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DETERMINATION OF TIME DEPENDENT FACTORS OF COEFFICIENTS IN FRACTIONAL DIFFUSION EQUATIONS

KENICHI FUJISHIRO[1)] AND YAVAR KIAN[2)]

Abstract. In the present paper, we consider initial-boundary value problems for partial differential equations with time-fractional derivatives which evolve in $Q = \Omega \times (0,T)$ where Ω is a bounded domain of $\mathbb{R}^d$ and $T > 0$. We study the stability of the inverse problems of determining the time-dependent parameter in a source term or a coefficient of zero-th order term from observations of the solution at a point $x_0 \in \overline{\Omega}$ for all $t \in (0,T)$.

1. Introduction

1.1. **Statement of the problem.** Let Ω be a bounded domain of $\mathbb{R}^d$, $d = 1,2,3$, with $\mathcal{C}^2$ boundary $\partial\Omega$. We set $\Sigma = \partial\Omega \times (0,T)$, $Q = \Omega \times (0,T)$ and we introduce $\mathcal{A}$ the uniformly elliptic differential operator defined by

$$\mathcal{A}u(x,t) := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x,t)\right),$$

where

$$a_{ij} = a_{ji}, \quad 1 \le i,j \le d, \quad \text{and} \quad \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \mu|\xi|^2, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^d,$$

for some $\mu > 0$. We associate to this elliptic operator a Robin boundary condition;

$$\mathcal{B}_\sigma u(x,t) := (1-\sigma(x))u(x,t) + \sigma(x)\partial_{\nu_A}u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T),$$

where

$$\partial_{\nu_A}u(x,t) = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}(x,t)\nu_j(x)$$

and $\nu = (\nu_1, \dots, \nu_d)$ is the outward unit normal vector to $\partial\Omega$. Here σ is a $\mathcal{C}^2$ function on $\partial\Omega$ satisfying

$$0 \le \sigma(x) \le 1, \quad x \in \partial\Omega$$

and the limit case $\sigma \equiv 0$ (resp. $\sigma \equiv 1$) corresponds to the Dirichlet (resp. Neumann) boundary condition. Recall that Robin boundary conditions are the mathematical formulation of the Newton's law of cooling where the heat transfer coefficient σ is utilized. The heat

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transfer coefficient is determined by details of the interface structure (sharpness, geometry) between two media. This law describes quite well the boundary between metals and gas and is good for the convective heat transfer.

We set also $\alpha \in (0,1)$ and we denote by ∂_t^α the Caputo fractional derivative with respect to t given by

$$\partial_t^\alpha g(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{dg}{ds}(s) ds.$$

Then, we introduce the following two initial-boundary value problems (IBVPs in short) for the fractional diffusion equations

$$\begin{cases} \partial_t^\alpha u(x,t) + \mathcal{A}u(x,t) = f(t)R(x,t), & (x,t) \in Q, \\ \mathcal{B}_\sigma u(x,t) = 0, & (x,t) \in \Sigma, \\ u(x,0) = 0, & x \in \Omega \end{cases} \tag{1.1}$$

and

$$\begin{cases} \partial_t^\alpha v(x,t) + \mathcal{A}v(x,t) + f(t)q(x,t)v(x,t) = 0, & (x,t) \in Q, \\ \mathcal{B}_\sigma v(x,t) = 0, & (x,t) \in \Sigma, \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

From now on, problems (1.1) and (1.2) will be respectively denoted by the source term problem and the reaction rate problem. In the present paper, we assume that $\mathcal{A}$, σ, R, v_0 and q are given and we consider the inverse problem of determining the parameter $\{f(t)\}_{t\in(0,T)}$ appearing in the source term problem (1.1) and the reaction rate problem (1.2) from observations at a point $x_0 \in \overline{\Omega}$ for all $t \in (0,T)$.

1.2. **Physical motivations.** Recall that diffusion equations with time fractional derivatives such as (1.1) and (1.2) are proposed as new models describing some sub-diffusive anomalous diffusion phenomena such as diffusion of ions in heterogeneous media or diffusion of fluid flow in inhomogeneous anisotropic porous media. Indeed, motivated by the work of Adams and Gelhar [1] related to models for highly heterogeneous aquifer, Hatano and Hatano [16] proposed a microscopic model of the diffusion of ions in heterogeneous media based on continuous-time random walk (CTRW in short). From the CTRW model, one can derive a fractional diffusion equation as a macroscopic model (see e.g., Metzler and Klafter [26] and Roman and Alemany [31]). We refer also to the work of [10] and the references therein for models of diffusion in porous media as well as other physical phenomena described by time fractional derivatives.

In particular, fractional diffusion equations can be associated to the diffusion of contaminants in a soil. In this context, our inverse problems correspond to the determination of a pollution source in (1.1) and the recovery of a reaction rate of pollutants in (1.2).

1.3. **Known results.** Let us recall that fractional derivatives for both ODEs and PDEs have attracted many attention. For detailed study of fractional calculus for instance we refer to books such as Matignon [25], Miller and Ross [27], Podlubny [29] and Samko, Kilbas and Marichev [35]. Concerning mathematical properties of partial differential equations with time fractional derivatives and related properties, we refer to Agarwal [3], Gejji and Jafari

[14], Gorenflo and Mainardi [15], Luchko [22], Matignon [24] and references therein. For the well-posedness of problems (1.1) and (1.2) with time-independent coefficients one can refer to [34].

Let us mention that the recovery of time-dependent parameters appearing in the diffusion equations (1.1)-(1.2) in the case $\alpha = 1$ has been considered by several authors. The recovery of time-dependent coefficients has been treated in Section 1.5 of Prilepko, Orlovsky and Vasin [30], Cannon and Esteva [7], Choulli and Kian [11] and Saitoh, Tuan and Yamamoto [32, 33]. Note also that using the strategy set by Bukhgeim and Klibanov [6] based on Carleman estimates, Gaitan and Kian [13] proved stable determination of time-dependent coefficients for diffusion equations in the special case of cylindrical domain Ω. The determination of time-dependent factor appearing in the source term has been studied by [7] and [32, 33], who derived respectively a logarithmic type and a Hölder type stability estimate.

In contrast to parabolic equations, few articles addressed inverse problems for (1.1)-(1.2) in the fractional case $0 < \alpha < 1$. In the one dimensional case $d = 1$, [8] proved unique determination of a time-independent coefficient and the fractional order α from Dirichlet boundary measurements. In the multidimensional case, [17] determined the fractional order α from measurements of the solutions at a point for any time. The recovery of a time-dependent factor appearing in the source term has been considered by Sakamoto and Yamamoto who proved in [34, Theorem 4.4] a result of stability for this problem. In the special case $\alpha = 1/2$ and $d = 1$, [9, 38] considered an approach similar to [6] based on Carleman estimates for equation (1.2). Using this approach, [9, 38] proved stability in the recovery of a time-independent coefficient of order zero from a single measurement on a subdomain. Moreover, [28] proved uniqueness in the recovery of a time-independent coefficient of order zero from measurements on a subdomain. In some recent work, [19] proved recovery of some general time-independent coefficients from the Dirichlet-to-Neumann map associated to a problem similar to (1.1) with an inhomogeneous Dirichlet boundary conditions of the form $\lambda(t)g(x)$ where λ is a strictly positive and analytic fixed function.

1.4. **Main results.** In order to state our two main results, let us first introduce some conditions that guaranty existence of sufficiently smooth solutions for the source term problem (1.1) and the reaction rate problem (1.2). For both (1.1)-(1.2), we assume that the coefficients a_{ij} and the parameter $\{f(t)\}_{t\in(0,T)}$ satisfy

$$\begin{cases} a_{ij} \in \mathcal{C}^1(\overline{\Omega}) & \text{if } \sigma \equiv 0, \\ a_{ij} \in \mathcal{C}^2(\overline{\Omega}) & \text{if } \sigma \not\equiv 0, \end{cases}$$

$$f \in L^\infty(0,T). \tag{1.3}$$

Note that the regularity of a_{ij} depends on whether $\sigma \equiv 0$ or not, which is due to condition (1.8) that will be introduced later.

For other given parameters in (1.1), we suppose

$$R \in L^p(0,T;H^2(\Omega)),\ \frac{8}{\alpha} < p \leq \infty \quad \text{and} \quad \mathcal{B}_\sigma R = 0 \quad \text{on } \Sigma. \tag{1.4}$$

Assuming these conditions, we prove in Section 3 that the IBVP (1.1) admits a unique solution $u \in \mathcal{C}([0,T]; H^2(\Omega))$ with $\partial_t^\alpha u \in L^p(0,T; H^s(\Omega))$ for some $s > d/2$. Therefore, using the Sobolev embedding theorem (e.g. [20, Theorem 9.8, chapter1]), for any $x_0 \in \overline{\Omega}$, we see that $u(x_0, \cdot) := t \mapsto \partial_t^\alpha u(x_0, t) \in L^p(0,T)$. Then, we can state our first result of stability in the recovery of the time-dependent factor appearing in the source term problem (1.1).

Theorem 1.1. *Let condition* (1.4) *be fulfilled and, for* $i = 1, 2$*, let* u_i *be the solution of* (1.1) *for* $f = f_i \in L^\infty(0,T)$*. We assume that there exist* $x_0 \in \overline{\Omega}$ *and* $\delta > 0$ *such that*

$$|R(x_0, t)| \geq \delta, \quad a.e.\ t \in (0,T). \tag{1.5}$$

Then, there exists a constant $C > 0$ *depending on* α*,* p*,* T*,* Ω*,* δ*,* $\mathcal{A}$*,* σ *and* $\|R\|_{L^p(0,T;H^2(\Omega))}$ *such that*

$$\|f_1 - f_2\|_{L^p(0,T)} \leq C\|\partial_t^\alpha u_1(x_0, \cdot) - \partial_t^\alpha u_2(x_0, \cdot)\|_{L^p(0,T)}, \tag{1.6}$$

$$\|\partial_t^\alpha u_1(x_0, \cdot) - \partial_t^\alpha u_2(x_0, \cdot)\|_{L^p(0,T)} \leq C\|f_1 - f_2\|_{L^\infty(0,T)}. \tag{1.7}$$

In particular, if we take $p = \infty$ *in* (1.4)*, then*

$$\begin{aligned} C^{-1}\|\partial_t^\alpha u_1(x_0, \cdot) - \partial_t^\alpha u_2(x_0, \cdot)\|_{L^\infty(0,T)} &\leq \|f_1 - f_2\|_{L^\infty(0,T)} \\ &\leq C\|\partial_t^\alpha u_1(x_0, \cdot) - \partial_t^\alpha u_2(x_0, \cdot)\|_{L^\infty(0,T)}. \end{aligned}$$

For the IBVP (1.2), we assume

$$\begin{cases} q \in L^\infty(0,T; H^2(\Omega)) \quad (\text{and } \partial_\nu q = 0 \text{ on } \Sigma \text{ if } \sigma \not\equiv 0), \\ v_0 \in H^4(\Omega) \quad \text{and} \quad \mathcal{B}_\sigma v_0 = \mathcal{B}_\sigma(\mathcal{A} v_0) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Then, we prove in Section 3 that the IBVP (1.2) admits a unique solution $v \in \mathcal{C}([0,T]; H^2(\Omega))$ with $\partial_t^\alpha v \in L^p(0,T; H^s(\Omega))$ for some $s > d/2$. Thus, we can consider $\partial_t^\alpha v(x_0, \cdot) \in L^p(0,T)$ and we can state our second result, concerning stability in the recovery of the time-dependent coefficient appearing in the reaction rate problem (1.2).

Theorem 1.2. *Let condition* (1.8) *be fulfilled and, for* $i = 1, 2$*, let* v_i *be the solution of* (1.2) *for* $f = f_i \in L^\infty(0,T)$ *with* $\|f_i\|_{L^\infty(0,T)} \leq M$*. We assume that there exist* $x_0 \in \overline{\Omega}$ *and* $\delta > 0$ *such that*

$$|q(x_0, t) v_2(x_0, t)| \geq \delta, \quad a.e.\ t \in (0,T). \tag{1.9}$$

Then, there exists a constant $C > 0$ *depending on* α*,* M*,* T*,* Ω*,* δ*,* $\mathcal{A}$*,* σ *and* $\|q\|_{L^\infty(0,T;H^2(\Omega))}$ *such that*

$$\begin{aligned} C^{-1}\|\partial_t^\alpha v_1(x_0, \cdot) - \partial_t^\alpha v_2(x_0, \cdot)\|_{L^\infty(0,T)} &\leq \|f_1 - f_2\|_{L^\infty(0,T)} \\ &\leq C\|\partial_t^\alpha v_1(x_0, \cdot) - \partial_t^\alpha v_2(x_0, \cdot)\|_{L^\infty(0,T)}. \end{aligned} \tag{1.10}$$

Note that in [34, Theorem 4.4], a problem similar to Theorem 1.1 has been considered. In contrast to [34, Theorem 4.4], Theorem 1.1 holds with a more general boundary condition and a factor $R(x,t)$ that depends also on the time variable t. Moreover, Theorem 1.1 weakened the regularity assumption for R of [34, Theorem 4.4]. These two extensions make Theorem 1.1 more suitable for application than [34, Theorem 4.4].

Let us observe that Theorem 1.2 is stated with solutions of the reaction rate problem (1.2) with a given initial condition v_0. The choice of this initial condition is related to condition (1.9). Indeed, condition (1.9) depends on the solutions of problem (1.2) that we want to determine. Nevertheless, combining some suitable maximum principle with the Hopf lemma one can derive (1.9) from conditions on the initial data and the coefficient q of the form $v_0 \geq \delta$, $f_2 q \geq 0$, $q(x_0, t) \geq \delta' > 0$, a.e. $t \in (0, T)$, $\inf_{x \in \partial\Omega} \sigma(x) > 0$. For the moment, such a result of maximum principle is only available for (1.2) with time-independent coefficients (e.g. [21, Theorem 1.1] and [23, Theorem 3]), but we believe that such a result can be extended to (1.2).

In our two main results, we assume conditions (1.5) and (1.9), which means that the observations at a point cannot be far from the source. However, the results for fractional diffusion equations without these conditions have not been obtained yet. Here we restrict ourselves to that case and we establish Lipschitz type stability estimates.

Let us remark that the results of this paper can be extended to the case $d \geq 4$. For this purpose additional conditions such as more regularity for a_{ij} and $\partial\Omega$ are required. In order to avoid technical difficulties, we only treat the case $d \leq 3$.

Let us mention that the arguments of Theorems 1.1 and 1.2 can be applied to parabolic equations (1.1)-(1.2) with $\alpha = 1$ for the recovery of a time-dependent parameter. It seems that, with [11, 13, 34], our results are one of the first results of stability in the recovery of a time-dependent parameter from a single measurement. In addition, even in the case $\alpha = 1$, to our best knowledge, with [34, Theorem 4.4], our results are the first results of determination of a time-dependent parameter from a single measurement at a point $x_0 \in \overline{\Omega}$ when $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$. So even in the context of stable recovery of a time-dependent parameter appearing in a parabolic equations, it seems that our results improve some previous known results. Nevertheless, our approach allows only recovery of parameters that depend only on the time variable t which is a restricted class of parameters. An interesting perspective will be to generalize our result to recovery of more general coefficients or source term parameters appearing in a fractional diffusion equation. For instance, we can consider recovery of parameters depending on the space variable x or both space and time variable. For this purpose, we need to consider a different approach and to overcome the difficulties that arise from fractional time derivatives (no exact formula of integration by parts, of Leibniz rule, of composition of derivatives...).

1.5. **Outline.** The remainder of this paper is composed of three sections. In Section 2, we study the forward problem for the IBVPs (1.1) and (1.2) and we prove the unique existence of sufficiently smooth solutions. In Sections 3 and 4, we complete the proof of our main results: Theorems 1.1 and 1.2.

2. Forward problem

This section is devoted to the well-posedness of the IBVPs (1.1) and (1.2). We prove existence and uniqueness of sufficiently smooth solutions of the source term problem (1.1) and the reaction rate problem (1.2). More precisely, we look for solutions u of (1.1) and v of (1.2)

such that u, $v \in \mathcal{C}([0,T]; H^2(\Omega))$ and $\partial_t^\alpha u, \partial_t^\alpha v \in \mathcal{C}([0,T]; H^{2\gamma}(\Omega))$ for some suitable values of $\gamma \in [0,1)$. The smoothness of these solutions are closely related to our inverse problem. Recall that for fractional diffusion equations with time-dependent coefficients or general boundary conditions, we cannot reduce the problems to fractional differential equations like [34]. To prove existence of sufficiently smooth solutions of (1.1)-(1.2), we consider the abstract evolution problem associated to (1.1)-(1.2) and we use some suitable fixed point theorem as in Beckers and Yamamoto [4]. We treat separately problems (1.1) and (1.2). We start with the source term problem (1.1).

2.1. **Smooth solutions of the source term problem.** In this subsection we study the well-posedness of problem (1.1). More precisely, we establish conditions that guaranty existence of sufficiently smooth solutions. The main result of this subsection can be stated as follows.

Proposition 2.1. *Let conditions* (1.3) *and* (1.4) *be fulfilled. Then, the IBVP* (1.1) *admits a unique solution* $u \in \mathcal{C}([0,T]; H^2(\Omega))$ *satisfying*

$$\mathcal{A}u \in \mathcal{C}([0,T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha u \in L^p(0,T; H^{2\gamma}(\Omega)) \tag{2.1}$$

for all $0 \le \gamma < 1 - 1/(p\alpha)$. *Moreover, we have*

$$\|\mathcal{A}u\|_{\mathcal{C}([0,T];H^{2\gamma}(\Omega))} + \|\partial_t^\alpha u\|_{L^p(0,T;H^{2\gamma}(\Omega))} \le C\|fR\|_{L^p(0,T;H^2(\Omega))} \tag{2.2}$$

with $C > 0$ *depending on* Ω, α, T, $\mathcal{A}$, σ *and* γ.

To prove this result we introduce the following IBVP.

$$\begin{cases} \partial_t^\alpha u(x,t) + \mathcal{A}u(x,t) = F(x,t), & (x,t) \in Q, \\ \mathcal{B}_\sigma u(x,t) = 0, & (x,t) \in \Sigma, \\ u(x,0) = 0, & x \in \Omega. \end{cases} \tag{2.3}$$

We also consider the following conditions

$$F \in L^p(0,T; H^2(\Omega)), \ \frac{8}{\alpha} < p \le \infty \quad \text{and} \quad \mathcal{B}_\sigma F = 0 \quad \text{on } \Sigma. \tag{2.4}$$

Note that if we set $F(x,t) = f(t)R(x,t)$, then conditions (1.3) and (1.4) are equivalent to (2.4). In order to solve the new problem (2.3) we will use an abstract formulation of the problem. For this purpose, we define the operator A as $\mathcal{A} + 1$ in $L^2(\Omega)$ equipped with the boundary condition $\mathcal{B}_\sigma h = 0$;

$$\begin{cases} D(A) := \{h \in H^2(\Omega);\ \mathcal{B}_\sigma h = 0 \text{ on } \partial\Omega\}, \\ Ah := \mathcal{A}h + h, \quad h \in D(A). \end{cases} \tag{2.5}$$

Recall that A is a selfadjoint and strictly positive operator with an orthonormal basis of eigenfunctions $(\phi_n)_{n\ge1}$ of finite order associated to a non-decreasing sequence of eigenvalues $(\lambda_n)_{n\ge1}$. From now on, $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and the inner product in $L^2(\Omega)$

respectively. We define the fractional power A^γ, $\gamma \geq 0$, of A by

$$\begin{aligned} D(A^\gamma) &:= \left\{ h \in L^2(\Omega);\ \sum_{n=1}^{\infty} \lambda_n^{2\gamma} \left|(h, \phi_n)\right|^2 < \infty \right\}, \\ A^\gamma h &:= \sum_{n=1}^{\infty} \lambda_n^\gamma (h, \phi_n)\phi_n, \quad h \in D(A^\gamma). \end{aligned} \tag{2.6}$$

Clearly, $D(A^\gamma)$ is a Hilbert space with the norm $\|\cdot\|_{D(A^\gamma)}$ defined by $\|h\|_{D(A^\gamma)} := \|A^\gamma h\|$. Since $D(A)$ is continuously embedded into $H^2(\Omega)$ with equivalent norm (e.g. [20, Theorem 5.4, Chapter 2]), we see by interpolation that

$$\begin{gathered} D(A^\gamma) \subset H^{2\gamma}(\Omega), \\ C^{-1}\|h\|_{H^{2\gamma}(\Omega)} \leq \|h\|_{D(A^\gamma)} \leq C\|h\|_{H^{2\gamma}(\Omega)}, \quad h \in D(A^\gamma), \end{gathered}$$

for $0 \leq \gamma \leq 1$.

We consider the following abstract problem in $L^2(\Omega)$;

$$\begin{cases} \partial_t^\alpha u(t) + Au(t) = F(t), & t \in (0, T), \\ u(0) = 0. \end{cases} \tag{2.7}$$

We define the operator valued function $\{S_A(t)\}_{t\geq 0}$ by

$$S_A(t)h = \sum_{n=1}^{\infty} (h, \phi_n) E_{\alpha,1}(-\lambda_n t^\alpha)\phi_n, \quad h \in L^2(\Omega),\ t \geq 0,$$

with $E_{\alpha,\beta}$, $\alpha > 0, \beta \in \mathbb{R}$, the Mittag-Leffler function given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

Recall that $S_A \in W^{1,1}(0, T; \mathcal{B}(L^2(\Omega)))$ (e.g. [4, 34]) and $\{S_A(t)\}_{t\geq 0}$ does not enjoy the properties of a semigroup: namely $S_A(t+s)h \neq S_A(t)S_A(s)h$. From now on, for all Banach space X and all $f \in L^1(0, T; \mathcal{B}(X))$, $g \in L^1(0, T; X)$, we define the causal convolution product $f * g$ given by

$$f * g(t) := \int_0^t f(t-s)g(s)ds, \quad t \in (0, T).$$

We introduce the operator valued function $\{\tilde{S}_A(t)\}_{t>0}$ given by

$$\tilde{S}_A(t)h = \sum_{n=1}^{\infty} (h, \phi_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)\phi_n, \quad h \in L^2(\Omega),\ t > 0.$$

In light of [6, formula (2.3)], for all $t \in (0, +\infty)$ we have $\tilde{S}_A(t) = -A^{-1}S_A'(t)$. Moreover, according to [34, Theorem 2.2], for any $F \in L^\infty(0, T; L^2(\Omega))$, problem (2.7) admits a unique solution

$$u(t) = \tilde{S}_A * F(t) = \int_0^t \tilde{S}_A(t-s)F(s)ds, \quad t \in (0, T). \tag{2.8}$$

In view of [4, Theorem 1], this solution is lying in $L^\infty(0,T;D(A^\gamma))$ for $0 \le \gamma < 1$, and we have

$$\|A^\gamma \tilde{S}_A(t)h\| \le Ct^{\alpha(1-\gamma)-1}\|h\|, \quad h \in L^2(\Omega),\ t > 0. \tag{2.9}$$

For additional information about the time behavior of fractional operators and some results on Mittag-Leffler functions, we refer to [24]. In particular, the mapping $t \mapsto \tilde{S}_A(t)$ belongs to $L^r(0,T;\mathcal{B}(L^2(\Omega)))$ if $r \in (1,\infty)$ satisfies $r(1-\alpha) < 1$.

We consider the following Young's inequality:

Lemma 2.2. *Let X be a Banach space, $f \in L^p(0,T;\mathcal{B}(X))$ and $g \in L^{p'}(0,T;X)$ with $1 \le p, p' \le \infty$ and $1/p + 1/p' = 1$. Then, $f * g$ is lying in $\mathcal{C}([0,T];X)$ and we have*

$$\|f * g(t)\|_X \le \|f\|_{L^p(0,t;\mathcal{B}(X))}\|g\|_{L^{p'}(0,t;X)}, \quad t \in [0,T].$$

We refer to [5, Exercise 4.30] and [36, Appendix A] for the proof of this lemma.

Let $p \in (1,\infty]$ be as in (2.4) and let $p' \in [1,\infty)$ be its conjugate index. Noting that $\mathcal{A}$ and $\tilde{S}_A(t)$ commute, we see from (2.8) that for $F \in L^p(0,T;D(A))$,

$$\mathcal{A}u(t) = \tilde{S}_A * \mathcal{A}F(t) = \int_0^t \tilde{S}_A(t-s)\mathcal{A}F(s)ds.$$

By $p > 1/\alpha$ and (2.9), the mapping $t \mapsto \tilde{S}_A(t)$ belongs to $L^{p'}(0,T;\mathcal{B}(L^2(\Omega)))$. Therefore, by Lemma 2.2, u belongs to $\mathcal{C}([0,T];D(A))$ and satisfies

$$\begin{aligned}\|\mathcal{A}u(t)\| &\le \int_0^t \|\tilde{S}_A(t-s)\mathcal{A}F(s)\|ds \le C\int_0^t (t-s)^{\alpha-1}\|F(s)\|_{D(A)}ds \\ &\le C\left(\int_0^t s^{(\alpha-1)p'}ds\right)^{1/p'} \|F\|_{L^p(0,t;D(A))} \le Ct^{\alpha-1/p}\|F\|_{L^p(0,T;D(A))}.\end{aligned}$$

Thus, we can define the map $\mathcal{H} : L^p(0,T;D(A)) \to \mathcal{C}([0,T];D(A))$ by

$$\mathcal{H}(w) := \tilde{S}_A * w, \quad w \in L^p(0,T;D(A)) \tag{2.10}$$

and for any $w \in L^p(0,T;D(A))$, it satisfies

$$\|\mathcal{H}(w)(t)\|_{D(A)} \le C\int_0^t (t-s)^{\alpha-1}\|w(s)\|_{D(A)}ds \le Ct^{\alpha-1/p}\|w\|_{L^p(0,T;D(A))}. \tag{2.11}$$

Using these properties we will prove Proposition 2.1.

Proof of Proposition 2.1. Let A be the operator defined by (2.5), then the IBVP (2.3) can be rewritten as

$$\begin{cases}\partial_t^\alpha u(t) + Au(t) = u(t) + F(t), & t \in (0,T), \\ u(0) = 0,\end{cases} \tag{2.12}$$

where $u(t) := u(\cdot,t)$ and $F(t) := F(\cdot,t)$. From now on, we divide the proof of Proposition 2.1 into three steps. First, using a fixed point argument we will prove that problem (2.12) admits a unique solution $u \in \mathcal{C}([0,T];H^2(\Omega))$. Then, we prove that u fulfills $\|u\|_{\mathcal{C}([0,T];H^2(\Omega))} \le C\|F\|_{L^p(0,T;D(A))}$. Finally, we prove (2.1) and estimate (2.2).

We start by proving existence and uniqueness of a solution $u \in \mathcal{C}([0,T];H^2(\Omega))$. Noting that $F \in L^p(0,T;D(A))$ by (2.4), we see from (2.8) that the solution u of (2.12) satisfies

$$u(t) = \mathcal{H}(u)(t) + \mathcal{H}(F)(t), \quad t \in (0,T), \tag{2.13}$$

where the map $\mathcal{H}$ is defined by (2.10). Therefore, we will look for a fixed point of the map $\mathcal{G} : \mathcal{C}([0,T];D(A)) \to \mathcal{C}([0,T];D(A))$ defined by

$$\mathcal{G}(w)(t) := \mathcal{H}(w)(t) + \mathcal{H}(F)(t), \quad w \in \mathcal{C}([0,T];D(A)),\ t \in (0,T).$$

Now we introduce the family $\{Y_r\}_{r>0}$ of functions on $(0,T)$, which is defined by

$$Y_r(t) = \frac{t^{r-1}}{\Gamma(r)}, \quad t > 0.$$

One can easily check that, for all $r, s > 0$, we have

$$Y_r * Y_s = Y_{r+s}. \tag{2.14}$$

In particular, noting that $Y_1(t) \equiv 1$, we also have

$$\int_0^t Y_r(t-\tau)d\tau = Y_r * Y_1(t) = Y_{r+1}(t). \tag{2.15}$$

By (2.11) and (2.15), for $w \in \mathcal{C}([0,T];D(A))$ and all $t \in [0,T]$, we have

$$\begin{aligned}\|\mathcal{H}(w)(t)\|_{D(A)} &\le C\int_0^t (t-s)^{\alpha-1}\|w(s)\|_{D(A)}ds \le C\left(\int_0^t Y_\alpha(t-s)ds\right)\|w\|_{\mathcal{C}([0,T];D(A))}\\ &= CY_{\alpha+1}(t)\|w\|_{\mathcal{C}([0,T];D(A))},\end{aligned}$$

and repeating the similar calculation, we get

$$\begin{aligned}\|\mathcal{H}^2 w(t)\|_{D(A)} &\le C\int_0^t (t-s)^{\alpha-1}\|\mathcal{H}(w)(s)\|_{D(A)}ds\\ &\le C\left(\int_0^t Y_\alpha(t-s)Y_{\alpha+1}(s)ds\right)\|w\|_{\mathcal{C}([0,T];D(A))}\\ &= CY_{2\alpha+1}(t)\|w\|_{\mathcal{C}([0,T];D(A))},\end{aligned}$$

where we have used (2.14) in the last equality. By induction, for all $w \in \mathcal{C}([0,T];D(A))$, we have

$$\|\mathcal{H}^n w(t)\|_{D(A)} \le CY_{n\alpha+1}(t)\|w\|_{\mathcal{C}([0,T];D(A))} \le \frac{CT^{n\alpha}}{\Gamma(n\alpha+1)}\|w\|_{\mathcal{C}([0,T];D(A))}, \quad t \in [0,T]. \tag{2.16}$$

Therefore, we obtain

$$\begin{aligned}\|\mathcal{G}^n(w_1) - \mathcal{G}^n(w_2)\|_{\mathcal{C}([0,T];D(A))} &= \|\mathcal{H}^n(w_1 - w_2)\|_{\mathcal{C}([0,T];D(A))}\\ &\le \frac{CT^{n\alpha}}{\Gamma(n\alpha+1)}\|w_1 - w_2\|_{\mathcal{C}([0,T];D(A))}\end{aligned}$$

for $w_1, w_2 \in \mathcal{C}([0,T];D(A))$. Thus, $\mathcal{G}^n$ is a contraction for sufficiently large $n \in \mathbb{N}$ and $\mathcal{G}$ admits a unique fixed point $u \in \mathcal{C}([0,T];D(A)) \subset \mathcal{C}([0,T];H^2(\Omega))$. Moreover, we have

$$u = \mathcal{G}(u) = \mathcal{G}^n(u) = \mathcal{H}^n(u) + \sum_{k=1}^n \mathcal{H}^k(F) \tag{2.17}$$

for any $n \in \mathbb{N}$.

Now let us show that this unique solution fulfills $\|u\|_{\mathcal{C}([0,T];H^2(\Omega))} \leq C\,\|F\|_{L^p(0,T;D(A))}$. For this purpose, we estimate each $\mathcal{H}^k(F)$. First, by (2.11), we have

$$\|\mathcal{H}(F)(t)\|_{D(A)} \leq CY_{\alpha+1-1/p}(t)\|F\|_{L^p(0,T;D(A))}, \quad t \in [0,T].$$

Applying (2.11) and (2.14), for all $t \in [0,T]$, we get

$$\begin{aligned}\|\mathcal{H}^2(F)(t)\|_{D(A)} &\leq C\int_0^t (t-s)^{\alpha-1}\|\mathcal{H}(F)(s)\|_{D(A)}ds \\ &\leq C\left(\int_0^t Y_\alpha(t-s)Y_{\alpha+1-1/p}(s)ds\right)\|F\|_{L^p(0,T;D(A))} \\ &= CY_{2\alpha+1-1/p}(t)\|F\|_{L^p(0,T;D(A))}.\end{aligned}$$

By induction, we obtain

$$\|\mathcal{H}^k(F)(t)\|_{D(A)} \leq CY_{k\alpha+1-1/p}(t)\|F\|_{L^p(0,T;D(A))} \leq \frac{CT^{k\alpha-1/p}\|F\|_{L^p(0,T;D(A))}}{\Gamma(k\alpha+1-1/p)}, \quad t \in [0,T].$$

Therefore, applying (2.16) and (2.17), we find

$$\begin{aligned}\|u\|_{\mathcal{C}([0,T];D(A))} &\leq \|\mathcal{H}^n(u)\|_{\mathcal{C}([0,T];D(A))} + \sum_{k=1}^n \|\mathcal{H}^k(F)\|_{\mathcal{C}([0,T];D(A))} \\ &\leq \frac{CT^{n\alpha}}{\Gamma(n\alpha+1)}\|u\|_{\mathcal{C}([0,T];D(A))} + C\left(\sum_{k=1}^n \frac{T^{k\alpha-1/p}}{\Gamma(k\alpha+1-1/p)}\right)\|F\|_{L^p(0,T;D(A))}\end{aligned}$$

and by taking sufficiently large $n \in \mathbb{N}$, we obtain

$$\|u\|_{\mathcal{C}([0,T];D(A))} \leq C\|F\|_{L^p(0,T;D(A))} \tag{2.18}$$

with C depending on T, α, Ω, $\mathcal{A}$ and σ.

We complete the proof of Proposition 2.1 by showing the estimates (2.1)-(2.2). We fix $0 \leq \gamma < 1-1/(p\alpha)$. Then, in light of (2.13), for all $t \in (0,T)$, we have $\mathcal{A}u(t) \in D(A^\gamma)$ with

$$A^\gamma(\mathcal{A}u)(t) = (A^\gamma\tilde{S}_A) * (\mathcal{A}u + \mathcal{A}F)(t)$$

and by (2.9), we have

$$\|A^\gamma\tilde{S}_A(t)\|_{\mathcal{B}(L^2(\Omega))} \leq Ct^{\mu-1},$$

where $\mu := \alpha(1-\gamma)$. Since $\mu > 1/p$, the mapping $t \mapsto A^\gamma\tilde{S}_A(t)$ belongs to $L^{p'}(0,T;\mathcal{B}(L^2(\Omega))$ where $p' \in [1,\infty)$ satisfies $1/p+1/p'=1$. Therefore, $\mathcal{A}u$ belongs to $\mathcal{C}([0,T];D(A^\gamma))$ and

$$\begin{aligned}\|\mathcal{A}u(t)\|_{D(A^\gamma)} &= \left\|\int_0^t A^\gamma\tilde{S}_A(t-s)\left(\mathcal{A}u(s)+\mathcal{A}F(s)\right)ds\right\| \\ &\leq C\int_0^t (t-s)^{\mu-1}\|u(s)\|_{D(A)}ds + C\int_0^t (t-s)^{\mu-1}\|F(s)\|_{D(A)}ds \\ &\leq C\left(\int_0^t (t-s)^{\mu-1}ds\right)\|u\|_{\mathcal{C}([0,T];D(A))} + C\left(\int_0^t s^{p'(\mu-1)}ds\right)^{1/p'}\|F\|_{L^p(0,t;D(A))} \\ &\leq CT^\mu\|u\|_{\mathcal{C}([0,T];D(A))} + CT^{\mu-1/p}\|F\|_{L^p(0,T;D(A))}\end{aligned} \tag{2.19}$$

Combining this with (2.18), we have

$$\|\mathcal{A}u(t)\|_{D(A^\gamma)} \le C\|F\|_{L^p(0,T;D(A))} \le C\|F\|_{L^p(0,T;H^2(\Omega))}.$$

Hence, we deduce that $\mathcal{A}u \in \mathcal{C}([0,T];H^{2\gamma}(\Omega))$ and

$$\|\mathcal{A}u\|_{\mathcal{C}([0,T];H^{2\gamma}(\Omega))} \le C\|F\|_{L^p(0,T;H^2(\Omega))}.$$

By the original equation $\partial_t^\alpha u = -\mathcal{A}u + F$, we see that $\partial_t^\alpha u$ belongs to $L^p(0,T;H^{2\gamma}(\Omega))$ with the estimate;

$$\begin{aligned}\|\partial_t^\alpha u\|_{L^p(0,T;H^{2\gamma}(\Omega))} &\le C\|\mathcal{A}u\|_{L^p(0,T;H^{2\gamma}(\Omega))} + C\|F\|_{L^p(0,T;H^{2\gamma}(\Omega))}\\ &\le C\|\mathcal{A}u\|_{\mathcal{C}([0,T];H^{2\gamma}(\Omega))} + C\|F\|_{L^p(0,T;H^2(\Omega))}\\ &\le C\|F\|_{L^p(0,T;H^2(\Omega))},\end{aligned}$$

which implies (2.2). Thus, we have completed the proof of the proposition. □

2.2. **Smooth solutions for the reaction rate problem.** In this subsection we study the well-posedness of problem (1.2). The main result of this subsection can be stated as follows.

Proposition 2.3. *Let conditions* (1.3) *and* (1.8) *be fulfilled. Then, the IBVP* (1.2) *admits a unique solution* $v \in \mathcal{C}([0,T];H^2(\Omega))$ *satisfying*

$$\mathcal{A}v \in \mathcal{C}([0,T];H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha v \in L^\infty(0,T;H^{2\gamma}(\Omega))$$

for all $0 \le \gamma < 1$. *Moreover, we have*

$$\|\mathcal{A}v\|_{\mathcal{C}([0,T];H^{2\gamma}(\Omega))} + \|\partial_t^\alpha v\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} \le C\|v_0\|_{H^4(\Omega)} \tag{2.20}$$

with C *depending on* Ω, α, T, $\|f\|_{L^\infty(0,T)}$, $\|q\|_{L^\infty(0,T;H^2(\Omega))}$, $\mathcal{A}$, σ *and* γ.

To prove this result we introduce the intermediate IBVP

$$\begin{cases}\partial_t^\alpha v(x,t) + \mathcal{A}v(x,t) + \tilde{q}(x,t)v(x,t) = F(x,t), & (x,t) \in Q,\\ \mathcal{B}_\sigma v(x,t) = 0, & (x,t) \in \Sigma,\\ v(x,0) = 0, & x \in \Omega.\end{cases} \tag{2.21}$$

If we assume $\tilde{q}(x,t) = f(t)q(x,t)$, then conditions (1.3) and (1.8) are equivalent to the condition

$$\begin{cases}1)\ \tilde{q} \in L^\infty(0,T;H^2(\Omega)) \quad (\text{and } \partial_\nu \tilde{q} = 0 \text{ on } \Sigma \text{ if } \sigma \not\equiv 0),\\ 2)\ v_0 \in H^4(\Omega) \quad \text{and} \quad \mathcal{B}_\sigma v_0 = \mathcal{B}_\sigma(\mathcal{A}v_0) = 0 \quad \text{on } \partial\Omega.\end{cases} \tag{2.22}$$

In order to prove Proposition 2.3, we consider an intermediate result related to the well-posedness of problem (2.21).

Lemma 2.4. *Let* $F \in L^\infty(0,T;H^2(\Omega))$ *satisfy* $\mathcal{B}_\sigma F = 0$ *and condition 1) of* (2.22) *be fulfilled. Then, the IBVP* (2.21) *admits a unique solution* $v \in \mathcal{C}([0,T];H^2(\Omega))$ *satisfying*

$$\mathcal{A}v \in \mathcal{C}([0,T];H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha v \in L^\infty(0,T;H^{2\gamma}(\Omega))$$

for all $0 \le \gamma < 1$. *Moreover, we have*

$$\|\mathcal{A}v\|_{\mathcal{C}([0,T];H^{2\gamma}(\Omega))} + \|\partial_t^\alpha v\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} \le C\|F\|_{L^\infty(0,T;H^2(\Omega))} \tag{2.23}$$

with C *depending on* Ω, T, α, $\|\tilde{q}\|_{L^\infty(0,T;H^2(\Omega))}$, $\mathcal{A}$, σ *and* γ.

Proof. Similarly to Proposition 2.1, the IBVP (2.21) can be rewritten as

$$\begin{cases} \partial_t^\alpha v(t) + Av(t) = (1-\tilde{q}(t))v(t) + F(t), \\ v(0) = 0, \end{cases} \tag{2.24}$$

where $v(t) := v(\cdot, t)$ and $F(t) := F(\cdot, t)$. Moreover, $\tilde{q}(t)$ denotes the multiplication operator by $\tilde{q}(x,t)$. Then, we can see that the solution v of (2.24) is a fixed point of the map $\mathcal{K} : \mathcal{C}([0,T]; D(A)) \to \mathcal{C}([0,T]; D(A))$ defined by

$$\mathcal{K}(w)(t) := (\mathcal{H}(1-\tilde{q}(t))w)(t) + \mathcal{H}(F)(t), \quad w \in \mathcal{C}([0,T]; D(A)),\ t \in (0,T).$$

Indeed, the Sobolev embedding theorem (e.g. [37, Theorem 2.1,Chapter II]) and condition 1) of (2.22) yields that $(1-\tilde{q})w$ belongs to $L^\infty(0,T; D(A))$ and satisfies

$$\|(1-\tilde{q}(t))w(t)\|_{D(A)} \le C\|w(t)\|_{D(A)}$$

with C depending on $\|\tilde{q}\|_{L^\infty(0,T;H^2(\Omega))}$. Thus we can see that $\mathcal{K}$ maps $\mathcal{C}([0,T]; D(A))$ and we find

$$\|\mathcal{K}^n(w_1) - \mathcal{K}^n(w_2)\|_{\mathcal{C}([0,T];D(A))} \le \frac{CT^{n\alpha}}{\Gamma(n\alpha+1)}\|w_1 - w_2\|_{\mathcal{C}([0,T];D(A))}, \\ w_1, w_2 \in \mathcal{C}([0,T]; D(A)),$$

which implies that $\mathcal{K}$ admits a unique fixed point $v \in \mathcal{C}([0,T]; D(A)) \subset \mathcal{C}([0,T]; H^2(\Omega))$. Then, we have

$$v = \mathcal{K}(v) = \mathcal{K}^n(v) = (\mathcal{H}(1-\tilde{q}(t)))^n(v) + \sum_{k=1}^{n}(\mathcal{H}(1-\tilde{q}(t)))^{k-1}(\mathcal{H}F). \tag{2.25}$$

Repeating the argument in Proposition 2.1, we deduce from (2.25) that

$$\|v\|_{\mathcal{C}([0,T];D(A))} \le C\|F\|_{L^\infty(0,T;D(A))} \tag{2.26}$$

with C depending on T, Ω, α, $\mathcal{A}$, σ and $\|\tilde{q}\|_{L^\infty(0,T;H^2(\Omega))}$.

Finally for $0 \le \gamma < 1$, using estimates similar to (2.19), we deduce that $\mathcal{A}v$ belongs to $\mathcal{C}([0,T]; H^{2\gamma}(\Omega))$ and satisfies

$$\|\mathcal{A}v\|_{\mathcal{C}([0,T];H^{2\gamma}(\Omega))} \le C\|F\|_{L^\infty(0,T;H^2(\Omega))}.$$

Moreover, combining this with the original equation, we get $\partial_t^\alpha v \in L^\infty(0,T; H^{2\gamma}(\Omega))$ and (2.23). □

Armed with Lemma 2.4, we are now in position to complete the proof of Proposition 2.3. For this purpose, we rewrite problem (1.2) as follows

$$\begin{cases} \partial_t^\alpha v(x,t) + \mathcal{A}v(x,t) + \tilde{q}(x,t)v(x,t) = 0, & (x,t) \in Q, \\ \mathcal{B}_\sigma v(x,t) = 0, & (x,t) \in \Sigma, \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{2.27}$$

We will prove the result of Proposition 2.3 for problem (2.27) by assuming (2.22) fulfilled.

Proof of Proposition 2.3. We split the solution v of (2.27) into two terms $v = w + v_0$ where w solves

$$\begin{cases} \partial_t^\alpha w(x,t) + \mathcal{A}w(x,t) + \tilde{q}(x,t)w(x,t) = F(x,t), & (x,t) \in Q, \\ \mathcal{B}_\sigma w(x,t) = 0, & (x,t) \in \Sigma, \\ w(x,0) = 0, & x \in \Omega \end{cases} \tag{2.28}$$

with $F(x,t) := -(\mathcal{A} + \tilde{q}(x,t))v_0(x)$[1]. Then, (2.22) implies $F \in L^\infty(0,T;D(A))$ with the estimate

$$\|F\|_{L^\infty(0,T;H^2(\Omega))} \le C\|v_0\|_{H^4(\Omega)}.$$

By Lemma 2.4, the IBVP (2.28) admits a unique solution $w \in \mathcal{C}([0,T];H^2(\Omega))$ satisfying

$$\mathcal{A}w \in \mathcal{C}([0,T];H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha w \in L^\infty(0,T;H^{2\gamma}(\Omega)).$$

Moreover, we find

$$\|\mathcal{A}w\|_{\mathcal{C}([0,T];H^{2\gamma}(\Omega))} + \|\partial_t^\alpha w\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} \le C\|F\|_{L^\infty(0,T;H^2(\Omega))} \le C\|v_0\|_{H^4(\Omega)}.$$

Therefore, the IBVP (2.27) admits a unique solution $v \in \mathcal{C}([0,T];H^2(\Omega))$ satisfying

$$\mathcal{A}v \in \mathcal{C}([0,T];H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha v \in L^\infty(0,T;H^{2\gamma}(\Omega)).$$

From the above estimate, we deduce (2.20). □

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. To this end, we prepare the following lemmata with Gronwall type inequalities;

Lemma 3.1. *Let $C, \alpha > 0$ and $h, d \in L^1(0,T)$ be nonnegative functions satisfying*

$$h(t) \le Cd(t) + C\int_0^t (t-s)^{\alpha-1}h(s)ds, \quad t \in (0,T),$$

then we have

$$h(t) \le Cd(t) + C\int_0^t (t-s)^{\alpha-1}d(s)ds, \quad t \in (0,T).$$

We refer to [18, Lemma 7.1.1, p.188] for the proof of this lemma. Let us also consider the following.

Lemma 3.2. *We take $2 \le p \le \infty$ and $\mu > 2/p$. Let $f \in L^\infty(0,T)$ and $w, S \in L^p(0,T)$ be non-negative functions satisfying the integral inequality*

$$f(t) \le w(t) + \int_0^t (t-s)^{\mu-1}f(s)S(s)ds, \quad a.e.\ t \in (0,T). \tag{3.1}$$

Then, we have

$$\|f\|_{L^p(0,T)} \le C\|w\|_{L^p(0,T)}, \tag{3.2}$$

where the constant C depends on p, μ, T and $\|S\|_{L^p(0,T)}$.

[1]Here we use the fact that $\partial_t^\alpha c = 0$ for any constant c. This result holds true for fractional derivatives in the Caputo sense which is being used here but for Riemann-Liouville fractional derivatives this would not be true (see [29, pages 78-81])

Proof. In order to prove this lemma, we will apply the Gronwall inequality to $b(t) := \|f\|^p_{L^p(0,t)}$. For this purpose, we will first derive an estimate of $b(t)$ of the form

$$b(t) \le C\|w\|^p_{L^p(0,T)} + C\int_0^t b(s)ds, \quad t \in (0,T). \tag{3.3}$$

Applying (3.1), we get

$$|f(s)|^p \le C|w(s)|^p + C\left|\int_0^s (s-\xi)^{\mu-1} f(\xi)S(\xi)d\xi\right|^p,$$

which implies

$$b(t) \le C\|w\|^p_{L^p(0,T)} + C\int_0^t \left|\int_0^s (s-\xi)^{\mu-1} f(\xi)S(\xi)d\xi\right|^p ds. \tag{3.4}$$

To obtain (3.3), we will estimate the right-hand side of the above inequality. By the Cauchy-Schwarz inequality, we get

$$\int_0^s |f(\xi)S(\xi)|^{p/2}d\xi = \int_0^s |f(\xi)|^{p/2}\cdot|S(\xi)|^{p/2}d\xi \le \left(\int_0^s |f(\xi)|^p d\xi\right)^{1/2}\left(\int_0^s |S(\xi)|^p d\xi\right)^{1/2},$$

which can also be written

$$\|fS\|_{L^{p/2}(0,s)} \le \|f\|_{L^p(0,s)}\|S\|_{L^p(0,s)}.$$

Therefore, for $p > 2$, an application of Lemma 2.2 yields

$$\left|\int_0^s (s-\xi)^{\mu-1} f(\xi)S(\xi)d\xi\right| \le \left(\int_0^s \xi^{r(\mu-1)}ds\right)^{1/r}\|fS\|_{L^{p/2}(0,s)} \le C\|f\|_{L^p(0,s)}\|S\|_{L^p(0,s)},$$

where $r \in [1,\infty)$ satisfies $2/p + 1/r = 1$. For $p = 2$, we also have

$$\left|\int_0^s (s-\xi)^{\mu-1} f(\xi)S(\xi)d\xi\right| \le s^{\mu-1}\|fS\|_{L^1(0,s)} \le C\|f\|_{L^2(0,s)}\|S\|_{L^2(0,s)}.$$

Thus, for any $p \ge 2$, we obtain

$$\left|\int_0^s (s-\xi)^{\mu-1} f(\xi)S(\xi)d\xi\right|^p \le Cb(s), \tag{3.5}$$

where C depends on T, p, μ and $\|S\|_{L^p(0,T)}$. By (3.4) and (3.5), we get (3.3) and by the Gronwall inequality, we get

$$b(t) \le C\|w\|^p_{L^p(0,T)}, \quad t \in (0,T)$$

with C depending on p, μ, T and $\|S\|_{L^p(0,T)}$. Thus, we have proved (3.2). □

Armed with Lemma 3.1 and 3.2, we are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. For $i = 1, 2$, let u_i be the solution to (1.1) with $f = f_i$ and set $u := u_1 - u_2$, $f := f_1 - f_2$. We recall that the regularity of these solutions are stated in Proposition 2.1 with $p \in (8/\alpha, +\infty]$ and $\gamma \in [0, 1-1/(p\alpha))$. Then, u solves (1.1) and satisfies

$$u(t) = \mathcal{H}(u + fR)(t) = \int_0^t \tilde{S}_A(t-s)(u(s) + f(s)R(s))ds,$$

where $u(t) := u(\cdot, t)$ and $R(t) := R(\cdot, t)$. Combining this representation with the Gronwall type inequality of Lemma 3.1 and 3.2, we will prove Theorem 1.1. For this purpose, we proceed in four steps. First, we set

$$d(t) := \int_0^t (t-s)^{\alpha-1}|f(s)|\|R(s)\|_{D(A)}ds.$$

and we prove that $\|u(t)\|_{D(A)}$ fulfills

$$\|u(t)\|_{D(A)} \leq Cd(t), \quad 0 < t < T. \tag{3.6}$$

Second, we establish the estimate

$$\|\mathcal{A}u(t)\|_{D(A^\gamma)} \leq C\int_0^t (t-s)^{\mu-1}|f(s)|\|R(s)\|_{D(A)}ds. \tag{3.7}$$

Third, applying estimates (3.6)-(3.7), we derive an estimate of $|\mathcal{A}u(x_0, t)|$. Combining this with Lemma 3.2, we prove (1.6). Forth, we show (1.7).

We start with estimate (3.6). By (2.11), we have

$$\begin{aligned}\|u(t)\|_{D(A)} &\leq C\int_0^t (t-s)^{\alpha-1}\|u(s)\|_{D(A)}ds + C\int_0^t (t-s)^{\alpha-1}|f(s)|\|R(s)\|_{D(A)}ds \\ &= C\int_0^t (t-s)^{\alpha-1}\|u(s)\|_{D(A)}ds + Cd(t). \end{aligned}\tag{3.8}$$

Then, applying Lemma 3.1 with $h(t) = \|u(t)\|_{D(A)}$, we get

$$\|u(t)\|_{D(A)} \leq Cd(t) + C\int_0^t (t-s)^{\alpha-1}d(s)ds, \quad 0 < t < T. \tag{3.9}$$

Using Fubini's theorem, for $\nu > 0$, we find

$$\begin{aligned}\int_0^t (t-s)^{\nu-1}d(s)ds &= \int_0^t (t-s)^{\nu-1}\left(\int_0^s (s-\xi)^{\alpha-1}|f(\xi)|\|R(\xi)\|_{D(A)}d\xi\right)ds \\ &= \int_0^t \left(\int_\xi^t (t-s)^{\nu-1}(s-\xi)^{\alpha-1}ds\right)|f(\xi)|\|R(\xi)\|_{D(A)}d\xi \\ &= B(\nu,\alpha)\int_0^t (t-\xi)^{\nu+\alpha-1}|f(\xi)|\|R(\xi)\|_{D(A)}d\xi \end{aligned}\tag{3.10}$$

where $B(\cdot,\cdot)$ is the Beta function. In particular, for $\nu = \alpha$, we obtain

$$\begin{aligned}\int_0^t (t-s)^{\alpha-1}d(s)ds &= B(\alpha,\alpha)\int_0^t (t-s)^{2\alpha-1}|f(s)|\|R(s)\|_{D(A)}ds \\ &\leq T^\alpha B(\alpha,\alpha)\int_0^t (t-s)^{\alpha-1}|f(s)|\|R(s)\|_{D(A)}ds \\ &\leq Cd(t). \end{aligned}$$

Combining this with (3.9), we deduce (3.6).

Next we estimate $\|\mathcal{A}u(t)\|_{D(A^\gamma)}$ for $d/4 < \gamma < 1 - 2/(p\alpha)$. Repeating the calculation in (2.19), we find

$$\|\mathcal{A}u(t)\|_{D(A^\gamma)} \le C \int_0^t (t-s)^{\mu-1} \left(\|u(s)\|_{D(A)} + |f(s)| \|R(s)\|_{D(A)} \right) ds, \quad \text{a.e. } t \in (0,T),$$

where $\mu = \alpha(1-\gamma)$. By (3.10) with $\nu = \mu$ and (3.6), we obtain

$$\begin{aligned}
\|\mathcal{A}u(t)\|_{D(A^\gamma)} &\le C \int_0^t (t-s)^{\mu-1} d(s) ds + C \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&= CB(\mu,\alpha) \int_0^t (t-s)^{\mu+\alpha-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\quad + C \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\le CT^\alpha B(\mu,\alpha) \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\quad + C \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\le C \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds.
\end{aligned}$$

This proves (3.7).

Now let us show (1.6). For this purpose, we first apply (3.6)-(3.7) to estimate $|\mathcal{A}u(x_0,t)|$. Since $\gamma > d/4$, the Sobolev embedding theorem yields

$$|\mathcal{A}u(x_0,t)| \le C\|\mathcal{A}u(\cdot,t)\|_{H^{2\gamma}(\Omega)} \le C\|\mathcal{A}u(t)\|_{D(A^\gamma)} \le C \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds. \tag{3.11}$$

From the original equation, we get

$$f(t)R(x_0,t) = \partial_t^\alpha u(x_0,t) + \mathcal{A}u(x_0,t), \quad \text{a.e. } t \in (0,T). \tag{3.12}$$

Combining this with (1.5) and (3.11), we find

$$\begin{aligned}
|f(t)| &\le \frac{1}{\delta} \left(|\partial_t^\alpha u(x_0,t)| + |\mathcal{A}u(x_0,t)| \right) \\
&\le C|\partial_t^\alpha u(x_0,t)| + C \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds, \quad \text{a.e. } t \in (0,T)
\end{aligned} \tag{3.13}$$

with C depending on δ, α, Ω, $\mathcal{A}$, σ and T. By Lemma 3.2, we see that

$$\|f\|_{L^p(0,T)} \le C\|\partial_t^\alpha u(x_0,\cdot)\|_{L^p(0,T)},$$

which implies (1.6).

Finally, we complete the proof by proving (1.7). According to (3.11) and (3.12), we have

$$\begin{aligned}
|\partial_t^\alpha u(x_0,t)| &\le |f(t)R(x_0,t)| + |\mathcal{A}u(x_0,t)| \\
&\le C|f(t)| \|R(\cdot,t)\|_{H^2(\Omega)} + C \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds
\end{aligned}$$

$$\leq C\|f\|_{L^\infty(0,T)}\|R(t)\|_{D(A)} + C\|f\|_{L^\infty(0,T)}\int_0^t (t-s)^{\mu-1}\|R(s)\|_{D(A)}ds$$
$$\leq C\|f\|_{L^\infty(0,T)}\|R(t)\|_{D(A)} + C\|f\|_{L^\infty(0,T)}T^{\mu-1/p}\|R\|_{L^p(0,T;D(A))}, \quad \text{a.e. } t \in (0,T).$$

Therefore, we find

$$\begin{aligned}\|\partial_t^\alpha u(x_0,\cdot)\|_{L^p(0,T)} &\leq C\|f\|_{L^\infty(0,T)}\|R\|_{L^p(0,T;D(A))} + C\|f\|_{L^\infty(0,T)}T^\mu\|R\|_{L^p(0,T;D(A))}\\ &\leq C\|f\|_{L^\infty(0,T)}.\end{aligned}$$

Thus, we have proved (1.7). □

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We first prepare the following generalized Gronwall's inequality;

Lemma 4.1. *Let $\mu, a, b > 0$ and $f \in L^1(0,T)$ be nonnegative function satisfying the integral inequality*

$$f(t) \leq a + b\int_0^t (t-s)^{\mu-1}f(s)ds, \quad a.e.\ t \in (0,T).$$

Then, we have

$$f(t) \leq aE_{\mu,1}\left((b\Gamma(\mu))^{1/\mu}t^\mu\right), \quad a.e.\ t \in (0,T).$$

We refer to [18, Lemma 7.1.2, p.189] for the proof of this result. In light of this result, we are now in position to prove Theorem 1.2.

Proof of Theorem 1.2. For $i = 1, 2$, let v_i be the solution to (1.2) with $f = f_i$ and set $v := v_1 - v_2$, $f := f_2 - f_1$. We recall that the smoothness of these solutions are given by Proposition 2.3 with $\gamma \in [0,1)$. Then, by setting $\tilde{q}(x,t) := f_1(t)q(x,t)$ and $R(x,t) := q(x,t)v_2(x,t)$, we see that v solves

$$\begin{cases}\partial_t^\alpha v(x,t) + \mathcal{A}v(x,t) + \tilde{q}(x,t)v(x,t) = f(t)R(x,t), & (x,t) \in Q,\\ \mathcal{B}_\sigma v(x,t) = 0, & (x,t) \in \Sigma,\\ v(x,0) = 0, & x \in \Omega.\end{cases}$$

Recall that v satisfies

$$\begin{aligned}v(t) &= \mathcal{H}((1-\tilde{q})v)(t) + \mathcal{H}(fR)(t)\\ &= \int_0^t \tilde{S}_A(t-s)(1-\tilde{q}(s))v(s)ds + \int_0^t \tilde{S}_A(t-s)f(s)R(s)ds,\end{aligned}$$

where we have set $v(t) := v(\cdot,t)$ and $R(t) := R(\cdot,t)$. Moreover, $\tilde{q}(t)$ denotes the multiplication operator by $\tilde{q}(x,t)$.

First we estimate $\|v(t)\|_{D(A)}$. Since $(1-\tilde{q}(t))v(t), R(t) \in D(A)$ by (1.8), we apply (2.11) to have

$$\|v(t)\|_{D(A)} \leq C\int_0^t (t-s)^{\alpha-1}\|(1-\tilde{q}(t))v(s)\|_{D(A)}ds + C\int_0^t (t-s)^{\alpha-1}|f(s)|\|R(s)\|_{D(A)}ds$$

$$\leq C\int_0^t (t-s)^{\alpha-1}\|v(s)\|_{D(A)}ds + C\int_0^t (t-s)^{\alpha-1}|f(s)|ds.$$

with C depending on Ω, α, M, $\mathcal{A}$, σ and $\|q\|_{L^\infty(0,T;H^2(\Omega))}$. Then, repeating the arguments used in Theorem 1.1, we obtain

$$\|v(t)\|_{D(A)} \leq C\int_0^t (t-s)^{\alpha-1}|f(s)|ds, \quad 0<t<T.$$

and from this estimate we also deduce that for any $0\leq\gamma<1$,

$$\|\mathcal{A}v(t)\|_{D(A^\gamma)} \leq C\int_0^t (t-s)^{\mu-1}|f(s)|ds, \quad 0<t<T,$$

where $\mu := \alpha(1-\gamma)$. Therefore, by taking $\gamma\in(d/4,1)$, we have

$$\begin{aligned}|\mathcal{A}v(x_0,t)+\tilde{q}(x_0,t)v(x_0,t)| &\leq C\|\mathcal{A}v(\cdot,t)+\tilde{q}(\cdot,t)v(\cdot,t)\|_{H^{2\gamma}(\Omega)}\\ &\leq C\|\mathcal{A}v(\cdot,t)\|_{H^{2\gamma}(\Omega)} + C\|v(\cdot,t)\|_{H^{2\gamma}(\Omega)}\\ &\leq C\|\mathcal{A}v(t)\|_{D(A^\gamma)} \leq C\int_0^t (t-s)^{\mu-1}|f(s)|ds. \end{aligned} \tag{4.1}$$

From the original equation, we have

$$f(t)R(x_0,t) = \partial_t^\alpha v(x_0,t) + \mathcal{A}v(x_0,t) + \tilde{q}(x_0,t)v(x_0,t), \quad \text{a.e. } t\in(0,T). \tag{4.2}$$

On the other hand, from (1.9), we deduce that

$$|R(x_0,t)| \geq c>0, \quad \text{a.e. } t\in(0,T)$$

with c depending on δ, Ω and T. Therefore, combining this with (4.1) and (4.2), we obtain

$$\begin{aligned}|f(t)| &\leq C|\partial_t^\alpha v(x_0,t)| + C|\mathcal{A}v(x_0,t)+\tilde{q}(x_0,t)v(x_0,t)|\\ &\leq C\|\partial_t^\alpha v(x_0,\cdot)\|_{L^\infty(0,T)} + C\int_0^t (t-s)^{\mu-1}|f(s)|ds, \quad \text{a.e. } t\in(0,T).\end{aligned}$$

Applying Lemma 4.1, we see that

$$|f(t)| \leq C\|\partial_t^\alpha v(x_0,\cdot)\|_{L^\infty(0,T)}.$$

Thus, we have proved the second inequality in (1.10). Moreover, by (4.2), we have

$$\begin{aligned}|\partial_t^\alpha v(x_0,t)| &\leq |f(t)R(x_0,t)| + |\mathcal{A}v(x_0,t)+\tilde{q}(x_0,t)v(x_0,t)|\\ &\leq |f(t)|\|R(\cdot,t)\|_{D(A)} + C\int_0^t (t-s)^{\mu-1}|f(s)|ds\\ &\leq C\left(\|R\|_{L^\infty(0,T;D(A))} + \frac{T^\mu}{\mu}\right)\|f\|_{L^\infty(0,T)}.\end{aligned}$$

Thus, we have proved the first inequality in (1.10). □

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