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A COMPACTNESS RESULT FOR AN ELLIPTIC EQUATION IN DIMENSION 2.

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ABSTRACT. We give a blow-up analysis for the solutions of an elliptic equation under some conditions on the prescribed curvature. Also, we derive a compactness result for this elliptic equation under a Lipschitz condition.

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1. INTRODUCTION AND MAIN RESULTS

We consider the following equation:

\[ (P_\epsilon) \begin{cases} -\Delta u - \epsilon(x_1\partial_1 u + x_2\partial_2 u) = Ve^u \text{ in } \Omega \\ u = 0 \text{ on } \partial \Omega. \end{cases} \]

Here, we assume that:

- \( \Omega \) starshaped,

and,

- \( u \in W^{1,1}_0(\Omega), \ e^u \in L^1(\Omega), \ 0 \leq V \leq b, \ \epsilon \geq 0. \)

When \( \epsilon = 0 \), the previous equation was studied by many authors, with or without the boundary condition, also for Riemann surfaces, see [1,18], we can find some existence and compactness results.

Among other results, we can see in [4] the following important Theorem,

**Theorem.** (Brezis-Merle [4]). If \((u_i)\) and \((V_i)\) are two sequences of functions relatively to the problem \((P_0)\) with, \( \epsilon = 0 \) and \( 0 < a \leq V_i \leq b < +\infty \), then, for all compact set \( K \) of \( \Omega \),

\[ \sup_K u_i \leq c = c(a, b, m, K, \Omega). \]

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We can find in [4] an interior estimate if we assume \( a = 0 \), but we need an assumption on the integral of \( e^{u_i} \).

If we assume \( V \) with more regularity, we can have another type of estimates, a \( \sup + \inf \) type inequalities. It was proved by Shafrir see [15], that, if \((u_i), (V_i)\) are two sequences of functions solutions of the previous equation without assumption on the boundary and, \( 0 < a \leq V_i \leq b < +\infty \), then we have the following interior estimate:

\[
C \left( \frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c(a, b, K, \Omega).
\]

We can see in [7] an explicit value of \( C \left( \frac{a}{b} \right) = \sqrt{\frac{a}{b}} \). In his proof, Shafrir has used the Stokes formula and an isoperimetric inequality, see [2]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose \((V_i)\) uniformly Lipschitzian with \( A \) the Lipschitz constant, then, \( C(a/b) = 1 \) and \( c = c(a, b, A, K, \Omega) \), see Brézis-Shafrir [5]. This result was extended for Hölderian sequences \((V_i)\) by Chen-Lin, see [7]. Also, we can see in [12], an extension of the Brezis-Shafrir result to compact Riemann surface without boundary. We can see in [13] explicit form, \( (\delta \pi m, m \in \mathbb{N}^* \text{ exactly}) \), for the numbers in front of the Dirac masses when the solutions blow-up. Here, the notion of isolated blow-up point is used. Also, we can see in [8] and [18] refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

Here we give the behavior of the blow-up points on the boundary and a new proof of Brezis-Merle conjecture with Lipschitz condition.

The Brezis-Merle Conjecture (see [4]) is:

**Conjecture.** Suppose that \( V_i \rightarrow V \) in \( C^0(\bar{\Omega}) \) with \( 0 \leq V_i \leq b \) for some positive constant \( b \). Also, we consider a sequence of solutions \((u_i)\) of \((P_0)\) relatively to \((V_i)\) such that,

\[
\int_{\Omega} e^{u_i} dx \leq C,
\]

is it possible to have:

\[
\|u_i\|_{L^\infty} \leq C = C(b, \Omega, C)?
\]

Here, we give a blow-up analysis on the boundary when the prescribed curvature are nonegative and bounded and on the other hand, if we add the assumption that these curvature are uniformly Lipschitzian, we have a compactness of the solutions of the problem \((P_\epsilon)\) for \( \epsilon \) small enough. (In particular we can take a sequence of \( \epsilon_i \) tending to 0):

For the behavior of the blow-up points on the boundary, the following condition is enough,
0 \leq V_i \leq b,

The condition $V_i \to V$ in $C^0(\bar{\Omega})$ is not necessary.

But for the compactness of the solutions we add the following condition:

$$||\nabla V_i||_{L^\infty} \leq A.$$ 

We have the following characterization of the behavior of the blow-up points on the boundary.

**Theorem 1.1.** Assume that $\max_{\Omega} u_i \to +\infty$, Where $(u_i)$ are solutions of the problem $(P_{\epsilon_i})$ with:

$$0 \leq V_i \leq b, \quad \text{and} \quad \int_{\Omega} e^{u_i} dx \leq C, \quad \epsilon_i \to 0,$$

then, after passing to a subsequence, there is a function $u$, there is a number $N \in \mathbb{N}$ and $N$ points $x_1, \ldots, x_N \in \partial\Omega$, such that,

$$\partial_{\nu} u_i \to \partial_{\nu} u + \sum_{j=1}^{N} \alpha_j \delta_{x_j}, \quad \alpha_j \geq 4\pi, \quad \text{weakly in the sense of measure } L^1(\partial \Omega).$$

$$u_i \to u \text{ in } C^{1}_{\text{loc}}(\bar{\Omega} - \{x_1, \ldots, x_N\}).$$

In the following theorem, we have a compactness result:

**Theorem 1.2.** Assume that $(u_i)$ are solutions of $(P_{\epsilon_i})$ relatively to $(V_i)$ with the following conditions:

$$||\nabla V_i||_{L^\infty} \leq A \quad \text{and} \quad \int_{\Omega} e^{u_i} \leq C, \quad \epsilon_i \to 0.$$

Then, we have:

$$||u_i||_{L^\infty} \leq c(b, A, C, \Omega),$$

2. **Proof of the Theorems**

**Proof of theorem 1.1:**

First of all, remark that, we have for two positive constants $C_q = C(q, \Omega)$ and $C_1 = C_1(\Omega)$:
\[ \|\nabla u_i\|_{L^q} \leq C_q \|\Delta u_i\|_{L^1} \leq (C'_q + \epsilon C_1 \|\nabla u_i\|_{L^1}), \quad \forall\; i \text{ and } 1 < q < 2. \]

Thus, if \( \epsilon > 0 \) is small enough and by the Holder inequality, we have the following estimate:

\[ \|\nabla u_i\|_{L^q} \leq C''_q, \quad \forall\; i \text{ and } 1 < q < 2. \]

**Step 1: interior estimate**

First of all remark that, if we consider the following equation:

\[
\begin{cases}
-\Delta w_i = -\epsilon_i (x_1 \partial_1 u_i + x_2 \partial_2 u_i) \in L^q, \quad 1 < q < 2 \text{ in } \Omega \\
w_i = 0 \text{ on } \partial \Omega.
\end{cases}
\]

We have by the elliptic estimates that \( w_i \in W^{2,1+\epsilon} \subset L^\infty \), and we can write the equation of the Problem as:

\[
\begin{cases}
-\Delta (u_i - w_i) = \tilde{V}_i e^{u_i - w_i}, \quad \text{in } \Omega \\
u_i - w_i = 0 \text{ on } \partial \Omega.
\end{cases}
\]

with,

\[ 0 \leq \tilde{V}_i = V_i e^{w_i} \leq \tilde{b}, \int_{\Omega} e^{u_i - w_i} \leq \tilde{C}. \]

We apply the Brezis-Merle theorem to \( u_i - w_i \) to have:

\[ u_i - w_i \in L^\infty_{\text{loc}}(\Omega), \]

and, thus:

\[ u_i \in L^\infty_{\text{loc}}(\Omega). \]

**Step 2: boundary estimate**

Now, we have:

\[ \int_{\partial \Omega} \partial_{\nu} u_i d\sigma \leq C, \]

\[ 4 \]
Without loss of generality, we can assume that \( \partial_{\nu} u_i > 0 \). Thus, (using the weak convergence in the space of Radon measures), we have the existence of a positive Radon measure \( \mu \) such that,

\[
\int_{\partial \Omega} \partial_{\nu} u_i \varphi d\sigma \to \mu(\varphi), \quad \forall \ \varphi \in C^0(\partial \Omega).
\]

We take an \( x_0 \in \partial \Omega \) such that, \( \mu(x_0) < 4\pi \). Without loss of generality, we can assume that the following curve, \( B(x_0, \epsilon) \cap \partial \Omega := I_\epsilon \) is an interval. (In this case, it is more simple to construct the following test function \( \eta_\epsilon \)). We choose a function \( \eta_\epsilon \) such that,

\[
\begin{cases}
\eta_\epsilon \equiv 1, \ & \text{on } I_\epsilon, \ 0 < \epsilon < \delta/2, \\
\eta_\epsilon \equiv 0, \ & \text{outside } I_{2\epsilon}, \\
0 \leq \eta_\epsilon \leq 1, \\
\|\nabla \eta_\epsilon\|_{L^\infty(I_{2\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}.
\end{cases}
\]

We take a \( \tilde{\eta}_\epsilon \) such that,

\[
\begin{cases}
-\Delta \tilde{\eta}_\epsilon = 0 \ & \text{in } \Omega \\
\tilde{\eta}_\epsilon = \eta_\epsilon \ & \text{on } \partial \Omega.
\end{cases}
\]

We use the following estimate, see [3, 11, 17],

\[ ||\nabla u_i||_{L^q} \leq C_q, \ \forall \ i \ \text{and} \ 1 < q < 2. \]

We deduce from the last estimate that, \( (u_i) \) converge weakly in \( W^{1,q}_0(\Omega) \), almost everywhere to a function \( u \geq 0 \) and \( \int_{\Omega} e^u < +\infty \) (by Fatou lemma). Also, \( V_i \) weakly converge to a nonnegative function \( V \) in \( L^\infty \). The function \( u \) is in \( W^{1,q}_0(\Omega) \) solution of:

\[
\begin{cases}
-\Delta u = V e^u \in L^1(\Omega) \ & \text{in } \Omega \\
u = 0 \ & \text{on } \partial \Omega.
\end{cases}
\]

According to the corollary 1 of Brezis-Merle result, see [4], we have \( e^{ku} \in L^1(\Omega), k > 1 \). By the elliptic estimates, we have \( u \in C^1(\Omega) \).

We can write,

\[
-\Delta((u_i - u)\tilde{\eta}_\epsilon) = (V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon - 2 < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon >. \quad (1)
\]

We use the interior estimate of Brezis-Merle, see [4].
**Step 1:** Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between $\tilde{\eta} \epsilon$ and $u$, we obtain,

$$\int_{\Omega} V e^{u} \tilde{\eta} d x = \int_{\partial \Omega} \partial_{\nu} u \eta d \sigma \leq 4 \epsilon \|\partial_{\nu} u\|_{L^\infty} = C \epsilon$$  \hspace{1cm} (2)

We have,

$$\begin{cases} 
-\Delta u_i = V_i e^{u_i} & \text{in } \Omega \\
 u_i = 0 & \text{on } \partial \Omega.
\end{cases}$$  \hspace{1cm} (3)

From (2) and (3) we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

$$\int_{\Omega} \left| (V_i e^{u_i} - V e^{u}) \tilde{\eta} \right| d x \leq 4 \pi - \epsilon_0 + C \epsilon$$  \hspace{1cm} (4)

**Step 2.1:** Estimate of integral of the second term of the right hand side of (1).

Let $\Sigma_\epsilon = \{ x \in \Omega, d(x, \partial \Omega) = \epsilon^2 \}$ and $\Omega_{\epsilon^2} = \{ x \in \Omega, d(x, \partial \Omega) \geq \epsilon^2 \}, \epsilon > 0$. Then, for $\epsilon$ small enough, $\Sigma_\epsilon$ is hypersurface.

The measure of $\Omega - \Omega_{\epsilon^2}$ is $k_2 \epsilon^2 \leq \mu_L(\Omega - \Omega_{\epsilon^2}) \leq k_1 \epsilon^2$.

**Remark:** for the unit ball $B(0, 1)$, our new manifold is $B(0, 1 - \epsilon^2)$.

We write,

$$\int_{\Omega} \left| \nabla (u_i - u) \right| \nabla \tilde{\eta} > |d x| = \int_{\Omega_{\epsilon^2}} \left| \nabla (u_i - u) \right| \nabla \tilde{\eta} > |d x| + \int_{\Omega - \Omega_{\epsilon^2}} \left| \nabla (u_i - u) \right| \nabla \tilde{\eta} > |d x|.$$  \hspace{1cm} (5)

**Step 2.1.1:** Estimate of $\int_{\Omega - \Omega_{\epsilon^2}} \left| \nabla (u_i - u) \right| \nabla \tilde{\eta} > |d x|$.

First, we know from the elliptic estimates that $\|\nabla \tilde{\eta}\|_{L^\infty} \leq C_1/\epsilon$, $C_1$ depends on $\Omega$.
We know that \((|\nabla u_i|)_i\) is bounded in \(L^q, 1 < q < 2\), we can extract from this sequence a subsequence which converge weakly to \(h \in L^q\). But, we know that we have locally the uniform convergence to \(|\nabla u|\) (by Brezis-Merle theorem), then, \(h = |\nabla u|\) a.e. Let \(q'\) be the conjugate of \(q\).

We have, \(\forall f \in L^{q'}(\Omega)\)

\[
\int_{\Omega} |\nabla u_i| f dx \to \int_{\Omega} |\nabla u| f dx
\]

If we take \(f = 1_{\Omega - \Omega_{\epsilon, 2}}\), we have:

for \(\epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, i \geq i_1, \int_{\Omega - \Omega_{\epsilon, 2}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon, 2}} |\nabla u| + \epsilon^2\).

Then, for \(i \geq i_1(\epsilon)\),

\[
\int_{\Omega - \Omega_{\epsilon, 2}} |\nabla u_i| \leq \text{mes}(\Omega - \Omega_{\epsilon, 2}) ||\nabla u||_{L^\infty} + \epsilon^2 = \epsilon^2 (k_1 ||\nabla u||_{L^\infty} + 1).
\]

Thus, we obtain,

\[
\int_{\Omega - \Omega_{\epsilon, 2}} |<\nabla (u_i - u)|\nabla \eta_\epsilon| dx \leq \epsilon C_1 (2k_1 ||\nabla u||_{L^\infty} + 1) \quad (6)
\]

The constant \(C_1\) does not depend on \(\epsilon\) but on \(\Omega\).

\textbf{Step 2.1.2:} Estimate of \(\int_{\Omega_{\epsilon, 2}} |<\nabla (u_i - u)|\nabla \eta_\epsilon| dx \).

We know that, \(\Omega_{\epsilon} \subset \subset \Omega\), and (because of Brezis-Merle’s interior estimates) \(u_i \to u\) in \(C^1(\Omega_{\epsilon, 2})\). We have,

\[
||\nabla (u_i - u)||_{L^\infty(\Omega_{\epsilon, 2})} \leq \epsilon^2, \text{ for } i \geq i_3 = i_3(\epsilon).
\]

We write,

\[
\int_{\Omega_{\epsilon, 2}} |<\nabla (u_i - u)|\nabla \eta_\epsilon| dx \leq ||\nabla (u_i - u)||_{L^\infty(\Omega_{\epsilon, 2})} ||\nabla \eta_\epsilon||_{L^\infty} \leq C_1 \epsilon \text{ for } i \geq i_3,
\]

For \(\epsilon > 0\), we have for \(i \in \mathbb{N}, i \geq \max\{i_1, i_2, i_3\} \).
\[ \int_{\Omega} \left| \nabla (u_i - u) \right| \nabla \tilde{\eta} \, dx \leq \epsilon C_1 (2k_1 \| \nabla u \|_{L^\infty} + 2) \quad (7) \]

From (4) and (7), we have, for \( \epsilon > 0 \), there is \( i_3 = i_3(\epsilon) \in \mathbb{N}, i_3 = \max \{ i_0, i_1, i_2 \} \) such that,

\[ \int_{\Omega} |\Delta [(u_i - u) \tilde{\eta}]| \, dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1 (2k_1 \| \nabla u \|_{L^\infty} + 2 + C) \quad (8) \]

We choose \( \epsilon > 0 \) small enough to have a good estimate of (1). Indeed, we have:

\[
\begin{cases}
-\Delta [(u_i - u) \tilde{\eta}] = g_{i, \epsilon} & \text{in } \Omega, \\
(u_i - u) \tilde{\eta} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

with \( \| g_{i, \epsilon} \|_{L^1(\Omega)} \leq 4\pi - \epsilon_0 \).

We can use Theorem 1 of [4] to conclude that there is \( q > 1 \) such that:

\[ \int_{V_i(x_0)} e^{q(u_i - u)} \, dx \leq \int_{\Omega} e^{q(u_i - u)\tilde{\eta}} \, dx \leq C(\epsilon, \Omega). \]

where, \( V_i(x_0) \) is a neigbourhood of \( x_0 \) in \( \bar{\Omega} \).

Thus, for each \( x_0 \in \partial \Omega - \{ \bar{x}_1, \ldots, \bar{x}_m \} \) there is \( \epsilon_{x_0} > 0, q_{x_0} > 1 \) such that:

\[ \int_{B(x_{00}, \epsilon_{x_0})} e^{q_{x_0}u_i} \, dx \leq C, \quad \forall i. \]

Now, we consider a cutoff function \( \eta \in C^\infty(\mathbb{R}^2) \) such that:

\[ \eta \equiv 1 \text{ on } B(x_0, \epsilon_{x_0}/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3). \]

We write,

\[ -\Delta (u_i \eta) = V_i e^{u_i \eta} - 2 < \nabla u_i \nabla \eta > -u_i \Delta \eta. \]

By the elliptic estimates, \( (u_i \eta)_i \) is uniformly bounded in \( W^{2,q_1}(\Omega) \) and also, in \( C^1(\Omega) \).

Finally, we have, for some \( \epsilon > 0 \) small enough,
\[ \|u_i\|_{C^1,B(x_0,\epsilon)} \leq c_3 \ \forall \ i. \]

We have proved that, there is a finite number of points \( \bar{x}_1, \ldots, \bar{x}_m \) such that the sequence \( (u_i)_i \) is locally uniformly bounded in \( \Omega - \{ \bar{x}_1, \ldots, \bar{x}_m \} \).

**Proof of theorem 1.2:**

The Pohozaev identity gives:

\[
\int_{\partial\Omega} x|\nu| (\partial_{\nu} u_i)^2 d\sigma + \epsilon \int_{\Omega} (x|\nabla u_i|)^2 dx + \epsilon \int_{\partial\Omega} x|\nu| V_i e^{u_i} d\sigma = \int_{\Omega} (x|\nabla V_i| + 2V_i) e^{u_i} dx
\]

We use the boundary condition and the fact that \( \Omega \) is starshaped and the fact that \( \epsilon > 0 \) to have that:

\[
\int_{\partial\Omega} (\partial_{\nu} u_i)^2 dx \leq c_0(b, A, C, \Omega).
\]

Thus we can use the weak convergence in \( L^2(\partial\Omega) \) to have a subsequence \( \partial_{\nu} u_i \), such that:

\[
\int_{\partial\Omega} \partial_{\nu} u_i \phi dx \to \int_{\partial\Omega} \partial_{\nu} u \phi dx, \ \forall \ \phi \in L^2(\partial\Omega),
\]

Thus, \( \alpha_j = 0, j = 1, \ldots, N \) and \( (u_i) \) is uniformly bounded.

**References**


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