The set of real numbers is denoted \mathbb{R} .

A matrix is stochastic if every entry is non-negative and each line sums up to 1. For a finite set Q (thought of as a set of states) and $E \subseteq \mathbb{R}$, we denote $\mathcal{M}_{Q \times Q}(E)$ the set of matrices over E . The restriction to stochastic matrices is denoted $\mathcal{S}_{Q \times Q}(E)$. We consider the ℓ_1 -norm $\|\cdot\|$ defined by $\|M\| = \max_j \sum_i M(i, j)$. It induces a topology on $\mathcal{M}_{Q \times Q}(\mathbb{R})$ and $\mathcal{S}_{Q \times Q}(\mathbb{R})$. The following classical properties will be useful:

Fact 1

- For every $M \in \mathcal{S}_{Q \times Q}(\mathbb{R})$, we have $\|M\| = 1$,
- For $M, M' \in \mathcal{M}_{Q \times Q}(\mathbb{R})$, we have $\|M \cdot M'\| \leq \|M\| \cdot \|M'\|$,
- The space $\mathcal{S}_{Q \times Q}(\mathbb{R})$ is compact (so also complete).

Definition 1 (Probabilistic automaton). A probabilistic automaton is given by a finite set of states Q , a transition function $\phi : A \rightarrow \mathcal{S}_{Q \times Q}(\{0, \frac{1}{2}, 1\})$, a stochastic vector of initial states I and a boolean vector of final states F .

A transition function $\phi : A \rightarrow \mathcal{S}_{Q \times Q}(\{0, \frac{1}{2}, 1\})$ naturally induces a morphism $\phi : A^* \rightarrow \mathcal{S}_{Q \times Q}(\{0, \frac{1}{2}, 1\})$. We denote by $P_{\mathcal{A}}(s \xrightarrow{w} t)$ the probability to go from state s to state t reading w on the automaton \mathcal{A} , i.e. $\phi(w)(s, t)$.

The *acceptance probability* of a word $w \in A^*$ by \mathcal{A} is $I \cdot \phi(w) \cdot F$, which we denote by $P_{\mathcal{A}}(w)$. In words, it is the probability that a run ends in a final state from (i.e. a state from F), starting from the initial distribution given by I .

Definition 2 (Value). The value of a probabilistic automaton \mathcal{A} , denoted by $\text{val}(\mathcal{A})$, is the supremum acceptance probability over all input words:

$$\text{val}(\mathcal{A}) = \sup_{w \in A^*} P_{\mathcal{A}}(w) .$$

We are interested in the following decision problem:

Problem 1 (Value 1 Problem). Given a probabilistic automaton \mathcal{A} , determine whether $\text{val}(\mathcal{A}) = 1$.

An equivalent formulation of the value 1 problem is as follows: given a probabilistic automaton \mathcal{A} , is it true that for all $\varepsilon > 0$, there exists a word w such that $P_{\mathcal{A}}(w) \geq 1 - \varepsilon$?

The value 1 problem can also be reformulated using the notion of *isolated cut-point* introduced by Rabin in his seminal paper [Rab63]: an automaton has value 1 if and only if the cut-point 1 is *not* isolated.

Unfortunately:

Theorem 2 ([GO10]). *The value 1 problem is undecidable.*

A series of papers ([GO10,CT12,FGO12,FGHO14,FGKO14]) tackled the following question, with different approaches and techniques:

“To what extent is the value 1 problem undecidable?”

One line of work was to construct algorithms to solve the problem on some subclass of probabilistic automata ([GO10,CT12,FGO12,FGKO14]). As proved in [FGKO14], the Markov Monoid algorithm is the most correct algorithm of all the algorithms proposed in these papers: all subclasses considered are included in the subclass of leaktight automata, for which the Markov Monoid algorithm correctly solves the value 1 problem.

Another route was to consider variants of the problem, by abstracting away the numerical values [FGHO14], but this does not lead to decidability.

In this paper, our aim is different. The objective is to draw a decidability barrier for the value 1 problem, through a precise understanding of both the Markov Monoid algorithm and the undecidability result.

3 The Prostochastic Theory

In this section, we develop a profinite theory for probabilistic automata. The main point here is to construct the free prostochastic monoid, which allows to reformulate the value 1 problem as an emptiness problem over prostochastic words. The prostochastic theory is then used as a formalism to prove the optimality of the Markov Monoid algorithm, in the next section.

3.1 The Free Prostochastic Monoid

A profinite monoid is a monoid for which two elements can be distinguished by a morphism into a finite monoid, *i.e.* by a finite automaton. To define prostochastic monoids, we use a stronger distinguishing feature, namely probabilistic automata, which correspond to stochastic matrices over the reals.

Definition 3 (Prostochastic Monoid). A monoid \mathcal{P} is prostochastic if for every $s \neq t \in \mathcal{P}$, there exists a morphism $\psi : \mathcal{P} \rightarrow \mathcal{S}_{Q \times Q}(\mathbb{R})$ such that $\psi(s) \neq \psi(t)$.

A prostochastic monoid \mathcal{P} is naturally equipped with the prostochastic topology, which is the smallest that makes continuous every morphism $\psi : \mathcal{P} \rightarrow \mathcal{S}_{Q \times Q}(\mathbb{R})$.

There are much more prostochastic monoids than profinite monoids. Indeed, $\mathcal{S}_{Q \times Q}(\mathbb{R})$ is prostochastic, but not profinite in general.

Lemma 1 (Prostochastic Monoids are Compact and Topological). Every prostochastic monoid \mathcal{P} is compact and topological, i.e. the product function

$$\begin{cases} \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P} \\ (s, t) \mapsto s \cdot t \end{cases}$$

is continuous.

Proof. Consider a prostochastic monoid \mathcal{P} , we show that it embeds into $M = \prod_{\psi: \mathcal{P} \rightarrow \mathcal{S}_{Q \times Q}(\mathbb{R})} \mathcal{S}_{Q \times Q}(\mathbb{R})$. We equip M with the product topology induced by the topology on $\mathcal{S}_{Q \times Q}(\mathbb{R})$, it is a compact space thanks to Tychonoff's theorem. Consider the map $\iota : \mathcal{P} \rightarrow M$ defined by $\iota(s) = (\psi(s))_{\psi: \mathcal{P} \rightarrow \mathcal{S}_{Q \times Q}(\mathbb{R})}$, it is a continuous injection. The continuity follows from the definition of both the prostochastic topology and the product topology, and the injectivity from the definition of the prostochastic monoid. Observe that $\iota(\mathcal{P})$ is closed, as it is equal to:

$$\bigcap_{\phi \neq \psi} \{m \in M \mid m_\phi = \phi(s) \text{ and } m_\psi = \psi(s) \text{ for some } s \in \mathcal{P}\}.$$

It follows that $\iota(\mathcal{P})$ is compact, and so is \mathcal{P} .

The continuity of the product in M follows from the continuity of the product in $\mathcal{S}_{Q \times Q}(\mathbb{R})$, which implies the continuity of the product in \mathcal{P} . ■

The main theorem of the prostochastic theory is the existence and uniqueness of a space, called the free prostochastic monoid, that satisfies a Universal Property. The statement is the same as in the profinite theory, replacing “profinite monoid” by “prostochastic monoid”.

Theorem 3 (Existence of the Free Prostochastic Monoid). For every finite alphabet A ,

1. There exists a prostochastic monoid $\mathcal{P}A^*$ and a continuous injection $\iota : A \rightarrow \mathcal{P}A^*$ such that every $\phi : A \rightarrow M$, where M is a prostochastic monoid, extends uniquely to a continuous morphism $\widehat{\phi} : \mathcal{P}A^* \rightarrow M$.
2. All prostochastic monoids satisfying this property are homeomorphic.

The unique prostochastic monoid satisfying the Universal Property stated in item 1. is called the free prostochastic monoid, and denoted $\mathcal{P}A^*$.

The uniqueness argument (item 2.) is a consequence of the Universal Property (item 1.), following standard arguments. The remainder of this subsection focuses on the existence part of this theorem (item 1.).

Before proceeding with the construction, we review the well-known situation in the profinite theory. There are several different constructions of the free profinite monoid, we consider two of them:

1. A simple yet abstract construction, as a subset of $\prod_{\phi:A \rightarrow M} M$,
2. A more pedestrian construction, as the completion of A^* equipped with an appropriate profinite distance,

Note that the uniqueness argument implies that the two constructions are equivalent. There are a number of extensions for classes of languages beyond regular languages. However, it has already been observed that these constructions lead to unappealing objects, for instance where the product function is not continuous.

To obtain a well-behaved prostochastic theory, we depart from this and construct a profinite theory of *probabilistic automata* rather than *probabilistic languages*, following the first construction. We further discuss in Subsection 3.2 this fine point, and how to construct the free prostochastic monoid following the second construction, as the completion of A^* equipped with a prostochastic distance.

We proceed with a construction of the free prostochastic monoid. Consider $X = \prod_{\phi:A \rightarrow M} M$, equipped with the product topology induced by the prostochastic topologies for each prostochastic monoid. Thanks to Tychonoff's theorem, it is compact. Denote $\iota : A \rightarrow X$ the map defined by $\iota(a) = (\phi(a))_{\phi:A \rightarrow M}$, it is a continuous injection.

Denote $\mathcal{P}A^* = \overline{\iota(A^*)}$, the closure of $\iota(A^*) \subseteq X$. Note that it is a monoid as the closure of a monoid.

We fix some notational conventions now. We denote sequences of finite words by $\mathbf{u}, \mathbf{v}, \mathbf{w}, \dots$. By definition, an element of $\mathcal{P}A^*$ is obtained as the limit of a sequence $\iota(\mathbf{u})$, in which case we say that \mathbf{u} induces \bar{u} . The elements of $\mathcal{P}A^*$ are denoted \bar{u}, \bar{v}, \dots . We sometimes implicitly assume that \bar{u} is induced by $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$, and the same for v, w, \dots .

Definition 4 (Converging Sequences and Equivalence). A sequence \mathbf{u} is converging if $\iota(\mathbf{u})$ converges (in X).

Two converging sequences \mathbf{u} and \mathbf{v} are equivalent if they induce the same element of $\mathcal{P}A^*$, i.e. if $\lim \iota(\mathbf{u}) = \lim \iota(\mathbf{v})$.

Unravelling the definitions, we obtain the following:

- a sequence \mathbf{u} is converging if, and only if, for every $\phi : A \rightarrow \mathcal{S}_{Q \times Q}(\mathbb{R})$, the sequence of stochastic matrices $\phi(\mathbf{u})$ converges (in $\mathcal{S}_{Q \times Q}(\mathbb{R})$),
- two converging sequences \mathbf{u} and \mathbf{v} are equivalent if, and only if, for every $\phi : A \rightarrow \mathcal{S}_{Q \times Q}(\mathbb{R})$, we have $\lim \phi(\mathbf{u}) = \lim \phi(\mathbf{v})$.

Observe that the second point implies that $\mathcal{P}A^*$ is a prostochastic monoid. Furthermore, the topology induced by X coincides with the prostochastic topology.

We now argue that $\mathcal{P}A^*$ satisfies the Universal Property. Indeed, for $\phi : A \rightarrow M$, define $\widehat{\phi} : \mathcal{P}A^* \rightarrow M$ by $\widehat{\phi}(\bar{u}) = \lim \phi(\mathbf{u})$, where \mathbf{u} is *some* sequence inducing \bar{u} . This is well-defined and induces a continuous morphism, extending ϕ . The uniqueness is clear.

This concludes the proof of Theorem 3.

3.2 Discussions

In this subsection, we discuss how to extend the other construction of the free profinite monoid. This reveals a subtlety: the construction above is *different* from the free profinite monoid with respect to the class of probabilistic languages.

We first describe a naïve approach to construct the free prostochastic monoid as the completion of A^* with an appropriate prostochastic distance, which will turn out to induce a different monoid with less appealing properties; for instance, its product is not continuous.

Recall that for the profinite distance, two words are *close* if there exists a *small* automaton that accepts one and rejects the other. One can define a prostochastic distance similarly.

Denote \mathcal{L} the class of probabilistic languages, i.e. the languages of the form $L^{>\frac{1}{2}}(\mathcal{A}) = \{w \in A^* \mid P_{\mathcal{A}}(w) > \frac{1}{2}\}$ for some probabilistic automaton \mathcal{A} . We say that two words u and v are N -separated with respect to \mathcal{L} if there exists a probabilistic automaton \mathcal{A} of size N such that $u \in L^{>\frac{1}{2}}(\mathcal{A})$ and $v \notin L^{>\frac{1}{2}}(\mathcal{A})$. Define $d_{\mathcal{L}}(u, v)$ as 2^{-N} , where N is minimal such that u and v are N -separated.

The function $d_{\mathcal{L}}$ is indeed a metric. One can define the completion of A^* equipped with the distance $d_{\mathcal{L}}$, as a candidate for the free prostochastic monoid. However, for this metric the product

$$\begin{cases} A^* \times A^* \rightarrow A^* \\ (u, v) \mapsto u \cdot v \end{cases}$$

is not continuous, hence in particular this space is not prostochastic. This is not a surprise:

Theorem 4 ([GGP10]). *Let \mathcal{L} be a Boolean algebra of languages. Consider the completion of A^* equipped with the distance $d_{\mathcal{L}}$, its product is continuous if, and only if, \mathcal{L} contains only regular languages.*

In light of this theorem, in order to obtain a well-behaved prostochastic theory, we need to move away from this construction relying on a *class of languages*. Instead of considering that a probabilistic automaton defines a language $L^{>\frac{1}{2}}(\mathcal{A})$, in other words a function $A^* \rightarrow \{0, 1\}$, we consider the function $P_{\mathcal{A}} : A^* \rightarrow [0, 1]$.

We quickly describe a different approach to construct the free prostochastic monoid as the completion of A^* with an appropriate prostochastic distance.

Definition 5 (Prostochastic Distance). *We say that two words u and v are (N, η) -separated if there exists $\phi : A \rightarrow \mathcal{S}_{Q \times Q}(\{0, \frac{1}{2}, 1\})$ such that*

$$|Q| \leq N \quad \text{and} \quad \|\phi(u) - \phi(v)\| \geq \eta .$$

Define $d(u, v)$ as 2^{-N} , where N is minimal such that u and v are $(N, 2^{-N})$ -separated.

Informally, two words are *close* if there exists a morphism into a *small* probabilistic automaton, that separates them by a *large* value. Hence the distance involves a threshold between the size of the automaton, which should be small, and the separation between the values, which should be large.

Unfortunately, the function d is not a metric, as it does not satisfy the triangle inequality, but only a weaker version:

$$d(u, v) \leq 2 \cdot \max\{d(u, w), d(w, v)\} .$$

Still, one can define the completion of A^* equipped with the distance d . The product function is uniformly continuous, and one can prove that this indeed gives rise to a prostochastic monoid satisfying the Universal Property, hence homeomorphic to the first construction.

3.3 Reformulation of the Value 1 Problem

The aim of this subsection is to reformulate the value 1 problem, which talks about sequences of finite words, into an emptiness problem over prostochastic words.

Definition 6 (Prostochastic Language of a Probabilistic Automaton). Let \mathcal{A} be a probabilistic automaton and \bar{u} a prostochastic word. We say that \bar{u} is accepted by \mathcal{A} if \bar{u} is induced by some converging sequence \mathbf{u} such that $\lim P_{\mathcal{A}}(\mathbf{u}) = 1$.

We denote by $L(\mathcal{A})$ the set of prostochastic words accepted by \mathcal{A} .

Note that \bar{u} is accepted by \mathcal{A} if, and only if, all sequences \mathbf{u} inducing \bar{u} satisfy $\lim P_{\mathcal{A}}(\mathbf{u}) = 1$: it does not depend on the chosen representative.

Theorem 5 (The Value 1 Problem and the Emptiness Problem over Prostochastic Words). Let \mathcal{A} be a probabilistic automaton. The following are equivalent:

- $\text{val}(\mathcal{A}) = 1$,
- $L(\mathcal{A})$ is non-empty.

Proof. Assume $\text{val}(\mathcal{A}) = 1$, then there exists a sequence of words \mathbf{u} such that $\lim P_{\mathcal{A}}(\mathbf{u}) = 1$. We see \mathbf{u} as a sequence of prostochastic words. By compactness of $\mathcal{P}A^*$ it contains a converging subsequence, which without loss of generality we assume is \mathbf{u} itself. The prostochastic word induced by \mathbf{u} belongs to $L(\mathcal{A})$.

Conversely, let \bar{u} be a prostochastic word accepted by \mathcal{A} . Consider a sequence \mathbf{u} inducing \bar{u} . By definition, we have $\lim P_{\mathcal{A}}(\mathbf{u}) = 1$, implying that $\text{val}(\mathcal{A}) = 1$. ■

4 Optimality of the Markov Monoid Algorithm

In this section, we use the prostochastic theory developed in the previous section to prove the optimality of the Markov Monoid algorithm. We first present the algorithm in Subsection 4.1, introduced in [FGO12]. We introduce two limit operators in Subsection 4.2, and our main technical tool, the fast sequences, in Subsection 4.3. We give in Subsection 4.4 a characterization of the Markov Monoid algorithm using polynomial prostochastic words, and the Subsection 4.5 shows an undecidability result for super-polynomial prostochastic words.

4.1 The Algorithm

The Markov Monoid algorithm was introduced in [FGO12]. The presentation that we give here is different yet equivalent. Consider \mathcal{A} a probabilistic automaton, the Markov Monoid algorithm consists in computing, through a saturation process, the Markov Monoid of \mathcal{A} .

It is a monoid of boolean matrices: all numerical values are projected away to boolean values. Formally, for $M \in \mathcal{S}_{Q \times Q}(\mathbb{R})$, define its boolean projection $\pi(M)$, as the boolean matrix such that $\pi(M)(s, t) = 1$ if $M(s, t) > 0$,

and $\pi(M)(s, t) = 0$ otherwise. Hence to define the Markov Monoid, one can consider the underlying non-deterministic automaton $\pi(\mathcal{A})$ instead of the probabilistic automaton \mathcal{A} .

The Markov Monoid of $\pi(\mathcal{A})$ contains the transition monoid of $\pi(\mathcal{A})$, which is the monoid generated by $\{\pi(\phi(a)) \mid a \in A\}$ and closed under (boolean matrix) products. Informally speaking, the transition monoid accounts for the boolean action of every finite word. Formally, for a word $w \in A^*$, the element $\langle w \rangle$ of the transition monoid of $\pi(\mathcal{A})$ satisfies the following: $\langle w \rangle(s, t) = 1$ if, and only if there exists a run from s to t reading w on $\pi(\mathcal{A})$.

The Markov Monoid generalizes the transition monoid by introducing a new operator, the stabilization. On the intuitive level first: let $M \in \mathcal{S}_{Q \times Q}(R)$, it can be interpreted as a Markov chain; its boolean projection $\pi(M)$ give the structural properties of this Markov chain. The stabilization $\pi(M)^\sharp$ accounts for $\lim_n M^n$, i.e. the behaviour of the Markov chain M in the limit. The formal definition of the stabilization operator relies on basic concepts from Markov chain theory.

Definition 7 (Stabilization). *Let M be a boolean matrix. It is idempotent if $M \cdot M = M$.*

Assume M is idempotent, then we say that $t \in Q$ is M -recurrent if for all $s \in Q$, if $M(s, t) = 1$, then $M(t, s) = 1$.

The stabilization operator is defined only on idempotent elements:

$$M^\sharp(s, t) = \begin{cases} 1 & \text{if } M(s, t) = 1 \text{ and } t \text{ is } M\text{-recurrent,} \\ 0 & \text{otherwise.} \end{cases}$$

The definition of the stabilization matches the intuition that in the Markov chain $\lim_n M^n$, the probability to be in non-recurrent states converges to 0. This will be made precise in Subsection 4.4.

Definition 8 (Markov Monoid). *The Markov Monoid of \mathcal{A} is the smallest set of boolean matrices containing $\{\pi(\phi(a)) \mid a \in A\}$ and closed under product and stabilization of idempotents.*

We give an equivalent presentation through ω -expressions, described by the following grammar:

$$E \quad \longrightarrow \quad a \quad | \quad E \cdot E \quad | \quad E^\omega .$$

We define an interpretation $\langle \cdot \rangle$ of ω -expressions into boolean matrices:

- $\langle a \rangle$ is $\pi(\phi(a))$,

- $\langle E_1 \cdot E_2 \rangle$ is $\langle E_1 \rangle \cdot \langle E_2 \rangle$,
- $\langle E^\omega \rangle$ is $\langle E \rangle^\sharp$, only defined if $\langle E \rangle$ is idempotent.

Then the Markov Monoid is $\{\langle E \rangle \mid E \text{ an } \omega\text{-expression}\}$.

The Markov Monoid algorithm computes the Markov Monoid, and looks for *value 1 witnesses*:

Definition 9 (Value 1 Witnesses). *A boolean matrix M is a value 1 witness if: for all $s \in I$, $t \in Q$, if $M(s, t) = 1$, then $t \in F$.*

The Markov Monoid algorithm answers “YES” if there exists a value 1 witness in the Markov Monoid, and “NO” otherwise. The following has been proved in [FGO12]:

Theorem 6 ([FGO12]).

- *If the Markov Monoid algorithm answers “YES” on input \mathcal{A} , then the probabilistic automaton \mathcal{A} has value 1,*
- *The converse does not hold in general: there exists a probabilistic automaton that has value 1, such that the Markov Monoid algorithm answers “NO”,*
- *The Markov Monoid algorithm can be implemented in PSPACE.*

4.2 Limit Operators for Prostochastic Words

We show in this subsection how to construct non-trivial prostochastic words. In particular, we want to define a limit operator that accounts for the stabilization operation from the Markov Monoid. To this end, we need to better understand *convergence speeds phenomena*: different limit behaviours can occur, depending on how fast the underlying Markov chains converge.

We will define two limit operators: an operator ω_P , where P stands for “polynomial”, and an operator ω_{SP} , where SP stands for “super-polynomial”. First, we analyze the automaton represented in Figure 2, which was introduced in [GO10].

As explained in [FGO12, FGKO14], if $x > \frac{1}{2}$, we have $\lim_n P_{\mathcal{A}}((ba^n)^{2^n}) = 1$, but $\lim_n P_{\mathcal{A}}((ba^n)^n) < 1$. This exhibits two different behaviours; the first one shall be accounted for by $(\mathbf{ba}^{\omega_P})^{\omega_{SP}}$, inducing a super-polynomial prostochastic word, the second by $(\mathbf{ba}^{\omega_P})^{\omega_P}$, inducing a polynomial prostochastic word.

Informally speaking, this automaton consists of two symmetric parts, left and right. The left part leads to the accepting state, and the right part to the rejecting sink. To reach the accepting state with arbitrarily high probability, one needs to “tip the scales” to the left. Consider the following experiment, which

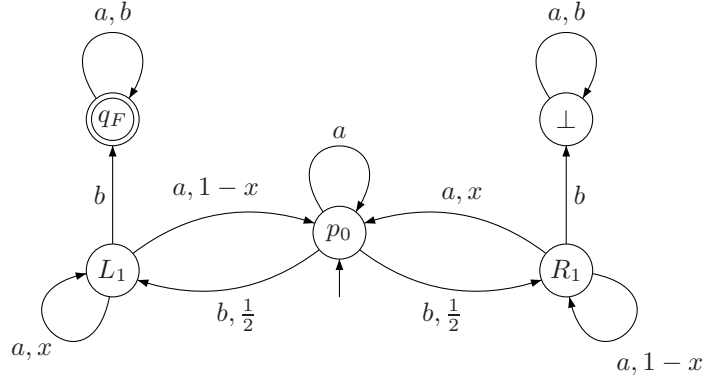


Fig. 2. Automaton accepting a super-polynomial prostochastic word but no polynomial ones.

consists in reading b and then a long sequence of a 's. It results in the following situation: with high probability, the current state is p_0 , with small probability it is L_1 , and with even smaller probability it is R_1 . To construct a sequence of words with arbitrarily high probability of being accepted, one has to play with this difference, and repeat the previous experiment many times. As shown by precise calculations, what matters is that this experiment is repeated exponentially more than the length of the experiment, leading to the sequence of words $((ba^n)^{2^n})_{n \in \mathbb{N}}$.

We now turn to the definitions of ω_P and ω_{SP} . Consider the two functions $f_P, f_{SP} : \mathbb{N} \rightarrow \mathbb{N}$ defined as follows:

- $f_P(n) = k!$, where k is maximal such that $k! \leq n$,
- $f_{SP}(n) = k!$, where k is maximal such that $k! \leq n^{\log(n)}$.

The function f_P grows linearly: roughly, $f_P(n) \sim n$, and the function f_E grows super-polynomially: roughly, $f_{SP}(n) \sim n^{\log(n)}$. Both choices of n and $n^{\log(n)}$ are arbitrary; one could replace n by any polynomial, and $n^{\log(n)}$ by any function both super-polynomial and sub-exponential.

The functions f_P and f_{SP} are *factorial-like*: for all $p \in \mathbb{N}$, there exists $k \in \mathbb{N}$, such that for all $n \geq k$, we have $p \mid f(n)$, i.e. p divides $f(n)$. The choice of factorial-like functions comes from the following classical result from Markov chain theory.

Lemma 2 (Powers of a Stochastic Matrix). *Let $M \in \mathcal{S}_{Q \times Q}(\mathbb{R})$. There exists $M^\infty \in \mathcal{S}_{Q \times Q}(\mathbb{R})$ such that:*

- for all $f : \mathbb{N} \rightarrow \mathbb{N}$ factorial-like, the sequence $(M^{f(n)})_{n \in \mathbb{N}}$ converges to M^∞ ,
- there exist two constants K and $C > 1$ such that
$$\|M^{f(n)} - M^\infty\| \leq K \cdot C^{-f(n)},$$
- if $\pi(M)$ is idempotent, then $\pi(M^\infty) = \pi(M)^\sharp$.

The two operators ω_P and ω_{SP} take as input a sequence of finite words, and output a sequence of finite words. Formally, let \mathbf{u} be a sequence of finite words, define:

$$\mathbf{u}^{\omega_P} = (u_n^{f_P(n \cdot |u_n|)})_{n \in \mathbb{N}} \quad ; \quad \mathbf{u}^{\omega_{SP}} = (u_n^{f_{SP}(n \cdot |u_n|)})_{n \in \mathbb{N}}.$$

It is not true in general that if \mathbf{u} converges, then \mathbf{u}^{ω_P} converges, nor $\mathbf{u}^{\omega_{SP}}$. In the next subsection, we will show that a sufficient condition is that \mathbf{u} is fast.

4.3 Fast Sequences

This subsection introduces fast sequences, as the *key* technical tool for the proofs to follow.

Definition 10 (Fast Sequences). *A sequence of finite words \mathbf{u} is fast if it converges (we denote \bar{u} the prostochastic word it induces), and for every $\phi : A \rightarrow \mathcal{S}_{Q \times Q}(\mathbb{R})$, there exist a polynomial P and $C > 1$ such that for every n ,*

$$\|\phi(u_n) - \widehat{\phi}(\bar{u})\| \leq P(|u_n|) \cdot C^{-|u_n|}.$$

A prostochastic word is fast if it is induced by *some* fast sequence. We denote by $\mathcal{P}A_f^*$ the set of fast prostochastic words. The next lemmas show the following:

- $\mathcal{P}A_f^*$ is a submonoid of $\widehat{\mathcal{P}A^*}$: in other words, the concatenation of two fast prostochastic words is a fast prostochastic word,
- ω_P is an operator $\mathcal{P}A_f^* \rightarrow \mathcal{P}A_f^*$,
- ω_{SP} is an operator $\mathcal{P}A_f^* \rightarrow \mathcal{P}A^*$.

Lemma 3 (Concatenation and Fast Sequences). *Let \mathbf{u}, \mathbf{v} be two fast sequences.*

The sequence $\mathbf{u} \cdot \mathbf{v} = (u_n \cdot v_n)_{n \in \mathbb{N}}$ is fast.

Proof. Let $\phi : A \rightarrow \mathcal{S}_{Q \times Q}(\mathbb{R})$ and n .

$$\begin{aligned} & \|\phi(u_n) \cdot \phi(v_n) - \widehat{\phi}(\bar{u}) \cdot \widehat{\phi}(\bar{v})\| \\ &= \|\phi(u_n) \cdot (\phi(v_n) - \widehat{\phi}(\bar{v})) - (\widehat{\phi}(\bar{u}) - \phi(u_n)) \cdot \widehat{\phi}(\bar{v})\| \\ &\leq \|\phi(u_n)\| \cdot \|\phi(v_n) - \widehat{\phi}(\bar{v})\| + \|\widehat{\phi}(\bar{u}) - \phi(u_n)\| \cdot \|\widehat{\phi}(\bar{v})\| \\ &= \|\phi(v_n) - \widehat{\phi}(\bar{v})\| + \|\widehat{\phi}(\bar{u}) - \phi(u_n)\| \end{aligned}$$

Since \mathbf{u} and \mathbf{v} are fast, the previous inequality implies that $\mathbf{u} \cdot \mathbf{v}$ is fast. ■

Lemma 4 (Limit Operators and Fast Sequences). *Let \mathbf{u}, \mathbf{v} be two equivalent fast sequences. Then:*

- \mathbf{u}^{ω_P} and \mathbf{v}^{ω_P} are fast and equivalent,
- $\mathbf{u}^{\omega_{SP}}$ and $\mathbf{v}^{\omega_{SP}}$ converge and are equivalent.

Proof. Let $\phi : A \rightarrow \mathcal{S}_{Q \times Q}(\mathbb{R})$ and $f \in \{f_P, f_{SP}\}$.

Thanks to Lemma 2, the sequence $(\widehat{\phi}(\bar{u})^{f(n \cdot |u_n|)})_{n \in \mathbb{N}}$ converges to $\widehat{\phi}(\bar{u})^\infty$, there exists two constants K and $C_1 > 1$ such that for every n , we have $\|\widehat{\phi}(\bar{u})^{f(n \cdot |u_n|)} - \widehat{\phi}(\bar{u})^\infty\| \leq K \cdot C_1^{-f(n \cdot |u_n|)}$.

We proceed in two steps, using the following inequality, which holds for every n :

$$\begin{aligned} & \|\phi(u_n^{f(n \cdot |u_n|)}) - \widehat{\phi}(\bar{u})^\infty\| \\ & \leq \|\phi(u_n)^{f(n \cdot |u_n|)} - \widehat{\phi}(\bar{u})^{f(n \cdot |u_n|)}\| \\ & \quad + \|\widehat{\phi}(\bar{u})^{f(n \cdot |u_n|)} - \widehat{\phi}(\bar{u})^\infty\|. \end{aligned}$$

For the left part, we rely on the following equality, where x and y may not commute:

$$x^N - y^N = \sum_{k=0}^{N-1} x^{N-k-1} \cdot (x - y) \cdot y^k.$$

Let $N = f(n \cdot |u_n|)$, this gives:

$$\begin{aligned} & \|\phi(u_n)^N - \widehat{\phi}(\bar{u})^N\| = \\ & \left\| \sum_{k=0}^{N-1} \phi(u_n)^{N-k-1} \cdot (\phi(u_n) - \widehat{\phi}(\bar{u})) \cdot \widehat{\phi}^k(\bar{u}) \right\| \\ & \leq \sum_{k=0}^{N-1} \|\phi(u_n)^{N-k-1}\| \cdot \|\phi(u_n) - \widehat{\phi}(\bar{u})\| \cdot \|\widehat{\phi}^k(\bar{u})\| \\ & \leq \sum_{k=0}^{N-1} \|\phi(u_n)\|^{N-k-1} \cdot \|\phi(u_n) - \widehat{\phi}(\bar{u})\| \cdot \|\widehat{\phi}(\bar{u})\|^k \\ & = N \cdot \|\phi(u_n) - \widehat{\phi}(\bar{u})\|. \end{aligned}$$

Since \mathbf{u} is fast, there exist a polynomial P and $C_2 > 1$ such that $\|\phi(u_n) - \widehat{\phi}(\bar{u})\| \leq P(|u_n|) \cdot C_2^{-|u_n|}$. Altogether, we have

$$\begin{aligned} & \|\phi(u_n^{f(n \cdot |u_n|)}) - \widehat{\phi}(\bar{u})^\infty\| \\ & \leq f(n \cdot |u_n|) \cdot P(|u_n|) \cdot C_2^{-|u_n|} + K \cdot C_1^{-f(n \cdot |u_n|)}. \end{aligned}$$

It follows that $(\phi(u_n^{f(n \cdot |u_n|)}))_{n \in \mathbb{N}}$ converges to $\widehat{\phi}(\bar{u})^\infty$, so both sequences \mathbf{u}^{ω_P} and \mathbf{u}^{ω_E} converge.

Furthermore, since \mathbf{u} and \mathbf{v} are equivalent, we have $\lim \phi(\mathbf{u}^{\omega_P}) = \lim \phi(\mathbf{v}^{\omega_P})$, implying that \mathbf{u}^{ω_P} and \mathbf{v}^{ω_P} are equivalent. The same goes for ω_{SP} .

For $f = f_P$, we have $f(n \cdot |u_n|) \leq n \cdot |u_n|$, so there exist some polynomial Q and $C > 1$ such that $\|\phi(u_n^{f(n \cdot |u_n|)}) - \widehat{\phi}(\bar{u})^\infty\| \leq Q(|u_n|) \cdot C^{-|u_n|}$, implying that \mathbf{u}^{ω_P} is fast. ■

We define an interpretation $\overline{\cdot}$ of ω -expressions into prostochastic words:

- \bar{a} is prostochastic word induced by the constant sequence of the one-letter word a ,
- $\overline{E_1 \cdot E_2} = \overline{E_1} \cdot \overline{E_2}$,
- $\overline{E^\omega} = \overline{E}^{\omega_P}$.

Definition 11 (Polynomial and Super-polynomial Prostochastic Words). *The set of polynomial prostochastic words is*

$$\{\overline{E} \mid E \text{ is an } \omega\text{-expression}\}.$$

The set of super-polynomial prostochastic words is

$$\{\overline{E}^{\omega_{SP}} \mid E \text{ is an } \omega\text{-expression}\}.$$

4.4 A Characterization with Polynomial Prostochastic Words

The aim of this subsection is to prove that for given a probabilistic automaton \mathcal{A} , for every ω -expression E , the element $\langle E \rangle$ of the Markov Monoid of \mathcal{A} is a value 1 witness if, and only if, the polynomial prostochastic word \overline{E} is accepted by \mathcal{A} . This implies the following characterization of the Markov Monoid algorithm:

The Markov Monoid algorithm answers “YES”
on input \mathcal{A}
if, and only if,
there exists a polynomial prostochastic word
accepted by \mathcal{A} .

This is the first item of Theorem 1. It follows from the following proposition.

Proposition 1. For all ω -expressions E , for every $\phi : A \rightarrow \mathcal{S}_{Q \times Q}(\mathbb{R})$, we have

$$\pi(\widehat{\phi}(\overline{E})) = \langle E \rangle .$$

Consequently, the element $\langle E \rangle$ of the Markov Monoid is a value 1 witness if, and only if, the polynomial prostochastic word \overline{E} is accepted by \mathcal{A} .

We prove the first part of Proposition 1 by induction on the ω -expression E , which now essentially amounts to gather the results from the previous sections. The second part is a direct corollary of the first part.

The base case is $a \in A$, clear.

The product case: let $E = E_1 \cdot E_2$, and $\phi : A \rightarrow \mathcal{S}_{Q \times Q}(\mathbb{R})$.

We prove that $\pi(\widehat{\phi}(\overline{E})) = \langle E \rangle$. Indeed, by definition $\widehat{\phi}(\overline{E}) = \widehat{\phi}(\overline{E}_1) \cdot \widehat{\phi}(\overline{E}_2)$ and $\langle E \rangle = \langle E_1 \rangle \cdot \langle E_2 \rangle$, so the conclusion follows from the induction hypothesis.

The iteration case: let $E = F^\omega$, and $\phi : A \rightarrow \mathcal{S}_{Q \times Q}(\mathbb{R})$.

We prove that $\pi(\widehat{\phi}(\overline{E})) = \langle E \rangle$. This follows from the definitions, the induction hypothesis and Lemma 2.

The proof of Proposition 1 is complete. It implies the first item of Theorem 1.

4.5 Undecidability for Super-polynomial Prostochastic Words

The aim of this subsection is to show that undecidability is around the corner:

The following problem is undecidable:
 given a probabilistic automaton \mathcal{A} ,
 determine whether there exists
 a super-polynomial prostochastic word accepted by \mathcal{A} .

This is the second item of Theorem 1.

Proof. We construct a reduction from the emptiness problem of probabilistic automata over finite words, proved to be undecidable in [Paz71]. Let \mathcal{A} be a probabilistic automaton, we ask if there exists a finite word w such that $P_{\mathcal{A}}(w) > \frac{1}{2}$. We construct a probabilistic automaton \mathcal{B} such that the following holds:

there exists a finite word w such that $P_{\mathcal{A}}(w) > \frac{1}{2}$
 if, and only if,
 there exists a super-polynomial prostochastic word
 accepted by \mathcal{B} .

The reduction is essentially as in [GO10], where they proved the undecidability of the value 1 problem. It is illustrated in Figure 3. In the original proof, it was enough to prove the existence of any prostochastic word accepted by \mathcal{B} . The challenge here is to improve the construction and the proof to show the existence of a super-polynomial prostochastic word accepted by \mathcal{B} .

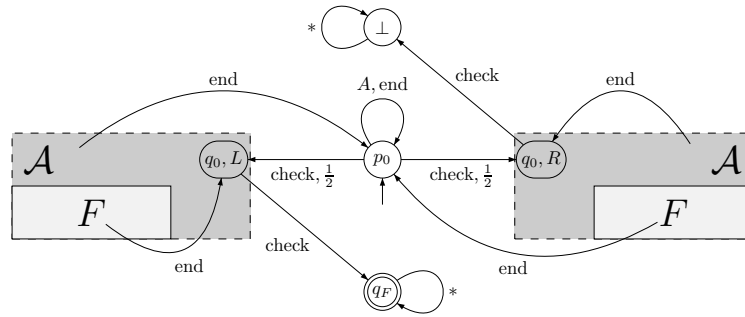


Fig. 3. Reduction.

The automaton \mathcal{B} is very similar to the one presented in Figure 2, except that the role of the letter a is now replaced by the simulation of a word in \mathcal{A} .

We fix the notations: the set of states of \mathcal{A} is Q , its transition function is ϕ , without loss of generality we assume that it has a unique initial state q_0 (which has no ingoing transitions), and the set of final states is F .

The alphabet of \mathcal{B} is $B = A \uplus \{\text{check}, \text{end}\}$, its set of states is $Q \times \{L, R\} \uplus \{p_0, \perp, q_F\}$, its transition function is ϕ' , the only initial state is p_0 and the only

final state is q_F . We define ϕ' as follows:

$$\left\{ \begin{array}{ll} \phi'(p_0, a) & = p_0 \text{ for } a \in A \\ \phi'(p_0, \text{end}) & = p_0 \\ \phi'(p_0, \text{check}) & = \frac{1}{2} \cdot (q_0, L) + \frac{1}{2} \cdot (q_0, R) \\ \phi'((q, d), a) & = (\phi(q, a), d) \text{ for } a \in A \\ \phi'((q_0, L), \text{check}) & = q_F \\ \phi'((q, L), \text{end}) & = q_0 \text{ if } q \in F \\ \phi'((q, L), \text{end}) & = p_0 \text{ if } q \notin F \\ \phi'((q_0, R), \text{check}) & = \perp \\ \phi'((q, R), \text{end}) & = p_0 \text{ if } q \in F \\ \phi'((q, R), \text{end}) & = q_0 \text{ if } q \notin F \\ \phi'(q_F, *) & = q_F \\ \phi'(\perp, *) & = \perp \end{array} \right.$$

Assume that there exists a finite word w such that $P_{\mathcal{A}}(w) > \frac{1}{2}$, then we claim that $(\text{check} \cdot (w \cdot \text{end})^{\omega_P})^{\omega_{SP}}$ is accepted by \mathcal{B} . Denote $x = P_{\mathcal{A}}(w)$.

We have

$$P_{\mathcal{A}}(p_0 \xrightarrow{\text{check} \cdot (w \cdot \text{end})^k} (q_0, L)) = \frac{1}{2} \cdot x^k,$$

and

$$P_{\mathcal{A}}(p_0 \xrightarrow{\text{check} \cdot (w \cdot \text{end})^k} (q_0, R)) = \frac{1}{2} \cdot (1 - x)^k.$$

We fix an integer N and analyze the action of reading $(\text{check} \cdot (w \cdot \text{end})^k)^N$: there are N “rounds”, each of them corresponding to reading $\text{check} \cdot (w \cdot \text{end})^k$ from p_0 . In a round, there are three outcomes: winning (that is, remaining in (q_0, L)) with probability $p_k = \frac{1}{2} \cdot x^k$, losing (that is, remaining in (q_0, R)) with probability $q_k = \frac{1}{2} \cdot (1 - x)^k$, or going to the next round (that is, reaching p_0) with probability $1 - (p_k + q_k)$. If a round is won or lost, then the next check leads to an accepting or rejecting sink; otherwise it goes on to the next round, for N rounds. Hence:

$$\begin{aligned} & P_{\mathcal{A}}((\text{check} \cdot (w \cdot \text{end})^k)^N) \\ &= \sum_{i=1}^{N-1} (1 - (p_k + q_k))^{i-1} \cdot p_k \\ &= p_k \cdot \frac{1 - (1 - (p_k + q_k))^{N-1}}{1 - (1 - (p_k + q_k))} \\ &= \frac{1}{1 + \frac{q_k}{p_k}} \cdot (1 - (1 - (p_k + q_k))^{N-1}) \end{aligned}$$

We now set $k = f_P(n \cdot (|w| + 1))$ and $N = f_E(n \cdot (1 + k \cdot (|w| + 1)))$. A simple calculation shows that the sequence $((1 - (p_k + q_k))^{N-1})_{n \in \mathbb{N}}$ converges to 0 as n goes to infinity. Furthermore, $\frac{q_k}{p_k} = (\frac{1-x}{x})^k$, which converges to 0 as n goes to infinity since $x > \frac{1}{2}$. It follows that the acceptance probability converges to 1 as n goes to infinity. Consequently:

$$\lim_n P_{\mathcal{A}}((\text{check} \cdot (w \cdot \text{end})^k)^N) = 1 ,$$

i.e. $\overline{(\text{check} \cdot (w \cdot \text{end})^{\omega_P})^{\omega_{SP}}}$ is accepted by \mathcal{B} .

Conversely, assume that for all finite words w , we have $P_{\mathcal{A}}(w) \leq \frac{1}{2}$. We claim that every finite word in B^* is accepted by \mathcal{B} with probability at most $\frac{1}{2}$. First of all, using simple observations we restrict ourselves to words of the form

$$w = \text{check} \cdot w_1 \cdot \text{end} \cdot w_2 \cdot \text{end} \cdots w_n \cdot \text{end} \cdot w' ,$$

with $w_i \in A^*$ and $w' \in B^*$. Since $P_{\mathcal{A}}(w_i) \leq \frac{1}{2}$ for every i , it follows that in \mathcal{B} , after reading the last letter end in w before w' , the probability to be in (q_0, L) is smaller or equal than the probability to be in (q_0, R) . This implies the claim. It follows that the value of \mathcal{B} is not 1, so \mathcal{B} accepts no prostochastic words thanks to Theorem 5. ■

This concludes the optimality argument for the Markov Monoid algorithm, which consisted in first characterizing its computations using polynomial prostochastic words, and then showing that considering super-polynomial prostochastic words leads to undecidability.

Conclusion and Perspectives

In this paper, we developed a profinite theory for probabilistic automata, called the prostochastic theory, and used it to formalize an optimality argument for the Markov Monoid algorithm. To the best of our knowledge, this is the first optimality argument for algorithms working on probabilistic automata.

This opens new perspectives. One of them is to further develop the prostochastic theory, for instance to better understand the class of fast prostochastic words, and another is to push our result, using the prostochastic theory to construct an optimal algorithm for approximating the value.

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