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CONSTRUCTIVE APPROXIMATION IN DE BRANGES–ROVNYAK SPACES

OMAR EL-FALLAH, EMMANUEL FRICAIN, KARIM KELLAY,
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ABSTRACT. In most classical holomorphic function spaces on the unit disk in which the polynomials are dense, a function f can be approximated in norm by its dilates $f_r(z) := f(rz)$ ($r < 1$). We show that this is *not* the case for the de Branges–Rovnyak spaces $\mathcal{H}(b)$. More precisely, we exhibit a space $\mathcal{H}(b)$ in which the polynomials are dense and a function $f \in \mathcal{H}(b)$ such that $\lim_{r \rightarrow 1^-} \|f_r\|_{\mathcal{H}(b)} = \infty$.

On the positive side, we prove the following approximation theorem for Toeplitz operators on general de Branges–Rovnyak spaces $\mathcal{H}(b)$. If (h_n) is a sequence in H^∞ such that $\|h_n\|_{H^\infty} \leq 1$ and $h_n(0) \rightarrow 1$, then $\|T_{h_n} f - f\|_{\mathcal{H}(b)} \rightarrow 0$ for all $f \in \mathcal{H}(b)$. Using this result, we give the first constructive proof that, if b is a non-extreme point of the unit ball of H^∞ , then the polynomials are dense in $\mathcal{H}(b)$.

1. INTRODUCTION

The de Branges–Rovnyak spaces are a family of subspaces $\mathcal{H}(b)$ of the Hardy space H^2 , parametrized by elements b of the closed unit ball of H^∞ . We shall give the precise definition in §2. In general $\mathcal{H}(b)$ is not closed in H^2 , but it carries its own norm $\|\cdot\|_{\mathcal{H}(b)}$ making it a Hilbert space.

The spaces $\mathcal{H}(b)$ were introduced by de Branges and Rovnyak in the appendix of [2] and further studied in their book [3]. The initial motivation was to provide canonical model spaces for certain types of contractions on Hilbert spaces. Subsequently it was realized that these spaces have several interesting connections with other topics in complex analysis and operator theory. For background information we refer to the books of de Branges and Rovnyak [3], Sarason [6], and the forthcoming monograph of Fricain and Mashreghi [5].

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The general theory of $\mathcal{H}(b)$ -spaces splits into two cases, according to whether b is an extreme point or a non-extreme point of the unit ball of H^∞ . For example, if b is non-extreme, then $\mathcal{H}(b)$ contains all functions holomorphic in a neighborhood of the closed unit disk $\overline{\mathbb{D}}$, whereas if b is extreme, then $\mathcal{H}(b)$ contains very few such functions. In particular, $\mathcal{H}(b)$ contains the polynomials if and only if b is non-extreme, and in this case, the polynomials are dense in $\mathcal{H}(b)$. Proofs of all these facts can be found in Sarason's book [6].

The density of polynomials in the non-extreme case is proved in [6] by showing that their orthogonal complement in $\mathcal{H}(b)$ is zero. The proof is non-constructive in the sense that it gives no clue how to find polynomial approximants to a given function. We know of no published work describing constructive methods of polynomial approximation in $\mathcal{H}(b)$, and it is surely of interest to have such methods available.

Perhaps the most natural approach is to try using dilations. Writing $f_r(z) := f(rz)$, the idea is to approximate f by f_r for some $r < 1$, and then f_r by the partial sums of its Taylor series. This idea works in many function spaces, but, as we shall see, it fails dismally in $\mathcal{H}(b)$, at least for certain choices of b . Indeed, it can happen that $\lim_{r \rightarrow 1^-} \|f_r\|_{\mathcal{H}(b)} = \infty$, even though $f \in \mathcal{H}(b)$. We shall prove this in §3, thereby answering a question posed in [1].

This phenomenon has other negative consequences, among them the surprising fact that the formula for $\|f\|_{\mathcal{H}(b)}$ in terms of the Taylor coefficients of f (see (8) below), previously known to hold for f holomorphic on a neighborhood of $\overline{\mathbb{D}}$, actually breaks down for general $f \in \mathcal{H}(b)$. It also shows that, in general, neither the Taylor partial sums of f , nor their Cesàro means need converge to f in $\mathcal{H}(b)$.

So, to construct polynomial approximants to functions in $\mathcal{H}(b)$, a different idea is needed. In §4, we establish a rather general approximation theorem for Toeplitz operators in $\mathcal{H}(b)$, valid for all b , both extreme and non-extreme. Then, in §5, we use this result to derive a constructive method for polynomial approximation in $\mathcal{H}(b)$ in the case when b is non-extreme.

2. BACKGROUND ON $\mathcal{H}(b)$ -SPACES

Given $\psi \in L^\infty(\mathbb{T})$, the corresponding Toeplitz operator $T_\psi : H^2 \rightarrow H^2$ is defined by

$$T_\psi f := P_+(\psi f) \quad (f \in H^2),$$

where $P_+ : L^2(\mathbb{T}) \rightarrow H^2$ denotes the orthogonal projection of $L^2(\mathbb{T})$ onto H^2 . Clearly T_ψ is a bounded operator on H^2 with $\|T_\psi\| \leq \|\psi\|_{L^\infty(\mathbb{T})}$. (In fact, by a theorem of Brown and Halmos, $\|T_\psi\| = \|\psi\|_{L^\infty(\mathbb{T})}$, but we do not need this.) If $h \in H^\infty$, then T_h is simply the operator of multiplication by h

and its adjoint is $T_{\bar{h}}$. Consequently, if $h, k \in H^\infty$, then $T_{\bar{h}}T_{\bar{k}} = T_{\overline{hk}} = T_{\bar{k}}T_{\bar{h}}$, a useful fact that we shall exploit frequently in what follows.

Definition 2.1. Let $b \in H^\infty$ with $\|b\|_{H^\infty} \leq 1$. The associated *de Branges–Rovnyak space* $\mathcal{H}(b)$ is the image of H^2 under the operator $(I - T_b T_{\bar{b}})^{1/2}$. We define a norm on $\mathcal{H}(b)$ making $(I - T_b T_{\bar{b}})^{1/2}$ a partial isometry from H^2 onto $\mathcal{H}(b)$, namely

$$\|(I - T_b T_{\bar{b}})^{1/2} f\|_{\mathcal{H}(b)} := \|f\|_{H^2} \quad (f \in H^2 \ominus \ker(I - T_b T_{\bar{b}})^{1/2}).$$

This is the definition of $\mathcal{H}(b)$ as given in [6]. The original definition of de Branges and Rovnyak, based on the notion of complementary space, is different but equivalent. An explanation of the equivalence can be found in [6, pp.7–8]. A third approach is to start from the positive kernel

$$k_w^b(z) := \frac{1 - \overline{b(w)}b(z)}{1 - \overline{w}z} \quad (z, w \in \mathbb{D}),$$

and to define $\mathcal{H}(b)$ as the reproducing kernel Hilbert space associated with this kernel.

As mentioned in the introduction, the theory of $\mathcal{H}(b)$ -spaces is pervaded by a fundamental dichotomy, namely whether b is or is not an extreme point of the unit ball of H^∞ . This dichotomy is illustrated by following result.

Theorem 2.2. *Let $b \in H^\infty$ with $\|b\|_{H^\infty} \leq 1$. The following statements are equivalent:*

- (i) b is a non-extreme point of the unit ball of H^∞ ;
- (ii) $\log(1 - |b|^2) \in L^1(\mathbb{T})$;
- (iii) $\mathcal{H}(b)$ contains all functions holomorphic in a neighborhood of $\overline{\mathbb{D}}$;
- (iv) $\mathcal{H}(b)$ contains all polynomials.

Proof. The equivalence between (i) and (ii) is proved in [4, Theorem 7.9]. That (i) implies (iii) is proved in [6, §IV-6]. Clearly (iii) implies (iv).

Finally, that (iv) implies (i) is proved as follows. If (i) fails, then b is extreme, and by [6, §V-1] the only functions in $\mathcal{H}(b)$ having a holomorphic continuation across the whole the unit circle are the rational functions in $\ker(T_{\bar{b}})$. Since $b \not\equiv 0$, it cannot be the case that $\ker(T_{\bar{b}})$ contains all polynomials, therefore neither does $\mathcal{H}(b)$. Hence (iv) fails too. \square

Henceforth we shall simply say that b is ‘extreme’ or ‘non-extreme’, it being understood that this relative to the unit ball of H^∞ .

From the equivalence between (i) and (ii), it follows that, if b is non-extreme, then there is an outer function a such that $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ a.e. on \mathbb{T} (see [6, §IV-1]). The function a is uniquely determined by b . We shall call (b, a) a *pair*. The following result gives a useful characterization of $\mathcal{H}(b)$ in this case.

Theorem 2.3 ([6, §IV-1]). *Let b be non-extreme, let (b, a) be a pair and let $f \in H^2$. Then $f \in \mathcal{H}(b)$ if and only if $T_{\bar{b}}f \in T_{\bar{a}}(H^2)$. In this case, there exists a unique function $f^+ \in H^2$ such that $T_{\bar{b}}f = T_{\bar{a}}f^+$, and*

$$(1) \quad \|f\|_{\mathcal{H}(b)}^2 = \|f\|_{H^2}^2 + \|f^+\|_{H^2}^2.$$

We end this section with an example that was studied in [7]. Let

$$(2) \quad b_0(z) := \frac{\tau z}{1 - \tau^2 z},$$

where $\tau := (\sqrt{5} - 1)/2$. The equivalence between (i) and (ii) in Theorem 2.2 shows that b_0 is non-extreme, and a calculation shows that the function a_0 making (b_0, a_0) a pair is given by

$$a_0(z) = \frac{\tau(1-z)}{1 - \tau^2 z}.$$

It was shown in [7] that b_0 has the special property that $\|f_r\|_{\mathcal{H}(b_0)} \leq \|f\|_{\mathcal{H}(b_0)}$ for all $f \in \mathcal{H}(b_0)$ and all $r < 1$. Using a standard argument of reproducing kernel Hilbert spaces, it is easy to see that this implies that $\lim_{r \rightarrow 1^-} \|f_r - f\|_{\mathcal{H}(b_0)} = 0$. As we shall see in the next section, this property is not shared by general b .

3. DILATION IN $\mathcal{H}(b)$

Our principal goal in this section is to prove the following theorem.

Theorem 3.1. *Let $b := b_0 B^2$, where b_0 is the function given by (2), and B is the Blaschke product with zeros at $w_n := 1 - 8^{-n}$ ($n \geq 1$). Let*

$$f(z) := \sum_{n \geq 1} 4^{-n} / (1 - w_n z) \quad (z \in \mathbb{D}).$$

Then b is non-extreme, $f \in \mathcal{H}(b)$, and we have

$$(3) \quad \lim_{r \rightarrow 1^-} |(f_r)^+(0)| = \infty \quad \text{and} \quad \lim_{r \rightarrow 1^-} \|f_r\|_{\mathcal{H}(b)} = \infty.$$

Notice that, by Theorem 2.3, if b is non-extreme and $f \in \mathcal{H}(b)$, then

$$\|f\|_{\mathcal{H}(b)} \geq \|f^+\|_{H^2} \geq |f^+(0)|.$$

Thus the second conclusion in (3) is actually a consequence of the first. We shall therefore concentrate our attention on the first conclusion.

To simplify the notation in what follows, we shall write $k_w(z) := 1/(1 - \bar{w}z)$, the Cauchy kernel. It is the reproducing kernel for H^2 in the sense that $f(w) = \langle f, k_w \rangle_{H^2}$ for all $f \in H^2$ and $w \in \mathbb{D}$. In particular, $\|k_w\|_{H^2}^2 = \langle k_w, k_w \rangle_{H^2} = k_w(w) = 1/(1 - |w|^2)$. We remark that k_w has the useful property that $T_{\bar{h}}(k_w) = \overline{h(w)}k_w$ for all $h \in H^\infty$. Indeed, given $g \in H^2$, we have

$$\langle g, T_{\bar{h}}(k_w) \rangle_{H^2} = \langle hg, k_w \rangle_{H^2} = h(w)g(w) = h(w)\langle g, k_w \rangle_{H^2} = \langle g, \overline{h(w)}k_w \rangle_{H^2}.$$

The proof of Theorem 3.1 depends on two lemmas. The first lemma provides a class of functions f for which $(f_r)^+(0)$ is readily identifiable.

Lemma 3.2. *Let b be non-extreme, let (b, a) be a pair and let $\phi := b/a$. Let*

$$f := \sum_{n \geq 1} c_n k_{w_n},$$

where $(w_n)_{n \geq 1}$ are zeros of b and $(c_n)_{n \geq 1}$ are scalars with $\sum_n |c_n| (1 - |w_n|)^{-1/2} < \infty$. Then $f \in \mathcal{H}(b)$ and

$$(4) \quad (f_r)^+(0) = \sum_{n \geq 1} c_n \overline{\phi(rw_n)} \quad (0 < r < 1).$$

Proof. The series defining f clearly converges absolutely in H^2 . Also, since $T_{\overline{b}} k_{w_n} = \overline{b(w_n)} k_{w_n} = 0$ for all n , we have $T_{\overline{b}} f = 0$, and consequently $f \in \mathcal{H}(b)$ by Theorem 2.3.

Now fix $r \in (0, 1)$ and consider

$$g := \sum_{n \geq 1} c_n \overline{\phi(rw_n)} k_{w_n}.$$

As $(\phi(rw_n))_{n \geq 1}$ is a bounded sequence, this series also converges absolutely in H^2 , and a simple calculation gives $T_{\overline{b}}(f_r) = T_{\overline{a}}(g_r)$. Thus $f_r \in \mathcal{H}(b)$ and $(f_r)^+ = g_r$. In particular (4) holds. \square

The second lemma is a technical result about Blaschke products.

Lemma 3.3. *Let B be an infinite Blaschke product whose zeros $(w_n)_{n \geq 1}$ lie in $(0, 1)$ and satisfy*

$$(5) \quad 0 < \alpha \leq \frac{1 - w_{n+1}}{1 - w_n} \leq \beta < \frac{1}{2} \quad (n \geq 1).$$

Then there exists a constant $C > 0$ such that

$$|B(rw_n)| \geq C \quad (w_n \leq r \leq w_{n+1}, n \geq 1).$$

Proof. We have

$$|B(rw_n)| = \prod_{k=1}^{\infty} \rho(w_k, rw_n),$$

where ρ is the pseudo-hyperbolic metric on \mathbb{D} , namely $\rho(z, w) := |z - w| / |1 - \overline{w}z|$. The condition (5) implies that $w_{n-1} < w_n$ for all n , and even that $w_{n-1} < w_n^2$. Indeed, we have

$$(6) \quad 1 - w_n^2 \leq 2(1 - w_n) \leq 2\beta(1 - w_{n-1}) < 1 - w_{n-1}.$$

It follows that, if $r \in [w_n, w_{n+1}]$, then $rw_n \in [w_n^2, w_n w_{n+1}] \subset (w_{n-1}, w_n)$, and consequently

(7)

$$|B(rw_n)| \geq \left(\prod_{k=1}^{n-2} \rho(w_k, w_{n-1}) \right) \times \rho(w_{n-1}, w_n^2) \times \rho(w_n, w_n w_{n+1}) \times \left(\prod_{k=n+1}^{\infty} \rho(w_k, w_n) \right).$$

Thus the lemma will be proved if we can show that each of the four terms on the right-hand side of (7) is bounded below by a positive constant independent of n .

By [4, Theorem 9.2], the condition (5) implies that the sequence (w_n) is uniformly separated, in other words, that there exists a constant $C' > 0$ such that

$$\prod_{k \neq j} \rho(w_k, w_j) \geq C' \quad (j \geq 1).$$

Applying this with $j = n - 1$ and $j = n$ takes care of the first and fourth terms in (7).

For the second term in (7), note that (6) gives $w_n^2 - w_{n-1} \geq (1 - 2\beta)(1 - w_{n-1})$, and clearly also $1 - w_n^2 w_{n-1} \leq 1 - w_{n-1}^2 \leq 2(1 - w_{n-1})$, whence

$$\rho(w_{n-1}, w_n^2) = \frac{w_n^2 - w_{n-1}}{1 - w_n^2 w_{n-1}} \geq \frac{1 - 2\beta}{2}.$$

Finally, for the third term in (7), we observe that $w_n - w_n w_{n+1} \geq w_1(1 - w_{n+1})$ and also $1 - w_n^2 w_{n+1} = (1 - w_n) + (w_n - w_n^2) + (w_n^2 - w_n^2 w_{n+1}) \leq 3(1 - w_n)$, whence

$$\rho(w_n, w_n w_{n+1}) = \frac{w_n - w_n w_{n+1}}{1 - w_n^2 w_{n+1}} \geq \frac{w_1}{3} \frac{1 - w_{n+1}}{1 - w_n} \geq \frac{w_1}{3} \alpha. \quad \square$$

Proof of Theorem 3.1. As remarked in §2, the function b_0 is non-extreme and the function a_0 making (b_0, a_0) a pair satisfies $\phi_0 := b_0/a_0 = z/(1 - z)$. As b and b_0 have the same outer factors, it follows that b is non-extreme and the function a making (b, a) a pair is just a_0 . Hence $\phi := b/a = B^2 b_0/a_0 = B^2 \phi_0$.

By Lemma 3.2, we have $f \in \mathcal{H}(b)$. The lemma also gives that

$$(f_r)^+(0) = \sum_{n \geq 1} 4^{-n} \overline{\phi(rw_n)} = \sum_{n \geq 1} 4^{-n} \overline{B(rw_n)}^2 \frac{rw_n}{1 - rw_n}.$$

As the terms in this series are non-negative, each one of them provides a lower bound for the sum. Given $r \in [w_1, 1)$, we choose n so that $w_n \leq r \leq w_{n+1}$. By Lemma 3.3 we have $|B(rw_n)| \geq C > 0$, where C is a constant independent of r and n . Thus

$$(f_r)^+(0) \geq 4^{-n} C^2 \frac{rw_n}{1 - rw_n} \geq 4^{-n} C^2 \frac{w_n^2}{1 - w_n^2} \asymp 2^n \asymp (1 - r)^{-1/3}.$$

In particular $(f_r)^+(0) \rightarrow \infty$ as $r \rightarrow 1^-$, as claimed. Finally, as remarked earlier, this implies that $\|f_r\|_{\mathcal{H}(b)} \rightarrow \infty$ as $r \rightarrow 1^-$. \square

We now present some consequences of this result.

Corollary 3.4. *Let b, f be as in Theorem 3.1. Then the Taylor partial sums $s_n(f)$ of f and their Cesàro means $\sigma_n(f)$ satisfy*

$$\limsup_{n \rightarrow \infty} \|s_n(f)\|_{\mathcal{H}(b)} = \infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} \|\sigma_n(f)\|_{\mathcal{H}(b)} = \infty.$$

Proof. Let us write $f(z) = \sum_{k=0}^{\infty} a_k z^k$. For each $z \in \mathbb{D}$, each $r \in (0, 1)$ and each $N \geq 0$,

$$\sum_{n=0}^N (1-r)r^n \sum_{k=0}^n a_k z^k = \sum_{k=0}^N a_k z^k \sum_{n=k}^N (1-r)r^n = \sum_{k=0}^N a_k z^k (r^k - r^{N+1}).$$

Letting $N \rightarrow \infty$, we deduce that $\sum_{n \geq 0} (1-r)r^n s_n(f) = f_r$, the convergence of the series being pointwise in \mathbb{D} . Now, if $\|s_n(f)\|_{\mathcal{H}(b)} \leq M$ for all n , then the series also converges in the norm of $\mathcal{H}(b)$, and we have

$$\|f_r\|_{\mathcal{H}(b)} = \left\| \sum_{n \geq 0} (1-r)r^n s_n(f) \right\|_{\mathcal{H}(b)} \leq \sum_{n \geq 0} (1-r)r^n M = M.$$

This contradicts the fact that $\|f_r\|_{\mathcal{H}(b)} \rightarrow \infty$ as $r \rightarrow 1^-$. We conclude that the sequence of norms $\|s_n(f)\|_{\mathcal{H}(b)}$ is unbounded.

The argument for $\sigma_n(f)$ is similar, this time using the elementary identity $f_r = \sum_{n \geq 0} (n+1)(1-r)^2 r^n \sigma_n(f)$. \square

Let b be non-extreme, let (b, a) be a pair and let $\phi := b/a$, say $\phi(z) = \sum_{j \geq 0} \widehat{\phi}(j) z^j$. It was shown in [1, Theorem 4.1] that, if f is holomorphic in a neighborhood of $\overline{\mathbb{D}}$, say $f(z) = \sum_{k \geq 0} \widehat{f}(k) z^k$, then the series $\sum_{j \geq 0} \widehat{f}(j+k) \overline{\widehat{\phi}(j)}$ converges absolutely for each k , and

$$(8) \quad \|f\|_{\mathcal{H}(b)}^2 = \sum_{k \geq 0} |\widehat{f}(k)|^2 + \sum_{k \geq 0} \left| \sum_{j \geq 0} \widehat{f}(j+k) \overline{\widehat{\phi}(j)} \right|^2.$$

It was left open whether the same formula holds for all $f \in \mathcal{H}(b)$. Using Theorem 3.1, we can now show that it does not.

Corollary 3.5. *Let b, f be as in Theorem 3.1. Then $\sum_{j \geq 0} \widehat{f}(j) \overline{\widehat{\phi}(j)}$ diverges.*

Proof. For $r \in (0, 1)$, the dilated function f_r is holomorphic in a neighborhood of $\overline{\mathbb{D}}$, and the argument in [1] that establishes the formula (8) shows that

$$(f_r)^+(0) = \sum_{j \geq 0} r^j \widehat{f}(j) \overline{\widehat{\phi}(j)}.$$

If b, f are as in Theorem 3.1, then $(f_r)^+(0) \rightarrow \infty$ as $r \rightarrow 1^-$, in other words,

$$\lim_{r \rightarrow 1^-} \sum_{j \geq 0} r^j \widehat{f}(j) \overline{\widehat{\phi}(j)} = \infty.$$

By Abel's theorem, it follows that the series $\sum_{j \geq 0} \widehat{f}(j) \overline{\widehat{\phi}(j)}$ diverges. \square

In Theorem 3.1, we chose b_0 so as to have a simple concrete example. With a more astute choice, we can prove more, obtaining examples where $\|f_r\|_{\mathcal{H}(b)}$ grows 'fast'. There is a limit on how fast it can grow: it was shown in [1, Theorem 5.2] that, if b is non-extreme and $f \in \mathcal{H}(b)$, then $\log^+ \|f_r\|_{\mathcal{H}(b)} = o((1-r)^{-1})$ as $r \rightarrow 1^-$. We now prove that this estimate is sharp.

Theorem 3.6. *Let $\gamma : (0, 1) \rightarrow (1, \infty)$ be a function such that $\log \gamma(r) = o((1-r)^{-1})$. Then there exist b non-extreme and $f \in \mathcal{H}(b)$ such that $\|f_r\|_{\mathcal{H}(b)} \geq \gamma(r)$ for all r in some interval $(r_0, 1)$.*

Proof. Let ϕ_1 be any function in the Smirnov class N^+ that is positive and increasing on $(0, 1)$. To say that $\phi_1 \in N^+$ means that we can write $\phi_1 = b_1/a_1$, where $a_1, b_1 \in H^\infty$ and a_1 is outer. Multiplying a_1 and b_1 by an appropriately chosen outer function, we may further ensure that $|a|^2 + |b|^2 = 1$ a.e. on \mathbb{T} and that $a_1(0) > 0$, in other words, that (b_1, a_1) is a pair. Repeating the proof of Theorem 3.1 with b_0 replaced by b_1 (but with the same B), we obtain $f \in \mathcal{H}(b)$ such that

$$(f_r)^+(0) \geq C^2(1-r)^{2/3} \phi_1(16r-15) \quad (w_1 < r < 1).$$

Since $\log \gamma(r) = o((1-r)^{-1})$, it is possible to choose ϕ_1 so that right-hand side exceeds $\gamma(r)$ for all r sufficiently close to 1. For these r , we therefore have

$$\|f_r\|_{\mathcal{H}(b)} \geq (f_r)^+(0) \geq \gamma(r). \quad \square$$

4. TOEPLITZ APPROXIMATION IN $\mathcal{H}(b)$

Our goal in this section is to establish the following approximation theorem for Toeplitz operators on $\mathcal{H}(b)$. It is valid for all b , both extreme and non-extreme.

Theorem 4.1. *Let $(h_n)_{n \geq 1} \subset H^\infty$ be such that $\|h_n\|_{H^\infty} \leq 1$ and $\lim_{n \rightarrow \infty} h_n(0) = 1$. Then, given b in the unit ball of H^∞ and $f \in \mathcal{H}(b)$, we have $T_{\overline{h_n}} f \in \mathcal{H}(b)$ for all n and*

$$\lim_{n \rightarrow \infty} \|T_{\overline{h_n}} f - f\|_{\mathcal{H}(b)} = 0.$$

The proof of this theorem requires a little more background on de Branges–Rovnyak spaces, which we now briefly summarize.

Let $b \in H^\infty$ with $\|b\|_{H^\infty} \leq 1$. We define the space $\mathcal{H}(\bar{b})$ in the same way as $\mathcal{H}(b)$, but with the roles of b and \bar{b} interchanged. Thus $\mathcal{H}(\bar{b})$ is the image of H^2 under the operator $(I - T_{\bar{b}}T_b)^{1/2}$, with norm defined by

$$\|(I - T_{\bar{b}}T_b)^{1/2}f\|_{\mathcal{H}(\bar{b})} := \|f\|_{H^2} \quad (f \in H^2 \ominus \ker(I - T_{\bar{b}}T_b)^{1/2}).$$

The spaces $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are related through the following theorem.

Theorem 4.2 ([6, §II-4]). *Let b be an element of the unit ball of H^∞ and let $f \in H^2$. Then $f \in \mathcal{H}(b)$ if and only if $T_{\bar{b}}f \in \mathcal{H}(\bar{b})$, and in this case*

$$\|f\|_{\mathcal{H}(b)}^2 = \|f\|_{H^2}^2 + \|T_{\bar{b}}f\|_{\mathcal{H}(\bar{b})}^2.$$

The advantage of $\mathcal{H}(\bar{b})$ over $\mathcal{H}(b)$, for our purposes at least, is that it has another description making it a little more amenable.

Theorem 4.3 ([6, §III-2]). *Let $b \in H^\infty$ with $\|b\|_{H^\infty} \leq 1$. Let $\rho := 1 - |b|^2$ on \mathbb{T} , let $H^2(\rho)$ be the closure of the polynomials in $L^2(\mathbb{T}, \rho d\theta/2\pi)$ and let $J_\rho : H^2 \rightarrow H^2(\rho)$ be the natural inclusion. Then J_ρ^* is an isometry of $H^2(\rho)$ onto $\mathcal{H}(\bar{b})$.*

Note that, in this last result, if b is an inner function, then $\rho \equiv 0$ and the whole situation is degenerate. However this causes no problem since, if b is inner, then we clearly have $\mathcal{H}(\bar{b}) = \{0\}$ anyway.

Using this result, we can prove a version of Theorem 4.1 for $\mathcal{H}(\bar{b})$.

Theorem 4.4. *Let $(h_n)_{n \geq 1} \subset H^\infty$ be such that $\|h_n\|_{H^\infty} \leq 1$ and $\lim_{n \rightarrow \infty} h_n(0) = 1$. Then, given b in the unit ball of H^∞ and $f \in \mathcal{H}(\bar{b})$, we have $T_{\bar{h}_n}f \in \mathcal{H}(\bar{b})$ for all n and*

$$\lim_{n \rightarrow \infty} \|T_{\bar{h}_n}f - f\|_{\mathcal{H}(\bar{b})} = 0.$$

Proof. Let $\rho := 1 - |b|^2$ on \mathbb{T} , and define $H^2(\rho)$ and J_ρ as in the preceding theorem. Given $h \in H^\infty$, let $M_h : H^2(\rho) \rightarrow H^2(\rho)$ be the operator of multiplication by h , namely $M_h g := hg$ ($g \in H^2(\rho)$). Note that $M_h J_\rho = J_\rho T_h$, so, taking adjoints, we have

$$(9) \quad J_\rho^* M_h^* = T_{\bar{h}} J_\rho^*.$$

Now let $f \in \mathcal{H}(\bar{b})$. By Theorem 4.3, there exists $g \in H^2(\rho)$ such that $f = J_\rho^* g$. Using (9), we have $T_{\bar{h}_n} f = T_{\bar{h}_n} J_\rho^* g = J_\rho^* M_{h_n}^* g$. As J_ρ^* is an isometry of $H^2(\rho)$ onto $\mathcal{H}(\bar{b})$, it follows that $T_{\bar{h}_n} f \in \mathcal{H}(\bar{b})$ and that

$$\|T_{\bar{h}_n} f - f\|_{\mathcal{H}(\bar{b})} = \|M_{h_n}^* g - g\|_{H^2(\rho)}.$$

It therefore remains to show that $\lim_{n \rightarrow \infty} \|M_{h_n}^* g - g\|_{H^2(\rho)} = 0$ for all $g \in H^2(\rho)$. It suffices to prove this when $g \in H^\infty$, because H^∞ is dense in $H^2(\rho)$ and the operators $M_{h_n}^*$ are uniformly bounded in norm (by 1). Now, given $g \in H^\infty$, we have

$$\begin{aligned} \|M_{h_n}^* g - g\|_{H^2(\rho)}^2 &= \|M_{h_n}^* g\|_{H^2(\rho)}^2 + \|g\|_{H^2(\rho)}^2 - 2 \operatorname{Re} \langle M_{h_n}^* g, g \rangle_{H^2(\rho)} \\ &\leq 2 \|g\|_{H^2(\rho)}^2 - 2 \operatorname{Re} \langle g, h_n g \rangle_{H^2(\rho)} \\ &= 2 \int_0^{2\pi} |g(e^{i\theta})|^2 (1 - \operatorname{Re} h_n(e^{i\theta})) \rho(e^{i\theta}) \frac{d\theta}{2\pi} \\ &\leq 2 \|g\|_{H^\infty}^2 \int_0^{2\pi} (1 - \operatorname{Re} h_n(e^{i\theta})) \frac{d\theta}{2\pi} \\ &= 2 \|g\|_{H^\infty}^2 (1 - \operatorname{Re} h_n(0)). \end{aligned}$$

As the right-hand side tends to zero, the proof is complete. \square

Finally, we deduce the corresponding result for $\mathcal{H}(b)$.

Proof of Theorem 4.1. Let $f \in \mathcal{H}(b)$. By Theorem 4.2 we have $T_{\bar{b}} f \in \mathcal{H}(\bar{b})$. For each n , we have $T_{\bar{b}} T_{h_n} f = T_{h_n} T_{\bar{b}} f \in \mathcal{H}(\bar{b})$, and hence $T_{h_n} f \in \mathcal{H}(b)$. Also, by Theorem 4.2 again,

$$\|T_{h_n} f - f\|_{\mathcal{H}(b)}^2 = \|T_{h_n} f - f\|_{H^2}^2 + \|T_{h_n} T_{\bar{b}} f - T_{\bar{b}} f\|_{\mathcal{H}(\bar{b})}^2.$$

The second term on the right-hand side tends to zero by Theorem 4.4. The first term can be written as $\|T_{h_n} f - f\|_{\mathcal{H}(\bar{b}_0)}^2$ where $b_0 \equiv 0$, so it too tends to zero by the same theorem. \square

Remark. Let $h \in H^\infty$ with $\|h\|_{H^\infty} \leq 1$. An inspection of the proof of Theorems 4.1 and 4.4 shows that, if $f \in \mathcal{H}(b)$, with say $T_{\bar{b}} f = J_\rho^* g$ where $g \in H^2(\rho)$, then, for all $\tilde{f}, \tilde{g} \in H^\infty$ we have

$$\|T_{\bar{h}} f - f\|_{\mathcal{H}(b)}^2 \leq 4 \|f - \tilde{f}\|_{H^2}^2 + 4 \|g - \tilde{g}\|_{H^2(\rho)}^2 + 2(\|\tilde{f}\|_{H^\infty}^2 + \|\tilde{g}\|_{H^\infty}^2)(1 - \operatorname{Re} h(0)).$$

In principle, this can be used to estimate the rate of convergence in Theorem 4.1. However, this estimate depends on how rapidly f and g can be approximated by bounded functions, and in practice this may be difficult to determine in the case of g , since g is not given explicitly.

5. POLYNOMIAL APPROXIMATION IN $\mathcal{H}(b)$

In this final section we present a constructive proof that polynomials are dense in $\mathcal{H}(b)$ when b is non-extreme.

Theorem 5.1. *Let b be non-extreme and let (b, a) be a pair. For $n \geq 1$, let h_n be the outer function satisfying $h_n(0) > 0$ and $|h_n| = \min\{1, n|a|\}$ a.e. on \mathbb{T} , explicitly:*

$$h_n(z) := \exp\left(\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(\min\{1, n|a(e^{i\theta})|\}) \frac{d\theta}{2\pi}\right) \quad (z \in \mathbb{D}).$$

Given $f \in \mathcal{H}(b)$, let (p_n) be a sequence of polynomials such that $\|f - p_n\|_{H^2} < 1/n^2$. Then $T_{h_n}^- p_n$ is a polynomial for each n , and

$$\lim_{n \rightarrow \infty} \|T_{h_n}^- p_n - f\|_{\mathcal{H}(b)} = 0.$$

There are many possible choices for the polynomials (p_n) . For example, they may be taken to be appropriately chosen Taylor partial sums of f .

The proof of Theorem 5.1 is based upon the Toeplitz approximation theorem of the previous section and the following lemma.

Lemma 5.2. *Let b be non-extreme and let (b, a) be a pair. If $h \in aH^\infty$, then $T_{\bar{h}}^-$ is a bounded operator from H^2 into $\mathcal{H}(b)$ and $\|T_{\bar{h}}^-\|_{H^2 \rightarrow \mathcal{H}(b)} \leq \|h/a\|_{H^\infty}$.*

Proof. Let $h = ah_0$, where $h_0 \in H^\infty$. For $f \in H^2$, we have

$$T_{\bar{h}}^- T_{\bar{h}}^- f = T_{\bar{h}}^- T_{\bar{a}}^- T_{\bar{h}_0}^- f = T_{\bar{a}}^- T_{\bar{h}_0}^- T_{\bar{b}}^- f,$$

so, by Theorem 2.3, $T_{\bar{h}}^- f \in \mathcal{H}(b)$ and $(T_{\bar{h}}^- f)^+ = T_{\bar{h}_0}^- T_{\bar{b}}^- f$. Consequently

$$\begin{aligned} \|T_{\bar{h}}^- f\|_{\mathcal{H}(b)}^2 &= \|T_{\bar{h}}^- f\|_{H^2}^2 + \|T_{\bar{h}_0}^- T_{\bar{b}}^- f\|_{H^2}^2 \\ &= \|T_{\bar{h}_0}^- T_{\bar{a}}^- f\|_{H^2}^2 + \|T_{\bar{h}_0}^- T_{\bar{b}}^- f\|_{H^2}^2 \\ &\leq \|h_0\|_{H^\infty}^2 (\|T_{\bar{a}}^- f\|_{H^2}^2 + \|T_{\bar{b}}^- f\|_{H^2}^2) \\ &\leq \|h_0\|_{H^\infty}^2 \|f\|_{H^2}^2, \end{aligned}$$

the last inequality coming from the fact that $|a|^2 + |b|^2 = 1$ a.e. on \mathbb{T} . \square

Proof of Theorem 5.1. For $k > \deg p_n$, we have $\langle T_{\bar{h}_n}^- p_n, z^k \rangle_{H^2} = \langle p_n, z^k h_n \rangle_{H^2} = 0$. It follows that $T_{\bar{h}_n}^- p_n$ is a polynomial (of degree no greater than that of p_n). Also

$$\|T_{\bar{h}_n}^- p_n - f\|_{\mathcal{H}(b)} \leq \|T_{\bar{h}_n}^- (p_n - f)\|_{\mathcal{H}(b)} + \|T_{\bar{h}_n}^- f - f\|_{\mathcal{H}(b)}.$$

Now, $|h_n| \leq n|a|$ a.e. on \mathbb{T} , and as both h_n and a are outer functions, it follows that $h_n/a \in H^\infty$ with $\|h_n/a\|_{H^\infty} \leq n$. Hence, by Lemma 5.2 and the choice of p_n , we have

$$\|T_{\bar{h}_n}^- (p_n - f)\|_{\mathcal{H}(b)} \leq \|h_n/a\|_{H^\infty} \|p_n - f\|_{H^2} \leq n(1/n^2),$$

and so $\|T_{\bar{h}_n}(p_n - f)\|_{\mathcal{H}(b)} \rightarrow 0$. Also $|h_n| \leq 1$ a.e. on \mathbb{T} , so again, since h_n is an outer function, we have $\|h_n\|_{H^\infty} \leq 1$. Further, by the dominated convergence theorem

$$h_n(0) = \exp\left(\int_0^{2\pi} \log(\min\{1, n|a(e^{i\theta})|\}) \frac{d\theta}{2\pi}\right) \rightarrow \exp(0) = 1 \quad \text{as } n \rightarrow \infty,$$

so by Theorem 4.1 we have $\|T_{\bar{h}_n}f - f\|_{\mathcal{H}(b)} \rightarrow 0$. Combining these facts, we obtain the desired conclusion that $\|T_{\bar{h}_n}p_n - f\|_{\mathcal{H}(b)} \rightarrow 0$. □

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