Density in $W^{s,p}(\Omega\,;N)$

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Abstract

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $0<s<\infty$ and $1\le p<\infty$. We prove that $C^\infty(\overline{\Omega};\mathbb{S}^1)$ is dense in $W^{s,p}(\Omega;\mathbb{S}^1)$ except when $1\le sp<2$ and $n\ge 2$. The main ingredient is a new approximation method for $W^{s,p}$-maps when $s<1$. With $0<s<1$, $1\le p<\infty$ and $sp<n$, Ω a ball, and N a general compact connected manifold, we prove that $C^\infty(\overline{\Omega};N)$ is dense in $W^{s,p}(\Omega;N)$ if and only if $\pi_{[sp]}(N)=0$. This supplements analogous results obtained by Bethuel when $s=1$, and by Bousquet, Ponce and Van Schaftingen when $s=2,3,\dots$ [General domains Ω have been treated by Hang and Lin when $s=1$; our approach allows to extend their result to $s<1$.] The case where $s>1$, $s\notin\mathbb{N}$, is still open.

1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n\ge 2$. [The questions we will consider are already interesting when Ω is a cube or a ball.] The first topic that we will address is whether $C^\infty(\overline{\Omega};\mathbb{S}^1)$ is dense in $W^{s,p}(\Omega;\mathbb{S}^1)$. Here, $s>0$ and $1\le p<\infty$, and we let

$$W^{s,p}(\Omega;\mathbb{S}^1)=\{u\in W^{s,p}(\Omega;\mathbb{R}^2);\ |u(x)|=1 \text{ a.e.}\};$$

for a set $N\subset\mathbb{R}^m$, we define $W^{s,p}(\Omega;N)$ similarly.

Of special interest to us is the case where $0<s<1$. Recall that in this case a standard norm on $W^{s,p}(\Omega)$ is $u\mapsto \|u\|_{L^p}+|u|_{W^{s,p}}$, where

$$|u|^p_{W^{s,p}}=\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy.$$

When $s>1$ is not an integer, we write $s=m+\sigma$, $m\in\mathbb{N}$, $0<\sigma<1$, and then a standard norm on $W^{s,p}$ is $u\mapsto \|u\|_{L^p}+\|D^m u\|_{W^{\sigma,p}}$.

In this direction, our main result is the following.

Theorem 1. $C^\infty(\overline{\Omega};\mathbb{S}^1)$ is dense in $W^{s,p}(\Omega;\mathbb{S}^1)$ when $sp<1$ or $sp\ge 2$.
If $1\le sp<2$, then $C^\infty(\overline{\Omega};\mathbb{S}^1)$ is not dense in $W^{s,p}(\Omega;\mathbb{S}^1)$.

Many special cases were already known (see the beginning of Section 2), but the case where $n\ge 3$, $s<1$ and $2\le sp<n$ was left open (see [12, Conjecture 2]). This is an interesting and unusual situation where density holds and lifting fails; more precisely, there exists some $u\in W^{s,p}(\Omega;\mathbb{S}^1)$ which cannot be written as $u=e^{\imath\varphi}$ with $\varphi\in W^{s,p}(\Omega;\mathbb{R})$ [5].

The proof of Theorem 1, which is presented in Section 2, relies on a new approximation result, valid only when $0<s<1$, which is discussed below (this is the content of Theorems 5 and 6). This original construction has its own interest and we believe that it might be useful in other contexts. An important feature of Theorem 5 is that it does not use any kind of smoothing or averaging. Hence it is especially appropriate in situations where maps take values into an arbitrary given set – not necessarily a manifold.

Remark 1.1. A completely different proof of Theorem 1 for the case $n \geq 3$, $s < 1$ and $2 \leq sp < n$ can be found in [13]. The main ingredient is the (non trivial) factorization theorem which asserts that each $u \in W^{s,p}(\Omega;\mathbb{S}^1)$ can be written as $u = e^{\imath\varphi}v$, with $\varphi \in W^{s,p}(\Omega;\mathbb{R})$ and $v \in W^{1,sp}(\Omega;\mathbb{S}^1)$ [21], [13].

Remark 1.2. In the case where $1 \leq sp < 2$, the reader may wonder what is the closure of $C^\infty(\overline{\Omega};\mathbb{S}^1)$ into $W^{s,p}$. This question is answered in [13]. Roughly speaking, we are able to define a distributional Jacobian Ju for every $u \in W^{s,p}(\Omega;\mathbb{S}^1)$ with $1 \leq sp < 2$, and then

$$\overline{C^\infty(\overline{\Omega};\mathbb{S}^1)}^{W^{s,p}} = \{u \in W^{s,p}(\Omega;\mathbb{S}^1);\ Ju = 0\}.$$

This is the $\mathbb{S}^1$ fractional counterpart of a result of Bethuel for maps in $H^1(B^3;\mathbb{S}^2)$ [2].

In the range $1 \leq sp < 2$, the substitute of $C^\infty(\overline{\Omega};\mathbb{S}^1)$ for density purposes is the following class, inspired by the important work of Bethuel and Zheng [4] and Bethuel [3]:

$$\mathscr{R}_{s,p} = \{u \in W^{s,p}(\Omega;\mathbb{S}^1);\ u \text{ is smooth outside some finite union of } (n-2)-\text{manifolds}\}.$$

For completeness, we recall the following known result.

Theorem 2. Let $n \geq 2$ and $s > 0$. Assume that $1 \leq sp < 2$. Then $\mathscr{R}_{s,p}$ is dense in $W^{s,p}(\Omega;\mathbb{S}^1)$.

If $s = 1$, Theorem 2 was obtained by Bethuel and Zheng [4] when $n = 2$ and by Bethuel [3] when $n \geq 3$. Other special cases were treated by Hardt, Kinderlehrer and Lin [17] and by Rivière [23]. In [6], Theorem 2 was proved for $s = \frac{1}{2}$ and $p = 2$; the argument in [6] extends readily to the full range $0 < s < 1$, $1 \leq sp < 2$; this is done in [13]. Finally, when $s > 1$ Theorem 2 was established by Bousquet [7].

We next consider the more general situation where the target space $\mathbb{S}^1$ is replaced by a compact connected manifold N without boundary, embedded in $\mathbb{R}^m$. To start with, we prove that when $n = 1$, $C^\infty(\overline{\Omega};N)$ is always dense in $W^{s,p}(\Omega;N)$; see Corollary 3.1. Our main result in Section 3 is a fractional version of a remarkable result of Bethuel [3], which asserts that, when $n \geq 2$ and $1 \leq p < n$, the class

$$\mathscr{R}_{1,p} = \{u \in W^{1,p}(\Omega;N);\ u \text{ is smooth outside some finite union of } (n-[p]-1)-\text{manifolds}\}$$

is dense in $W^{1,p}(\Omega;N)$ (with [ ] denoting the integer part). When $0 < s < 1$, we prove

Theorem 3. Assume that $n \geq 2$, $0 < s < 1$ and $sp < n$. Then

$$\mathscr{R}_{s,p} = \{u \in W^{s,p}(\Omega;N)\ ;\ u \text{ is continuous outside a finite union of } (n-[sp]-1)-\text{ manifolds}\}$$

is dense in $W^{s,p}(\Omega;N)$.

Remark 1.3. Let $n \geq 2$ and $s > 0$. Assume that either $sp < 1$ or $sp \geq n$. Then $C^\infty(\overline{\Omega};N)$ is dense in $W^{s,p}(\Omega;N)$. For the case $sp < 1$, see Section 3.2; the case $sp \geq n$ is handled as in [25], [14]. On the other hand, given any $s > 0$ and $p \geq 1$ such that $1 \leq sp < n$, there exists some manifold N such that $C^\infty(\overline{\Omega};N)$ is not dense in $W^{s,p}(\Omega;N)$; it suffices to take $N = \mathbb{S}^{[sp]}$ and apply Theorem 4 below.

Remark 1.4. With more work, it is possible to improve the conclusion of Theorem 3 by replacing, in the definition of the class $\mathscr{R}_{s,p}$, "u continuous" by "u smooth". This requires a smoothing procedure. Such a procedure with $s = 1$ (in the spirit of the proof of the $H = W$ theorem of Meyers and Serrin) is described in [10]. This can be adapted to arbitrary s, but will not be detailed here.

Remark 1.5. When $1 < p < \infty$ and $s = 1 - \frac{1}{p}$, Theorem 3 was proved by Mucci [22], using a method inspired by Bethuel [3] and completely different from ours. It is not clear whether this kind of method might lead to a proof of Theorem 3.

Recall the following result due to Bethuel [3]: Assume that Ω is a ball (or a cube). For $p < n$, $C^\infty(\overline{\Omega};N)$ is dense in $W^{1,p}(\Omega;N)$ if and only if $\pi_{[p]}(N) = 0$. The extension of this result to $s = 2,3,\dots$ can be found in Bousquet, Ponce and Van Schaftingen [8]. A partial analog in our situation is

Theorem 4. Assume that $0 < s < 1$, $sp < n$ and that Ω is a ball. Then $C^\infty(\overline{\Omega};N)$ is dense in $W^{s,p}(\Omega;N)$ if and only if $\pi_{[sp]}(N) = 0$.

For special target manifolds N, Theorem 4 was obtained by Bousquet, Ponce and Van Schaftingen [9].

When Ω is more complicated, one may still give necessary and sufficient conditions for the density of $C^\infty(\overline{\Omega};N)$ in $W^{s,p}(\Omega;N)$. Indeed, when $s = 1$ such conditions (depending on $[p]$) were discovered by Hang and Lin [16, Theorem 6.3]. The proof of Theorem 4 shows that the same conditions govern the case $s < 1$, provided we replace $[p]$ by $[sp]$.

Two natural questions remain open:

Open Problem 1. Assume that $s > 1$ is not an integer and that $sp < n$. Is it true that $\mathscr{R}_{s,p}$ is dense in $W^{s,p}(\Omega;N)$?

By Theorem 2, the answer is positive when $N = \mathbb{S}^1$. This is also the case when N is arbitrary and $s = 2,3,\dots$ (Bousquet, Ponce and Van Schaftingen [8]). However, the general case is still open even for simple targets such as $N = \mathbb{S}^2$.

Open Problem 2. Assume that $s > 1$ is not an integer, $sp < n$ and that Ω is a ball. Is it true that $C^\infty(\overline{\Omega};N)$ is dense in $W^{s,p}(\Omega;N)$ if and only if $\pi_{[sp]}(N) = 0$?

The main idea for the proof of Theorem 1. We describe here, without proof, the basic tool, namely approximation by piecewise j-homogeneous maps.

For simplicity, we explain our construction first in 3-d. Let $Q = [-1,1]^3$ and let $g : \partial Q \to \mathbb{R}^m$. We may extend g to a map $h : Q \to \mathbb{R}^m$ through the formula $h(x) = g\left(\dfrac{x}{|x|}\right)$, where $|\ |$ stands for the sup norm. The map h is the "homogeneous" extension of g.

Let now K be the 1-dimensional skeleton (=union of edges) of Q and let $g : K \to \mathbb{R}^m$. One may extend g to Q in two steps: first, by homogeneous extension on each face of ∂Q, next by homogeneous extension from ∂Q to Q. This extension will be again called "homogeneous".

Similarly, given a map defined on the 0-skeleton (=union of vertices) of Q, one may extend it in three steps "homogeneously" to Q.

More generally, if K is the j-skeleton of the cube $Q = [-1,1]^n$ and $g : K \to \mathbb{R}^m$, then g has a "homogeneous" extension $h : Q \to \mathbb{R}^m$, obtained in $(n-j)$ steps. Such a map will be called j-homogeneous.

One can also consider the more general situation where the cube is replaced by a finite mesh $\mathscr{C} = \cup_i Q_i$ and extend maps defined on the j-skeleton of $\mathscr{C}$ to "piecewise j-homogeneous" maps on $\mathscr{C}$.

We may now state our main approximation result.

Let $F \subset \mathbb{R}^m$ be an arbitrary set, $0 < s < 1$, $sp < n$. Let Ω, ω be two smooth open bounded subsets of $\mathbb{R}^n$ such that $\overline{\Omega} \subset \omega$ and let $f \in W^{s,p}(\omega;F)$.

Theorem 5. Assume that $0 < s < 1$ and $sp < n$. Let j be an integer such that $[sp] \le j \le n-1$. Then there exists a sequence $\{\mathscr{C}^k\}$ of finite meshes, such that $\overline{\Omega} \subset \mathscr{C}^k \subset \omega$, and a sequence of maps $f_k : \mathscr{C}^k \to F$ such that:

a) Each f_k is piecewise j-homogeneous on $\mathscr{C}^k$, i.e., f_k is the j-homogeneous extension of its restriction to the j-dimensional skeleton $\mathscr{S}^k$ of $\mathscr{C}^k$.

b) Each f_k belongs to $W^{s,p}(\mathscr{C}^k;F)$.

c) $f_k \to f$ in $W^{s,p}(\Omega)$ as $k \to \infty$.

When $j = n-1$, the main ingredient in the proof of Theorem 5 is presented in Section 4; Section 5 treats the case where $j \le n-2$ and contains the proof of Theorem 5.
Of special interest to us will be the case where $j = [sp]$. When sp is not an integer, the restriction of f_k to $\mathscr{S}^k$ is continuous. In particular, each f_k is continuous on $\mathscr{C}^k$ outside some finite union of ℓ-dimensional cubes, with $\ell = n - [sp] - 1$. This need not be the case when sp is an integer.
When F is a compact manifold and $j = [sp]$, Theorem 5 can be considerably improved:

Theorem 6. Assume that $0 < s < 1$, $1 \le sp < n$ and that F is a compact manifold without boundary. Let $j = [sp]$. Then there exist sequences $\{\mathscr{C}^k\}$ and $\{f_k\}$ such that a)-c) hold and, in addition,

d) For each k, the restriction to $\mathscr{S}^k$ of f_k is Lipschitz.

The proof, presented in Section 8, uses tools developed in Sections 6 and 7.

Remark 1.6. We emphasize the fact that these approximation results are specific to the case where $0 < s < 1$. For example, the map $u(x_1, x_2) = x_1$ cannot be approximated in $W^{1,1}((0,1)^2)$ by piecewise 1-homogeneous maps associated to meshes contained in $(-1,2)^2$; see Lemma 4.9 in Section 4. One may extend the argument given there in order to prove that, for any p and j, non constant smooth maps cannot be approximated in $W^{1,p}(\Omega)$ by piecewise j-homogeneous maps associated to meshes contained in ω.

On the other hand, although the use of homogeneous extensions appears already in Bethuel [3] in connection with the approximation of $W^{1,p}$ maps, our method is different from Bethuel's one. His method involves smoothing of u on a set $A \subset \Omega$ such that $\Omega \setminus A$ is small. Homogeneous extensions are used only in $\Omega \setminus A$. In our approach, homogeneous extensions are used everywhere.

The main results of this paper have been mentioned in personal communications starting in 2003 and a sketch of proof can be found in [19] and [20]. Since then, several papers have addressed related questions.

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Contents

2 Proof of Theorem 1 using Theorem 6

We start by presenting more details about the cases already known.

a) Assume that $1 \le sp < 2$ and that $0 \in \Omega \subset \mathbb{R}^2$. Let $u(x) = \dfrac{x}{|x|}$; here, $|\ |$ stands for the Euclidean norm. One may check that $u \in W^{s,p}(\Omega;\mathbb{S}^1)$. Indeed, assume first that $s < 1$. We have $u \in W^{1,q} \cap L^\infty$, for each $q < 2$. To obtain that $u \in W^{s,p}$, we take $sp < q < 2$ and use the Gagliardo-Nirenberg-Sobolev embedding $W^{1,q} \cap L^\infty \subset W^{s,p}$. Assume next that $s > 1$. Since ∇u is homogeneous of degree -1 and smooth outside the origin, we have $\nabla u \in W^{\sigma,q}$ whenever $(1+\sigma)q < 2$; this is obtained by arguing as in [24, proof of Lemma 1 (ii), p. 44]. In particular, $\nabla u \in W^{s-1,p}$, so that $u \in W^{s,p}$.

We claim that there is no sequence $\{u_k\} \subset C^\infty(\overline{\Omega};\mathbb{S}^1)$ such that $u_k \to u$ in $W^{s,p}$. Argue by contradiction as in [25]. Then there is some small $r > 0$ such that, possibly after passing to a subsequence, $u_k \to u$ in $W^{s,p}(C(0,r))$; here, $C(0,r)$ is the circle of radius r centered at the origin.

If $sp > 1$, this implies uniform convergence of u_k to u on $C(0,r)$. Therefore, $\deg(u_k, C(0,r)) \to \deg(u, C(0,r)) = 1$. However, $\deg(u_k, C(0,r)) = 0$ since u_k is smooth in Ω.

When $sp = 1$, convergence need not be uniform anymore. However, we know that $W^{s,p}(C(0,r)) \subset$ VMO with continuous embedding, see e. g. [14]. We conclude as above using the continuity of the degree under BMO convergence [14].

When $\Omega \subset \mathbb{R}^n$, with $n \ge 3$, one argues similarly using the map $u(x) = \dfrac{(x_1, x_2)}{|(x_1, x_2)|}$, $x = (x_1, \dots, x_n)$.

b) Assume that $sp < 1$. Let $u \in W^{s,p}(\Omega;\mathbb{S}^1)$. By [5], one may write $u = e^{\imath\varphi}$, with $\varphi \in W^{s,p}(\Omega;\mathbb{R})$. If $\{\varphi_k\} \subset C^\infty(\overline{\Omega};\mathbb{R})$ converges to φ in $W^{s,p}$, it is immediate that $u_k := e^{\imath\varphi_k} \to u$ in $W^{s,p}$ (see e.g. [6, proof of (5.43)]).

c) Assume that $s \ge 1$ and $sp \ge 2$. Then we may write $u = ve^{\imath\varphi}$, with $v \in C^\infty(\overline{\Omega};\mathbb{S}^1)$ and $\varphi \in W^{s,p} \cap W^{1,sp}(\Omega;\mathbb{R})$ [12, Cases 2 and 3, pp. 128-129]. Let now $\{\varphi_k\} \subset C^\infty(\overline{\Omega};\mathbb{R})$ converge to φ in $W^{s,p} \cap W^{1,sp}$. Then $e^{\imath\varphi_k} \to e^{\imath\varphi}$ in $W^{s,p}$ [11, Theorem 1.1'], which immediately implies that $ve^{\imath\varphi_k} \to ve^{\imath\varphi} = u$ in $W^{s,p}$.

d) Assume that $sp \ge n$. Then density of $C^\infty(\overline{\Omega};\mathbb{S}^1)$ in $W^{s,p}(\Omega;\mathbb{S}^1)$ is well-known [25] via the Sobolev embeddings $W^{s,p} \subset C^0$ when $sp > n$ and $W^{s,p} \subset$VMO when $sp = n$. For further use, we note that density holds also when $\mathbb{S}^1$ is replaced by an arbitrary compact manifold.

We next turn to the case $0 < s < 1$ and $2 \le sp < n$, which is the only one really new.

Proof of Theorem 1. We assume that $0 < s < 1$ and $2 \le sp < n$. Let $q = sp$. Recall the Gagliardo-Nirenberg type embedding [5, Appendix D]

$$W^{1,q} \cap L^\infty \subset W^{s,p} \tag{2.1}$$

(valid since $q > 1$). This embedding is continuous in the sense that

$$\text{if } f_k \to f \text{ in } W^{1,q} \text{ and } \|f_k\|_{L^\infty} \le C, \text{ then } f_k \to f \text{ in } W^{s,p}. \tag{2.2}$$

On the other hand, since $q \ge 2$, a result of Bethuel and Zheng [4] asserts that $C^\infty(\overline{\Omega};\mathbb{S}^1)$ is dense in $W^{1,q}(\Omega;\mathbb{S}^1)$. Combining this with (2.1)-(2.2), we find that

$$W^{1,q}(\Omega;\mathbb{S}^1) = \overline{C^\infty(\overline{\Omega};\mathbb{S}^1)}^{W^{1,q}} \subset \overline{C^\infty(\overline{\Omega};\mathbb{S}^1)}^{W^{s,p}}.$$

Let now $u \in W^{s,p}(\Omega;\mathbb{S}^1)$. We start by extending u to a neighborhood ω of $\overline{\Omega}$; this is achieved via reflections and yields a map $f \in W^{s,p}(\omega;\mathbb{S}^1)$.
We next claim that the maps f_k given by Theorem 6 are in $W^{1,r}$ for each $r < [sp]+1$. In particular, we have $f_k \in W^{1,q}$. To establish this fact we rely on the following

Lemma 2.1. Assume that $n \geq 2$ and let $U \subset \mathbb{R}^n$ be an open set. Let K be a closed subset of U such that $\mathscr{H}^{n-1}(K) = 0$. Let $u \in W^{1,1}_{loc}(U \setminus K)$ be such that $\int_{U\setminus K} |\nabla u| < \infty$. Then $u \in W^{1,1}_{loc}(U)$ and the Sobolev gradient of u is the Sobolev gradient of $u_{|U\setminus K}$.

This result is proved in [12, Lemma 2.15]. [For similar results, see e.g. [18, Lemma 3], [15, Introduction].] We apply this lemma with $U = \mathscr{C}^k$ and $K = \Sigma^k$, the set of discontinuity points of f_k; this is the "dual skeleton" of $\mathscr{S}^k$. Then Σ^k is a finite union of $(n-j-1)$-dimensional cubes, and thus $\mathscr{H}^{n-1}(K) = 0$. On the other hand, a straightforward calculation yields

$$|\nabla f_k(x)| \leq \frac{C_k}{\operatorname{dist}(x,\Sigma^k)}, \ \forall\, x \in \mathscr{C}^k \setminus \Sigma^k, \tag{2.3}$$

and thus $\nabla f_k \in L^r$ when $r < j+1$. We find that $f_k \in W^{1,1}_{loc}(\mathscr{C}^k)$, and actually $f_k \in W^{1,r}(\mathscr{C}^k)$ when $r < j+1$.

Thus

$$W^{s,p}(\Omega;\mathbb{S}^1) \subset \overline{W^{1,q}(\Omega;\mathbb{S}^1)}^{W^{s,p}} \subset \overline{C^\infty(\overline{\Omega};\mathbb{S}^1)}^{W^{s,p}}. \qquad \square$$

3 The case of a general target manifold

Here we will address several questions related to the space $W^{s,p}(\Omega;N)$, where Ω is a smooth bounded domain in $\mathbb{R}^n$ and N is a compact manifold without boundary embedded in $\mathbb{R}^m$.

3.1 Proof of Theorem 3 using Theorems 5 and 6

We start by extending a map $u \in W^{s,p}(\Omega;N)$ to a map $f \in W^{s,p}(\omega;N)$.
If $sp < 1$, then the maps f_k given by Theorem 5 are piecewise constant, and thus in $\mathscr{R}_{s,p}$, and we are done.
Assume next that $sp \geq 1$. Let q be such that $sp < q < [sp]+1$. Note that $1 < q < n$ and that $[q] = [sp]$. As in the proof of Theorem 1, (2.1) and (2.2) hold (since $q > 1$ and $q \geq sp$). Combining (2.1) and (2.2) with Bethuel's density result for the class $\mathscr{R}_{1,q}$ (valid since $q < n$), we find that $\mathscr{R}_{1,q} \subset \mathscr{R}_{s,p}$ and

$$W^{1,q}(\Omega;N) = \overline{\mathscr{R}_{1,q}}^{W^{1,q}} \subset \overline{\mathscr{R}_{1,q}}^{W^{s,p}} \subset \overline{\mathscr{R}_{s,p}}^{W^{s,p}}.$$

Since the maps f_k given by Theorem 6 are in $W^{1,q}$ (this uses the fact that $q < [sp]+1$), we obtain

$$W^{s,p}(\Omega;N) \subset \overline{W^{1,q}(\Omega;N)}^{W^{s,p}} \subset \overline{\mathscr{R}_{s,p}}^{W^{s,p}}. \qquad \square$$

3.2 Proof of Theorem 4 using Theorems 5 and 6

We start with the case $sp < 1$; here, the topological condition is that N is connected, which is satisfied by assumption. As we will see, in this case Ω could be any smooth domain.

If N is a curve, then N is diffeomorphic to $\mathbb{S}^1$, and a straightforward argument reduces the problem to the one of the density of $C^\infty(\overline{\Omega};\mathbb{S}^1)$ in $W^{s,p}(\Omega;\mathbb{S}^1)$, which follows from Theorem 1.
Assume next that $\dim N \geq 2$. Let $u \in W^{s,p}(\Omega;N)$. We first extend it near $\overline{\Omega}$, next we consider

a map f_k as in Theorem 5. It suffices to prove that such a map, which is piecewise constant, can be approximated by smooth N-valued maps. Now f_k assumes ony finitely many values, say $a_1,\dots,a_l$. Let $\Gamma\subset N$ be a smooth simple curve that contains $a_1,\dots,a_l$. Then $f_k\in W^{s,p}(\Omega;\Gamma)$. By our discussion on curves, f_k may be approximated by Γ-valued (thus N-valued) smooth maps.

We now turn to the case $1\le sp<n$.

Condition $\pi_{[sp]}(N)=0$ is necessary. Let $j=[sp]$. Argue by contradiction and let $v\in C^\infty(\mathbb{S}^j;N)$ such that v is not homotopic to a constant. Assume that Ω is the unit ball and let $u:\Omega\to N$, $u(x)=v\left(\frac{(x_1,\dots,x_{j+1})}{|(x_1,\dots,x_{j+1})|}\right)$; here, $|\ |$ stands for the Euclidean norm. It is easy to see that $u\in W^{1,q}$ for each $q<j+1$, and thus $u\in W^{s,p}$. As in the proof of a) in Section 2, the stability of the homotopy class under uniform (or BMO) convergence implies that there is no sequence $\{u_k\}$ of smooth N-valued maps such that $u_k\to u$ in $W^{s,p}$.

Condition $\pi_{[sp]}(N)=0$ is sufficient. It suffices to prove that each map f_k given by Theorem 6 can be approximated by smooth maps. Let q be such that $sp<q<[sp]+1$, so that $[q]=[sp]$. Then $C^\infty(\overline{\Omega};N)$ is dense in $W^{1,q}(\Omega;N)$, since $\pi_{[q]}(N)=0$ and Ω is a ball [3], [16]. The proof of Theorem 3 implies that $C^\infty(\overline{\Omega};N)$ is dense in $W^{s,p}(\Omega;N)$. □

Corollary 3.1. If I is a bounded interval, then $C^\infty(\overline{I};N)$ is dense in $W^{s,p}(I;N)$ for each s and p.

Proof. When $sp<1$, density follows from Theorem 4. When $sp\ge 1$, we are in case d) discussed in Section 2 and we still have density. □

4 Approximation by homogeneous extensions

At the end of Section 5, we will present two proofs of Theorem 5. The first one is quite long, but covers all the possible cases and has the advantage of introducing several calculations which will prove useful in Sections 6-8.

The second proof, much shorter, is valid under the additional assumption $j\ge 1$. It relies on two rather short calculations and on interpolation. While the same strategy could serve to prove some of the auxiliary results in later sections, e.g. Lemma 8.1, it is unclear whether this approach could be used in obtaining Lemmas 6.2 and 7.3, which are at the heart of the proof of Theorem 6. If interpolation could help in obtaining Lemmas 6.2 and 7.3, then this approach would lead to significantly shorter proofs of Theorems 5 and 6.

For the convenience of the reader, the "long proof" of Theorem 5 is split into two parts: this section is devoted to approximation by piecewise $(n-1)$-homogeneous maps. Section 5 treats the case of piecewise j-homogeneous maps, with $j\le n-2$. The proofs of Theorem 5 are presented at the end of Section 5.

Throughout the remaining sections, C will denote a constant depending only on n, s and p. If necessary, we will enhance the dependence on the parameters by denoting $C=C(n,s,p)$, etc.

If $f:\mathbb{R}^n\to\mathbb{R}^m$, one may associate to f a family $\{f_{T,\varepsilon}\}_{T\in\mathbb{R}^n,\varepsilon>0}$ of piecewise $(n-1)$-homogeneous maps as follows: for each $T\in\mathbb{R}^n$, there exists exactly one horizontal (=with faces parallel to the coordinate hyperplanes) mesh of size 2ε having T as one of its centers. [The mesh consists of the cubes $2\varepsilon K+(-\varepsilon,\varepsilon)^n$, with $K\in\mathbb{Z}^n$.] We restrict f to the boundary of this mesh, next extend homogeneously this restriction to the cubes of the mesh. The map obtained by this procedure will be denoted $f_{T,\varepsilon}$ or simply f_T when ε is fixed.
Analytically, $f_{T,\varepsilon}$ is defined as follows: let $|\ |$ denote the sup norm in $\mathbb{R}^n$. For $\varepsilon>0$, let

$$Q_\varepsilon(X)=\{Y\in\mathbb{R}^n\ ;\ |Y-X|<\varepsilon\},\ Q_\varepsilon=Q_\varepsilon(0).$$

For a.e. $X\in\mathbb{R}^n$, there exists a unique $K\in\mathbb{Z}^n$ such that $X\in Q_\varepsilon(T+2\varepsilon K)$. Then

$$f_T(X)=f_{T,\varepsilon}(X)=f\left(T+2\varepsilon K+\varepsilon\frac{X-T-2\varepsilon K}{|X-T-2\varepsilon K|}\right).$$

This section is essentially devoted to the proof of

Lemma 4.1. Let $0 < s < 1, 1 \le p < \infty$ be such that $sp < n$. For each $f \in W^{s,p}(\mathbb{R}^n;\mathbb{R}^m)$ there are sequences $\varepsilon_k \to 0$ and $\{T_k\} \subset \mathbb{R}^n$ such that $f_{T_k,\varepsilon_k} \to f$ in $W^{s,p}(\mathbb{R}^n)$.

Proof. We will establish the following estimate

$$\frac{1}{\varepsilon^n}\int\limits_{Q_\varepsilon} \|f - f_{T,\varepsilon}\|^p_{W^{s,p}} dT \le a(\varepsilon) + b(\varepsilon), \tag{4.1}$$

where

$$a(\varepsilon) \to 0 \text{ as } \varepsilon \to 0 \text{ and } \int\limits_0^1 \frac{b(\varepsilon)}{\varepsilon} d\varepsilon < \infty. \tag{4.2}$$

Assume (4.1) proved for the moment. Then (4.2) implies that, for a sequence $\varepsilon_k \to 0$, we have $a(\varepsilon_k) + b(\varepsilon_k) \to 0$. The conclusion of Lemma 4.1 is then an immediate consequence of (4.1).

We next turn to the proof of (4.1). A warning about notation. The calculations below will involve multiple integrals. In order to make these calculations easier to follow, an integral of the form $\int_{A\times B} f(X,Y)dXdY$ will be denoted $\int_A dX \int_B dY\, f(X,Y)$.

For the convenience of the reader, we split the proof of (4.1) into several steps.

Step 1. We have

$$A := \frac{1}{\varepsilon^n}\int\limits_{Q_\varepsilon} \|f - f_{T,\varepsilon}\|^p_{L^p} dT \to 0 \text{ as } \varepsilon \to 0. \tag{4.3}$$

Indeed, since $(Q_\varepsilon(T + 2\varepsilon K))_{K\in\mathbb{Z}^n}$ is an a.e. partition of $\mathbb{R}^n$ and $f_T = f_{T+2\varepsilon K}$ for $T \in \mathbb{R}^n$ and $K \in \mathbb{Z}^n$, we have

$$\begin{aligned} A =& \frac{1}{\varepsilon^n}\int\limits_{|T|<\varepsilon} dT \sum_{K\in\mathbb{Z}^n} \int\limits_{Q_\varepsilon(T+2\varepsilon K)} |f(X) - f_T(X)|^p dX = \frac{1}{\varepsilon^n}\int\limits_{\mathbb{R}^n} dX \int\limits_{Q_\varepsilon(X)} |f(X) - f_T(X)|^p dT = \\ =& \frac{1}{\varepsilon^n}\int\limits_{\mathbb{R}^n} dX \int\limits_{Q_\varepsilon} \left|f(X) - f\left(X + Y - \varepsilon\frac{Y}{|Y|}\right)\right|^p dY = \frac{1}{\varepsilon^n}\int\limits_{Q_\varepsilon} \left\|f(\cdot) - f\left(\cdot + Y - \varepsilon\frac{Y}{|Y|}\right)\right\|^p_{L^p} dY. \end{aligned} \tag{4.4}$$

We note that $Y \in Q_\varepsilon \implies Y - \varepsilon\dfrac{Y}{|Y|} \in Q_\varepsilon$. Therefore,

$$A \le 2^n \sup\{\|f(\cdot) - f(\cdot + Y)\|^p_{L^p}; |Y| \le \varepsilon\}. \tag{4.5}$$

Inequality (4.5) implies (4.3) and completes Step 1.

In order to complete the proof of Lemma 4.1, it remains to estimate

$$\begin{aligned} B :=& \frac{1}{\varepsilon^n}\int\limits_{Q_\varepsilon} |f - f_T|^p_{W^{s,p}}\, dT \\ =& \frac{1}{\varepsilon^n}\int\limits_{Q_\varepsilon} dT \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} dXdY\, \frac{|[f(X) - f_T(X)] - [f(Y) - f_T(Y)]|^p}{|X - Y|^{n+sp}}. \end{aligned}$$

and more specifically to obtain an upper bound of the form $B \le a(\varepsilon)+b(\varepsilon)$, with a and b as in (4.2).

To this end, we use the inequalities

$$|[f(X)-f_T(X)]-[f(Y)-f_T(Y)]|^p \le \begin{cases} C(|f(X)-f_T(X)|^p+|f(Y)-f_T(Y)|^p), & \text{if } |X-Y|>\varepsilon \\ C(|f(X)-f(Y)|^p+|f_T(X)-f_T(Y)|^p), & \text{if } |X-Y|\le\varepsilon \end{cases}.$$

We find that

$$B \le C(I+J+D), \tag{4.6}$$

where

$$I = \frac{1}{\varepsilon^n}\int\limits_{Q_\varepsilon} dT \iint\limits_{|X-Y|>\varepsilon} dXdY\ \frac{|f(X)-f_T(X)|^p}{|X-Y|^{n+sp}},$$

$$J = \frac{1}{\varepsilon^n}\int\limits_{Q_\varepsilon} dT \iint\limits_{|X-Y|<\varepsilon} dXdY\ \frac{|f(X)-f(Y)|^p}{|X-Y|^{n+sp}}$$

and

$$D = \frac{1}{\varepsilon^n}\int\limits_{Q_\varepsilon} dT \iint\limits_{|X-Y|<\varepsilon} dXdY\ \frac{|f_T(X)-f_T(Y)|^p}{|X-Y|^{n+sp}}. \tag{4.7}$$

Thus our purpose is to establish the estimates

$$I \le a(\varepsilon)+b(\varepsilon),\ J \le a(\varepsilon)+b(\varepsilon),\ D \le a(\varepsilon)+b(\varepsilon), \tag{4.8}$$

with $a(\varepsilon)$ and $b(\varepsilon)$ as in (4.2).

Clearly,

$$J = 2^n \iint\limits_{|X-Y|<\varepsilon} dXdY\ \frac{|f(X)-f(Y)|^p}{|X-Y|^{n+sp}} \to 0 \text{ as } \varepsilon \to 0.$$

Therefore, it remains to estimate I and D.

Step 2. Estimate of I

We have

$$I = \frac{C}{\varepsilon^n}\int\limits_{Q_\varepsilon} dT \int\limits_{\mathbb{R}^n} dX\ \frac{|f(X)-f_T(X)|^p}{\varepsilon^{sp}} = \frac{C}{\varepsilon^{n+sp}}\int\limits_{Q_\varepsilon} dT \int\limits_{\mathbb{R}^n} dX\ |f(X)-f_T(X)|^p.$$

As in the proof of (4.4), we find that

$$I = \frac{C}{\varepsilon^{n+sp}}\int\limits_{\mathbb{R}^n} dX \int\limits_{Q_\varepsilon} dY\ \left|f(X)-f\left(X+Y-\varepsilon\frac{Y}{|Y|}\right)\right|^p.$$

We next introduce a change of variables widely used in what follows. We write $Y=\delta\omega$ (or $Y=r\omega$ or $Y=\lambda\omega$ elsewhere), where $\delta=|Y|=\max\{|Y_1|,\dots,|Y_n|\}$ and $|\omega|=1$. We will denote the new variables δ and ω as polar coordinates.

In polar coordinates, the expression of I becomes

$$\begin{aligned} I &= \frac{C}{\varepsilon^{n+sp}}\int\limits_{\mathbb{R}^n} dX \int\limits_0^\varepsilon \delta^{n-1}d\delta \int\limits_{|\omega|=1} d\omega\ |f(X)-f(X+\delta\omega-\varepsilon\omega)|^p = \\ &= \frac{C}{\varepsilon^{n+sp}}\int\limits_{\mathbb{R}^n} dX \int\limits_0^\varepsilon (\varepsilon-\lambda)^{n-1}d\lambda \int\limits_{|\omega|=1} d\omega\ |f(X)-f(X-\lambda\omega)|^p = \\ &= \frac{C}{\varepsilon^{n+sp}}\int\limits_{\mathbb{R}^n} dX \int\limits_{|Y|<\varepsilon} dY\ \frac{(\varepsilon-|Y|)^{n-1}}{|Y|^{n-1}}\ |f(X)-f(X-Y)|^p. \end{aligned} \tag{4.9}$$

Since clearly

$$\frac{(\varepsilon-|Y|)^{n-1}}{\varepsilon^{n+sp}|Y|^{n-1}} \le \frac{1}{|Y|^{n+sp}} \text{ if } |Y|<\varepsilon\le 1, \tag{4.10}$$

we find that

$$I \le C \iint\limits_{|X-Z|<\varepsilon} dXdZ\ \frac{|f(X)-f(Z)|^p}{|X-Z|^{n+sp}} \to 0 \text{ as } \varepsilon\to 0,$$

and thus I satisfies (4.8).

Step 3. Estimate of D
We start by noting that, if $X\in Q_\varepsilon(T+2\varepsilon K)$ and $|Y-X|<\varepsilon$, then

$$Y \in \bigcup_{\substack{L\in\mathbb{Z}^n\\ |L|\le 1}} Q_\varepsilon(T+2\varepsilon(K+L)).$$

Therefore,

$$D \le \frac{1}{\varepsilon^n}\int\limits_{Q_\varepsilon} dT \sum_{\substack{K,L\in\mathbb{Z}^n\\ |L|\le 1}} \int\limits_{Q_\varepsilon(T+2\varepsilon K)} dX \int\limits_{Q_\varepsilon(T+2\varepsilon(K+L))} dY\ \frac{|f_T(X)-f_T(Y)|^p}{|X-Y|^{n+sp}}.$$

For $L\in\mathbb{Z}^n$, set

$$D_L = \frac{1}{\varepsilon^n}\int\limits_{Q_\varepsilon} dT \sum_{K\in\mathbb{Z}^n} \int\limits_{Q_\varepsilon(T+2\varepsilon K)} dX \int\limits_{Q_\varepsilon(T+2\varepsilon(K+L))} dY\ \frac{|f_T(X)-f_T(Y)|^p}{|X-Y|^{n+sp}}, \tag{4.11}$$

so that

$$D \le \sum_{\substack{L\in\mathbb{Z}^n\\ |L|\le 1}} D_L.$$

We estimate separately each D_L. We consider two cases: $L=0$ and $|L|=1$.

Step 3.1. Estimate of D_0
Since $f_T = f_{T+2\varepsilon K}$, $\forall\, T\in\mathbb{R}^n$, $\forall\, K\in\mathbb{Z}^n$, we have

$$\begin{aligned} D_0 =& \frac{1}{\varepsilon^n}\int\limits_{Q_\varepsilon} dT \sum_{K\in\mathbb{Z}^n} \int\limits_{Q_\varepsilon(T+2\varepsilon K)} dX \int\limits_{Q_\varepsilon(T+2\varepsilon K)} dY\ \frac{|f_T(X)-f_T(Y)|^p}{|X-Y|^{n+sp}} = \\ =& \frac{1}{\varepsilon^n}\int\limits_{\mathbb{R}^n} dU \int\limits_{Q_\varepsilon(U)} dX \int\limits_{Q_\varepsilon(U)} dY\ \frac{|f_U(X)-f_U(Y)|^p}{|X-Y|^{n+sp}}. \end{aligned}$$

In polar coordinates, we obtain

$$D_0 = \frac{1}{\varepsilon^n}\int\limits_{\mathbb{R}^n} dU \int\limits_0^\varepsilon \delta^{n-1}d\delta \int\limits_0^\varepsilon \lambda^{n-1}d\lambda \int\limits_{|\omega|=1} d\omega \int\limits_{|\sigma|=1} d\sigma\ \frac{|f_U(U+\delta\omega)-f_U(U+\lambda\sigma)|^p}{|\delta\omega-\lambda\sigma|^{n+sp}}.$$

Since $f_U(U+\delta\omega)=f(U+\varepsilon\omega)$ and $f_U(U+\lambda\sigma)=f(U+\varepsilon\sigma)$, we find that

$$D_0 = \frac{1}{\varepsilon^n}\int\limits_{\mathbb{R}^n} dU \int\limits_{|\omega|=1} d\omega \int\limits_{|\sigma|=1} d\sigma\ |f(U+\varepsilon\omega)-f(U+\varepsilon\sigma)|^p k(\omega,\sigma), \tag{4.12}$$

where

$$k(\omega,\sigma)=\int_0^{\varepsilon}\delta^{n-1}d\delta\int_0^{\varepsilon}\lambda^{n-1}\,d\lambda\frac{1}{|\delta\omega-\lambda\sigma|^{n+sp}}.$$

In order to complete Step 3.1, we will use

Lemma 4.2. Assume that $sp<n$. Then, for $|\omega|=|\sigma|=1$, we have

$$k(\omega,\sigma)\le\frac{C\varepsilon^{n-sp}}{|\omega-\sigma|^{n+sp-1}}. \tag{4.13}$$

Remark 4.3. In the proof of Lemma 4.1, the condition $sp<n$ is used only to obtain (4.13) and its more general form (4.17).

Proof of Lemma 4.2. We have $k(\omega,\sigma)=\overline{k}(\omega,\sigma)+\tilde{k}(\omega,\sigma)$, where $\overline{k}(\omega,\sigma)=\iint\limits_{\lambda\le\delta}\dots$, $\tilde{k}(\omega,\sigma)=\iint\limits_{\delta\le\lambda}\dots$ We will establish (4.13) for $k(\omega,\sigma)$ replaced by $\overline{k}(\omega,\sigma)$; a similar inequality holds for $\tilde{k}(\omega,\sigma)$. We have

$$\begin{aligned}\overline{k}(\omega,\sigma)&=\int_0^{\varepsilon}\delta^{n-1}d\delta\int_0^{\delta}\lambda^{n-1}d\lambda\,\frac{1}{|\delta\omega-\lambda\sigma|^{n+sp}}=\\&=\int_0^{1}(t\varepsilon)^{n-1}\varepsilon dt\int_0^{1}(t\tau\varepsilon)^{n-1}t\varepsilon d\tau\,\frac{1}{t^{n+sp}\varepsilon^{n+sp}|\omega-\tau\sigma|^{n+sp}}.\end{aligned}$$

Thus,

$$\overline{k}(\omega,\sigma)=\varepsilon^{n-sp}\int_0^{1}t^{n-sp-1}dt\int_0^{1}\tau^{n-1}\,\frac{d\tau}{|\omega-\tau\sigma|^{n+sp}}\le C\varepsilon^{n-sp}\int_0^{1}\tau^{n-1}\,\frac{d\tau}{|\omega-\tau\sigma|^{n+sp}}$$

(here, we use the fact that $sp<n$).

We complete the proof of Lemma 4.2 by establishing the following inequality.

$$F:=\int_0^{1}\tau^{n-1}\frac{d\tau}{|\omega-\tau\sigma|^{n+sp}}\le\frac{C}{|\omega-\sigma|^{n+sp-1}}\ \text{if}\ |\omega|=|\sigma|=1. \tag{4.14}$$

Indeed, if $|\omega-\sigma|\ge\dfrac{1}{20}$, inequality (4.14) is clear, since in this case we have $|\omega-\tau\sigma|\ge C$, for $0\le\tau\le1$.

Let now $|\omega-\sigma|<\dfrac{1}{20}$. We split $F=F_1+F_2$, where $F_1=\int\limits_0^{1-3|\omega-\sigma|}\dots$, $F_2=\int\limits_{1-3|\omega-\sigma|}^{1}\dots$

On the one hand, we have

$$|\omega-\tau\sigma|\ge C|\omega-\sigma|\ \text{if}\ |\omega|=|\sigma|=1\ \text{and}\ \tau\in\mathbb{R}.$$

Therefore,

$$F_2\le\frac{C|\omega-\sigma|}{|\omega-\sigma|^{n+sp}}\le\frac{C}{|\omega-\sigma|^{n+sp-1}}. \tag{4.15}$$

On the other hand, when $0\le\tau\le1$ we have

$$|\omega-\tau\sigma|=|(1-\tau)\omega+\tau(\omega-\sigma)|\ge1-\tau-\tau|\omega-\sigma|=1-\tau(1+|\omega-\sigma|).$$

Thus

$$F_1 \le \int_0^{1-3|\omega-\sigma|} \tau^{n-1}\,\frac{d\tau}{[1-\tau(1+|\omega-\sigma|)]^{n+sp}} = \frac{1}{(1+|\omega-\sigma|)^n}\int_{2|\omega-\sigma|+3|\omega-\sigma|^2}^{1}\frac{(1-t)^{n-1}}{t^{n+sp}}dt$$
$$\le \int_{2|\omega-\sigma|+3|\omega-\sigma|^2}^{1}\frac{dt}{t^{n+sp}} \le \frac{C}{|\omega-\sigma|^{n+sp-1}}. \tag{4.16}$$

We obtain (4.14) when $|\omega-\sigma| < \frac{1}{20}$ combining (4.15) with (4.16).
The proof of Lemma 4.2 is complete. □

Remark 4.4. For further use, we note that the proof of Lemma 4.2 shows that (4.14) holds under more general assumptions on ω and σ. More specifically, if $sp < n$ then we have

$$\int_0^1 \tau^{n-1}\,\frac{d\tau}{|\omega-\tau\sigma|^{n+sp}} \le \frac{C}{|\omega-\sigma|^{n+sp-1}} \text{ if } |\sigma|=1 \text{ and } 1\le|\omega|\le 3. \tag{4.17}$$

Step 3.1 continued. Recall that we want to establish an estimate of the form $D_0 \le a(\varepsilon)+b(\varepsilon)$.
By (4.12) and Lemma 4.2, we have

$$\begin{aligned} D_0 &\le \frac{C}{\varepsilon^{sp}}\int_{\mathbb{R}^n} dU \int_{|\omega|=1} d\omega \int_{|\sigma|=1} d\sigma\,\frac{|f(U+\varepsilon\omega)-f(U+\varepsilon\sigma)|^p}{|\omega-\sigma|^{n+sp-1}} \\ &= \frac{C}{\varepsilon^{n-1}}\int_{\mathbb{R}^n} dU \int_{|\omega|=\varepsilon} d\omega \int_{|\sigma|=\varepsilon} d\sigma\,\frac{|f(U+\omega)-f(U+\sigma)|^p}{|\omega-\sigma|^{n+sp-1}} \\ &= \frac{C}{\varepsilon^{n-1}}\int_{\mathbb{R}^n} dU \int_{|\omega|=\varepsilon} d\omega \int_{|\sigma|=\varepsilon} d\sigma\,\frac{|f(U+\omega-\sigma)-f(U)|^p}{|\omega-\sigma|^{n+sp-1}} \\ &= \frac{C}{\varepsilon^{n-1}}\int_{\mathbb{R}^n} dU \int_{|\omega|=\varepsilon} d\omega \int_{|\lambda-\omega|=\varepsilon} d\lambda\,\frac{|f(U+\lambda)-f(U)|^p}{|\lambda|^{n+sp-1}}. \end{aligned} \tag{4.18}$$

Here is another lemma needed in Step 3.1.

Lemma 4.5. Let $G(\lambda)\ge 0$ be any measurable function. Then

$$H := \int_{|\omega|=\varepsilon} d\omega \int_{|\lambda-\omega|=\varepsilon} d\lambda G(\lambda) \le C\left(\varepsilon^{n-2}H_0 + \varepsilon^{n-1}\sum_{j=1}^{n}\sum_{q=-1}^{1} H_{j,q}\right), \tag{4.19}$$

where

$$H_0 := \int_{|\lambda|\le 2\varepsilon} d\lambda G(\lambda),\ H_{j,q} := \int_{\substack{|\hat\lambda_j|\le 2\varepsilon \\ \lambda_j = 2q\varepsilon}} d\hat\lambda_j G(\lambda).$$

Here, we use the standard notation $\hat\lambda_j = (\lambda_1,\dots,\lambda_{j-1},\lambda_{j+1},\dots,\lambda_n)$.

Proof of Lemma 4.5. We have

$$H = \sum_{j=1}^{n}\sum_{l=1}^{n}\int_{\substack{|\hat\omega_j|\le\varepsilon \\ |\omega_j|=\varepsilon}} d\hat\omega_j \int_{\substack{\widehat{|\lambda-\omega_l|}\le\varepsilon \\ |(\lambda-\omega)_l|=\varepsilon}} d\hat\lambda_l\, G(\lambda) := \sum_{j=1}^{n}\sum_{l=1}^{n} E_{j,l}. \tag{4.20}$$

We first estimate $E_{j,l}$ for $j \neq l$. Assume e.g. $j=1$, $l=n$. Then

$$\begin{aligned} E_{1,n} &= \int\limits_{\substack{|\hat{\omega}_1|\le\varepsilon \\ \omega_1=\pm\varepsilon}} d\hat{\omega}_1 \int\limits_{\substack{|\widehat{\lambda-\omega_n}|\le\varepsilon \\ \lambda_n=\omega_n\pm\varepsilon}} d\hat{\lambda}_n G(\lambda) \\ &\le 2 \int\limits_{|\lambda|\le 2\varepsilon} d\lambda G(\lambda) \int\limits_{|\omega_k|\le\varepsilon,\, \forall\, k\in[\![2,n-1]\!]} d\omega_2 \dots d\omega_{n-1} \le C\varepsilon^{n-2} \int\limits_{|\lambda|\le 2\varepsilon} d\lambda G(\lambda). \end{aligned} \tag{4.21}$$

Let now $j=l$. Assume e.g. $j=l=n$. Since $\omega_n = \pm\varepsilon$ and $\lambda_n = \omega_n \pm \varepsilon$, we have $\lambda_n \in \{-2\varepsilon, 0, 2\varepsilon\}$. Therefore,

$$\begin{aligned} E_{n,n} &= 2 \int\limits_{|\hat{\omega}_n|\le\varepsilon} d\hat{\omega}_n \int\limits_{\substack{|\widehat{\lambda-\omega_n}|\le\varepsilon \\ \lambda_n\in\{0,\pm 2\varepsilon\}}} d\hat{\lambda}_n G(\lambda) \\ &\le C \int\limits_{\substack{|\hat{\lambda}_n|\le 2\varepsilon \\ \lambda_n\in\{0,\pm 2\varepsilon\}}} d\hat{\lambda}_n \int\limits_{|\hat{\omega}_n|\le\varepsilon} d\hat{\omega}_n G(\lambda) = C\varepsilon^{n-1} \sum_{q=-1}^{1} \int\limits_{\substack{|\hat{\lambda}_n|\le 2\varepsilon \\ \lambda_n = 2q\varepsilon}} d\hat{\lambda}_n G(\lambda). \end{aligned} \tag{4.22}$$

Lemma 4.5 follows from (4.20)-(4.22). □

Step 3.1 continued. Recall that we look for an estimate of the form $D_0 \le a(\varepsilon) + b(\varepsilon)$.

By (4.18) and Lemma 4.5 applied with

$$G(\lambda) = G(U,\lambda) = \frac{|f(U+\lambda) - f(U)|^p}{|\lambda|^{n+sp-1}},$$

we find that

$$\begin{aligned} D_0 &\le C\frac{1}{\varepsilon} \int\limits_{\mathbb{R}^n} dU \int\limits_{|\lambda|\le 2\varepsilon} d\lambda\, G(U,\lambda) + C \sum_{j=1}^{n} \sum_{q=-1}^{1} \int\limits_{\mathbb{R}^n} dU \int\limits_{\substack{|\hat{\lambda}_j|\le 2\varepsilon \\ \lambda_j = 2q\varepsilon}} d\hat{\lambda}_j G(U,\lambda) \\ &:= C\Big(P_0 + \sum_{j=1}^{n} \big(P_{j,0} + P_{j,2\varepsilon} + P_{j,-2\varepsilon}\big)\Big). \end{aligned} \tag{4.23}$$

In view of the above, we will establish estimates of the form $P \le a(\varepsilon) + b(\varepsilon)$, where P is one of the P_0, $P_{j,0}$, $P_{j,\pm 2\varepsilon}$.

Estimate of P_0. We have

$$\begin{aligned} \int\limits_0^1 d\varepsilon \frac{P_0}{\varepsilon} &= \int\limits_{\mathbb{R}^n} dU \int\limits_0^1 \frac{d\varepsilon}{\varepsilon^2} \int\limits_{|\lambda|\le 2\varepsilon} d\lambda\, \frac{|f(U+\lambda)-f(U)|^p}{|\lambda|^{n+sp-1}} \\ &= \int\limits_{\mathbb{R}^n} dU \int\limits_{|\lambda|\le 2} d\lambda\, \frac{|f(U+\lambda)-f(U)|^p}{|\lambda|^{n+sp-1}} \int\limits_{|\lambda|/2}^1 \frac{d\varepsilon}{\varepsilon^2} \\ &\le C \int\limits_{\mathbb{R}^n} dU \int\limits_{|\lambda|\le 2} d\lambda\, \frac{|f(U+\lambda)-f(U)|^p}{|\lambda|^{n+sp}} \\ &= C \iint\limits_{|X-Y|<2} dX dY\, \frac{|f(X)-f(Y)|^p}{|X-Y|^{n+sp}} < \infty. \end{aligned} \tag{4.24}$$

Estimate of $P_{j,2\varepsilon}$. (A similar estimate holds for $P_{j,-2\varepsilon}$.) Assume e.g. $j=n$. Then

$$P_{n,2\varepsilon}=\int_{\mathbb{R}^n} dU \int_{|\hat\lambda_n|\le 2\varepsilon} d\hat\lambda_n \, \frac{|f(U+(\hat\lambda_n,2\varepsilon))-f(U)|^p}{(2\varepsilon)^{n+sp-1}},$$

so that

$$\begin{aligned}\int_0^1 d\varepsilon \frac{P_{n,2\varepsilon}}{\varepsilon} &= C\int_0^1 \frac{d\varepsilon}{\varepsilon^{n+sp}} \int_{\mathbb{R}^n} dU \int_{|\hat\lambda_n|\le 2\varepsilon} d\hat\lambda_n \, |f(U+(\hat\lambda_n,2\varepsilon))-f(U)|^p \\ &= C\int_{\mathbb{R}^n} dU \int_{|\lambda|=\lambda_n\le 2} d\lambda \, \frac{|f(U+\lambda)-f(U)|^p}{|\lambda|^{n+sp}} \\ &\le C\int_{\mathbb{R}^n} dU \int_{|\lambda|\le 2} d\lambda \, \frac{|f(U+\lambda)-f(U)|^p}{|\lambda|^{n+sp}} <\infty.\end{aligned} \tag{4.25}$$

Estimate of $P_{j,0}$. Assume $j=n$. Then

$$P_{n,0}=\int_{\mathbb{R}^n} dU \int_{|\hat\lambda_n|\le 2\varepsilon} d\hat\lambda_n \, \frac{|f(U+(\hat\lambda_n,0))-f(U)|^p}{|\hat\lambda_n|^{n+sp-1}}. \tag{4.26}$$

In order to estimate $P_{n,0}$, we rely on a variant of a well-known lemma due to Besov [1, proof of Lemma 7.44, p. 208], more precisely

Lemma 4.6. We have, for $1\le l\le n$,

$$\begin{aligned}R_l &:= \int_{\mathbb{R}^n} dU \int_{|\lambda_k|\le\varepsilon,\,\forall\, k\le l} d\lambda_1\dots d\lambda_l \, \frac{\left|f\left(U+\sum_{k=1}^l \lambda_k e_k\right)-f(U)\right|^p}{|(\lambda_1,\dots,\lambda_l)|^{l+sp}} \\ &\le C\int_{\mathbb{R}^n} dU \int_{|\lambda|\le 2\varepsilon} d\lambda \, \frac{|f(U+\lambda)-f(U)|^p}{|\lambda|^{n+sp}}.\end{aligned} \tag{4.27}$$

The standard form of Lemma 4.6 corresponds corresponds to $l=1$. The proof we present below for arbitrary l is essentially the same as for $l=1$.

Proof of Lemma 4.6. For $\lambda'=(\lambda_1,\dots,\lambda_l)\in\mathbb{R}^l$ and $U\in\mathbb{R}^n$, let $W=W_{\lambda',U}:=U+\sum_{k=1}^j \lambda_k e_k$. Let $Q=Q_{\lambda',U}$ be the cube centered at the midpoint of the segment $[U,W]$ and of sidelength $\frac{1}{4}|\lambda'|$. For any $V\in Q$, we have

$$\left|f\left(U+\sum_{k=1}^l \lambda_k e_k\right)-f(U)\right|^p \le C(|f(V)-f(U)|^p+|f(V)-f(W)|^p). \tag{4.28}$$

By taking the average integral of (4.28) in V over Q , we find that

$$\begin{aligned}R_l &\le C\int_{\mathbb{R}^n} dU \int_{|\lambda'|\le\varepsilon} d\lambda' \int_Q \frac{dV}{|\lambda'|^{l+sp}} \{|f(V)-f(U)|^p+|f(V)-f(W)|^p\} \\ &= 2C\int_{\mathbb{R}^n} dU \int_{|\lambda'|\le\varepsilon} d\lambda' \int_Q dV \, \frac{|f(V)-f(U)|^p}{|\lambda'|^{l+sp+n}}.\end{aligned}$$

Noting that $|V-U| \le 2|\lambda'|$, we obtain

$$R_l \le C \int_{\mathbb{R}^n} dU \int_{|V-U|\le 2\varepsilon} dV\ |f(V)-f(U)|^p \int_{1/2|V-U|\le|\lambda'|\le\varepsilon} \frac{d\lambda'}{|\lambda'|^{l+sp+n}}$$
$$\le C \int_{\mathbb{R}^n} dU \int_{|V-U|\le 2\varepsilon} dV\ \frac{|f(V)-f(U)|^n}{|V-U|^{n+sp}}.$$

The proof of Lemma 4.6 is complete. □

Step 3.1. completed. Lemma 4.6 and (4.26) imply that

$$P_{j,0} \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0,\ \forall\, j \in [\![1,n]\!].$$

Step 3.1 is now complete.

Step 3.2. Estimate of D_L when $L \in \mathbb{Z}^n$ and $|L| = 1$. Similarly to Step 3.1, we will establish an estimate of the form $D_L \le a(\varepsilon) + b(\varepsilon)$.
Recall that

$$D_L = \frac{1}{\varepsilon^n} \int_{Q_\varepsilon} dT \sum_{K\in\mathbb{Z}^n} \int_{Q_\varepsilon(T+2\varepsilon K)} dX \int_{Q_\varepsilon(T+2\varepsilon(K+L))} dY\ \frac{|f_T(X)-f_T(Y)|^p}{|X-Y|^{n+sp}}.$$

If we set $V = V_U = U + 2\varepsilon L$, $X_U = U + \dfrac{X-U}{|X-U|}$ and $Y_V = V + \dfrac{Y-V}{|Y-V|}$, then we have

$$D_L = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} dU \int_{Q_\varepsilon(U)} dX \int_{Q_\varepsilon(V)} dY\ \frac{|f(X_U)-f(Y_V)|^p}{|X-Y|^{n+sp}}.$$

In polar coordinates, we obtain

$$D_L = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} dU \int_0^\varepsilon \delta^{n-1} d\delta \int_0^\varepsilon \lambda^{n-1} d\lambda \int_{|\omega|=1} d\omega \int_{|\sigma|=1} d\sigma\ \frac{|f(U+\varepsilon\omega)-f(V+\varepsilon\sigma)|^p}{|\delta\omega-\lambda\sigma-2\varepsilon L|^{n+sp}}$$
$$= \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} dU \int_{|\omega|=1} d\omega \int_{|\sigma|=1} d\sigma\ |f(U+\varepsilon\omega)-f(V+\varepsilon\sigma)|^p\, k(\omega,\sigma),$$

where

$$k(\omega,\sigma) = \int_0^\varepsilon \delta^{n-1} d\delta \int_0^\varepsilon \lambda^{n-1} d\lambda\ \frac{1}{|\delta\omega-\lambda\sigma-2\varepsilon L|^{n+sp}}.$$

To estimate D_L, we rely on a variant of Lemma 4.2 (which formally corresponds to $L = 0$ in (4.29)).

Lemma 4.7. Assume that $sp < n$. For $|\omega| = |\sigma| = 1$ and $L \in \mathbb{Z}^n$ with $|L| = 1$ we have

$$k(\omega,\sigma) \le \frac{C\varepsilon^{n-sp}}{|\omega-\sigma-2L|^{n+sp-1}}. \tag{4.29}$$

Proof of Lemma 4.7. We have

$$k(\omega,\sigma)=\varepsilon^{n-sp}\int_0^1 t^{n-1}dt\int_0^1 \tau^{n-1}d\tau\,\frac{1}{|t\omega-\tau\sigma-2L|^{n+sp}}. \tag{4.30}$$

We claim that

$$|t\omega-\tau\sigma-2L|\geq C|t\omega-\sigma-2L|. \tag{4.31}$$

Indeed, when $0\leq\tau\leq 1/2$, inequality (4.31) is clear, since in this case we have $|t\omega-\sigma-2L|\leq 4$ and $|t\omega-\tau\sigma-2L|\geq 2-t-\tau\geq 1/2$.
Assume now $\tau\geq 1/2$. We consider the map

$$\varphi:\overline{Q}_1\cup(\overline{Q}_1(2L)\smallsetminus Q_{1/2}(2L))\longrightarrow\mathbb{R}^n,$$

defined by

$$\varphi(X)=\begin{cases}X, & \text{if } X\in\overline{Q}_1\\ 2L+\dfrac{X-2L}{|X-2L|}, & \text{if } X\in\overline{Q}_1(2L)\smallsetminus Q_{1/2}(2L)\end{cases}.$$

It is easy to check that φ is well-defined, in the sense that

$$X=2L+\frac{X-2L}{|X-2L|}\text{ for every }X\in\overline{Q}_1\cap\overline{Q}_1(2L).$$

Note that, in $\overline{Q}_1(2L)\smallsetminus Q_{1/2}(2L)$, φ is the radial projection centered at $2L$ on $\partial Q_1(2L)$. Clearly, φ is Lipschitz. Inequality (4.31) for $1/2\leq\tau\leq 1$ is now obvious, since it reads

$$|\varphi(t\omega)-\varphi(\tau\sigma+2L)|\leq\frac{1}{C}|t\omega-(\tau\sigma+2L)|.$$

Combining (4.30) and (4.31), we obtain

$$k(\omega,\sigma)\leq C\varepsilon^{n-sp}\int_0^1 t^{n-1}\,\frac{dt}{|t\omega-\sigma-2L|^{n+sp}}. \tag{4.32}$$

Applying (4.17) with ω replaced by $\sigma+2L$ and σ replaced by ω (here, we use $sp<n$), we obtain (4.29) from (4.17) and (4.32).

The proof of Lemma 4.7 is complete. □

Step 3.2 continued. We continue our way to an estimate of the form $D_L\leq a(\varepsilon)+b(\varepsilon)$.

By Lemma 4.7 we obtain

$$\begin{aligned}D_L&\leq\frac{C}{\varepsilon^{sp}}\int_{\mathbb{R}^n}dU\int_{|\omega|=1}d\omega\int_{|\sigma|=1}d\sigma\,\frac{|f(U+\varepsilon\omega)-f(U+\varepsilon\sigma+2\varepsilon L)|^p}{|\omega-\sigma-2L|^{n+sp-1}}\\&=\frac{C}{\varepsilon^{n-1}}\int_{\mathbb{R}^n}dU\int_{|\omega|=\varepsilon}d\omega\int_{|\sigma|=\varepsilon}d\sigma\,\frac{|f(U)-f(U+2\varepsilon L+\sigma-\omega)|^p}{|2\varepsilon L+\sigma-\omega|^{n+sp-1}}.\end{aligned}$$

Thus

$$D_L\leq\frac{C}{\varepsilon^{n-1}}\int_{\mathbb{R}^n}dU\int_{|\omega|=\varepsilon}d\omega\int_{|\lambda+\omega-2\varepsilon L|=\varepsilon}d\lambda\,\frac{|f(U)-f(U+\lambda)|^p}{|\lambda|^{n+sp-1}}. \tag{4.33}$$

We combine (4.33) with the following straightforward variant of Lemma 4.5, whose proof is left to the reader:

Lemma 4.8. Let $G(\lambda) \geq 0$ be any measurable function. Then for $L \in \mathbb{Z}^n$ with $|L| = 1$ we have

$$H := \int\limits_{|\omega|=\varepsilon} d\omega \int\limits_{|\lambda+\omega-2\varepsilon L|=\varepsilon} d\lambda\, G(\lambda) \leq C\left(\varepsilon^{n-2} H_0 + \varepsilon^{n-1} \sum_{j=1}^{n} \sum_{q=-2}^{2} H_{j,q}\right), \tag{4.34}$$

where

$$H_0 := \int\limits_{|\lambda|\leq 4\varepsilon} d\lambda\, G(\lambda),\ H_{j,q} := \int\limits_{\substack{|\hat{\lambda}_j|\leq 4\varepsilon \\ \lambda_j = 2q\varepsilon}} d\hat{\lambda}_j\, G(\lambda).$$

Step 3.2 completed. By (4.33) and Lemma 4.8, we obtain

$$\begin{aligned} D_L \leq & \frac{C}{\varepsilon} \int\limits_{\mathbb{R}^n} dU \int\limits_{|\lambda|\leq 4\varepsilon} \frac{|f(U+\lambda)-f(U)|^p}{|\lambda|^{n+sp-1}} d\lambda \\ & + C \sum_{j=1}^{n} \sum_{q=-2}^{2} \int\limits_{\substack{|\hat{\lambda}_j|\leq 4\varepsilon \\ \lambda_j = 2q\varepsilon}} d\hat{\lambda}_j \frac{|f(U+\lambda)-f(U)|^p}{|\lambda|^{n+sp-1}}. \end{aligned} \tag{4.35}$$

Estimate (4.35) is similar to (4.23) and we handle it in the same way.

The proof of Lemma 4.1 is complete. □

We end this section by proving that, in $W^{1,p}$, approximation by piecewise homogeneous maps fails. The special case we treat below ($p = 1$, $n = 2$) is easily generalized to any dimension or p.

Lemma 4.9. Let $u(x_1, x_2) = x_1$. Then there is no sequence $\{u_k\}$ of piecewise 1-homogeneous maps associated to meshes contained in $(-1,2)^2$ such that $u_k \to u$ in $W^{1,1}((0,1)^2)$.

Note that the conclusion of the lemma is that not only the estimates given by Lemma 4.1 do not hold when $s = 1$, but also that any possible approximation method relying on piecewise homogeneous maps fails.

Proof. We argue by contradiction and assume that there exists a sequence $\{u_k\}$ of piecewise 1-homogeneous maps associated to meshes contained in $(-1,2)^2$ such that $u_k \to u$ in $W^{1,1}$. Let u_k be piecewise 1-homogeneous on the mesh $\mathscr{C}^k$, with $(0,1)^2 \subset \mathscr{C}^k \subset (-1,2)^2$. Let $2l_k$ (with $l_k \leq 2$) be the size of the squares in $\mathscr{C}_k$ and set

$$\mathscr{D}_k = \{Q \in \mathscr{C}_k\ ;\ Q \subset (0,1)^2\}.$$

Clearly, there is some $l^0 > 0$ such that

$$\text{if } l_k < l^0, \text{ then } \Big|\bigcup_{Q\in\mathscr{D}_k} Q\Big| \geq 1/2. \tag{4.36}$$

We distinguish two possibilities:

Case 1. $l_k < l^0$

Let S be the center of $Q = Q_{l_k}(S) \in \mathscr{D}_k$. For $X \in Q \setminus \{S\}$, set $V = V(X) = (X-S)/\|X-S\|$; here, $\|\ \|$ stands for the Euclidean norm. Since u_k is constant along the segment $[S,X]$, we have $\dfrac{\partial u_k}{\partial V}(X) = 0$ a.e. in X. Therefore,

$$|(\nabla u_k - \nabla u)(X)| \geq \left|\frac{\partial u_k}{\partial V}(X) - \frac{\partial u}{\partial V}(X)\right| = |V_1|. \tag{4.37}$$

We find that

$$\int_Q |\nabla u_k - \nabla u| \geq \int_Q |V_1| \geq C l_k^2 = C|Q|;$$

the last inequality follows by scaling. Using (4.36), we find that

$$\|u_k - u\|_{W^{1,1}((0,1)^2)} \geq C. \tag{4.38}$$

Thus, for large k, we are in

Case 2. $l_k \geq l^0$
Possibly after passing to a subsequence, we may assume that:

a) $l_k \to l$ for some $l \geq l^0$.

b) All the meshes $\mathscr{C}_k$ contain the same number of squares, say m.

c) The centers of the squares $Q_{1,k},\dots,Q_{m,k}$ in $\mathscr{C}_k$, say $S_{1,k},\dots,S_{m,k}$, converge respectively to $S_1,\dots,S_m$.

Set $Q_j = Q_l(S_j)$. By (4.37) and dominated convergence, we have

$$\lim_k \int_{(0,1)^2} |\nabla(u_k - u)| \geq \lim_k \sum_j \int_{Q_{j,k}\cap(0,1)^2} \frac{|(X - S_{j,k})_1|}{|X - S_{j,k}|} = \sum_j \int_{Q_j\cap(0,1)^2} \frac{|(X - S_j)_1|}{|X - S_j|} > 0.$$

This contradiction completes the proof of Lemma 4.9. □

5 A more general approximation method

The approximation method described in Section 4 goes as follows: fix some $\varepsilon > 0$ and $T \in \mathbb{R}^n$. Consider the mesh $\mathscr{C}_n$ of n-dimensional cubes of sidelength 2ε having T as one of the centers. Let $\mathscr{C}_{n-1}$ be the $(n-1)$-dimensional skeleton associated to this mesh, i.e., $\mathscr{C}_{n-1}$ is the union of the boundaries of the cubes in $\mathscr{C}_n$. Let H_n be the mapping that associates to every $g : \mathscr{C}_{n-1} \to \mathbb{R}^m$ its homogeneous extension (on each cube of $\mathscr{C}_n$) to $\mathbb{R}^n$. Lemma 4.1 asserts that, if $0 < s < 1, sp < n$ and $f \in W^{s,p}(\mathbb{R}^n;\mathbb{R}^m)$, then $H_n(f|_{\mathscr{C}_{n-1}}) \to f$ in $W^{s,p}(\mathbb{R}^n)$ for some suitable choice of $\varepsilon_k \to 0$ and $T_k \in \mathbb{R}^n$.

We will describe below a more general situation. We start by defining the lower dimensional skeletons associated to $\mathscr{C}_n$. This is done by backward induction: $\mathscr{C}_{n-2}$ is the union of the $(n-2)$-dimensional boundaries of the cubes in $\mathscr{C}_{n-1}$, and so on. For $g : \mathscr{C}_j \to \mathbb{R}^m$, let $H_{j+1}(g)$ be its homogeneous extension to $\mathscr{C}_{j+1}$.

Let $0 \leq j < n$. For $\varepsilon > 0$ and $T \in \mathbb{R}^n$, we associate to each map $f : \mathbb{R}^n \to \mathbb{R}^m$ a map $f_T = f_{T,\varepsilon} : \mathbb{R}^n \to \mathbb{R}^m$ through the formula

$$f_T = H_n(H_{n-1}(\cdots(H_{j+1}(f|_{\mathscr{C}_j}))\cdots)).$$

We start by deriving a useful formula for f_T. It will be convenient to denote by X^j the projection of $X \in Q_\varepsilon$ onto the j-skeleton of Q_ε; in particular, we have $X^n = X$. In order to keep notation easier to follow, we will often denote a point in $\mathbb{R}^n$ by X^n rather than X.
Let $K \in \mathbb{Z}^n$ and set $U = T + 2\varepsilon K$. We consider the following a.e. partition of Q_ε:

$$Q_\varepsilon = \bigcup_{q\in\{-1,1\}^{n-j}} \bigcup_{\sigma\in S_{n-j,n}} Q_{\varepsilon,q,\sigma}. \tag{5.1}$$

Here, $S_{n-j,n} = \{\sigma : \{1,\dots,n-j\} \to \{1,\dots,n\};\ \sigma \text{ into }\}$.
A point $X^n \in Q_\varepsilon$ belongs to $Q_{\varepsilon,q,\sigma}$ provided:

a) The $\sigma(i)^{th}$ coordinate of X^n, denoted $(X^n)_{\sigma(i)}$, has the sign of q_i, for $i \in [\![1, n-j]\!]$.

b) In absolute value, the largest coordinate of X^n is $(X^n)_{\sigma(1)}$, the second largest is $(X^n)_{\sigma(2)}$,..., the $(n-j)^{th}$ largest is $(X^n)_{\sigma(n-j)}$.

Analytically, this means that $Q_{\varepsilon,q,\sigma}$ is defined by the inequalities

$$q_1(X^n)_{\sigma(1)} \geq \cdots \geq q_{n-j}(X^n)_{\sigma(n-j)} \geq |(X^n)_k|, \forall\, k \neq \sigma(1), \ldots, \sigma(n-j).$$

Let, for $K \in \mathbb{Z}^n$, $U = T + 2\varepsilon K$. It is easy to check that, for a.e. $X^n \in Q_{\varepsilon,q,\sigma}$, the radial projection of $U + X^n$ onto $\mathscr{C}_{n-1}$ is $U + X^{n-1}$, where

$$(X^{n-1})_{\sigma(1)} = \varepsilon q_1,\ (X^{n-1})_l = \varepsilon \frac{(X^n)_l}{|(X^n)_{\sigma(1)}|},\ \forall\, l \neq \sigma(1).$$

Similary, the projection of $U + X^{n-1}$ onto $\mathscr{C}_{n-2}$ is $U + X^{n-2}$, with

$$(X^{n-2})_{\sigma(1)} = \varepsilon q_1, (X^{n-2})_{\sigma(2)} = \varepsilon q_2,\ (X^{n-2})_l = \varepsilon \frac{(X^n)_l}{|(X^n)_{\sigma(2)}|},\ \forall\, l \neq \sigma(1), \sigma(2),$$

and so on. In particular, we have $f_T(U + X^n) = f(U + X^j)$, $X^n \in Q_{\varepsilon,q,\sigma}$, where

$$(X^j)_{\sigma(k)} = \varepsilon q_{\sigma(k)},\ \forall\, k \in [\![1, n-j]\!],\ (X^j)_l = \varepsilon \frac{(X^n)_l}{|(X^n)_{\sigma(n-j)}|},\ \forall\, l \neq \sigma(1), \ldots, \sigma(n-j).$$

This section is essentially devoted to the proof of the following generalization of Lemma 4.1.

Lemma 5.1. Let $0 \leq j < n, 0 < s < 1, sp < j + 1$ and let $f \in W^{s,p}(\mathbb{R}^n; \mathbb{R}^m)$. Then there are sequences $\varepsilon_k \to 0$ and $\{T_k\} \subset \mathbb{R}^n$ such that $f_{T_k,\varepsilon_k} \to f$ in $W^{s,p}(\mathbb{R}^n)$.

Note that Lemma 4.1 corresponds to $j = n - 1$.

Proof of Lemma 5.1. The proof of Lemma 5.1 is similar to that of Lemma 4.1, some computations being essentially identical. An additional difficulty appears in the estimate of D (for the definition of D, see (4.7)). In order to facilitate the presentation we use the same notation as in Section 4, and follow the steps in Section 4. Let us recall that our goal is to obtain estimates of the form

$$I \leq a(\varepsilon) + b(\varepsilon),\ J \leq a(\varepsilon) + b(\varepsilon),\ D_L \leq a(\varepsilon) + b(\varepsilon),$$

with I, J, D_L analogous to the quantities introduced in the previous section, and $a(\varepsilon)$ and $b(\varepsilon)$ satisfying (4.2).

Step 1. We have

$$A = \frac{1}{\varepsilon^n} \int\limits_{Q_\varepsilon} \|f - f_{T,\varepsilon}\|_{L^p}^p dT \to 0 \text{ as } \varepsilon \to 0. \tag{5.2}$$

Indeed, as in proof of (4.3) we find that

$$A = \frac{1}{\varepsilon^n} \int\limits_{Q_\varepsilon} \|f(\cdot) - f(\cdot + X^n - X^j)\|_{L^p}^p dX^n.$$

Since $X^n \in Q_\varepsilon \Rightarrow X^n - X^j \in Q_\varepsilon$, the argument used in the proof of (4.3) yields (5.2).

Step 2. Estimate of I
In our situation, I is given by

$$I = \frac{C}{\varepsilon^{n+sp}} \int\limits_{\mathbb{R}^n} dU \int\limits_{Q_\varepsilon} dX^n |f(U) - f(U + X^n - X^j)|^p. \tag{5.3}$$

It is convenient to split the integral $\int_{Q_\varepsilon} \dots$ in (5.3) as

$$\int_{Q_\varepsilon} \dots = \sum_{q\in\{-1,1\}^{n-j}} \sum_{\sigma\in S_{n-j,n}} \int_{Q_{\varepsilon,q,\sigma}} \dots$$

We estimate, e.g., the integral $\overline{I}$ corresponding to $q_i = 1$, $\sigma(i) = i$, $\forall\, i \in [\![1, n-j]\!]$, the other terms being similar. If we set $Z := X^j - X^n$, then

$$\overline{I} = \frac{C}{\varepsilon^{n+sp}} \int_{\mathbb{R}^n} dU \int_{\substack{|(X^n)_l| \le (X^n)_{n-j},\, \forall\, l > n-j \\ (X^n)_{n-j} \le (X^n)_{n-j-1} \le \dots \le (X^n)_1 \le \varepsilon}} dX^n\, |f(U) - f(U-Z)|^p$$

and

$$Z_l = \begin{cases} \varepsilon - (X^n)_l, & \text{if } l \le n-j \\ \left(\dfrac{\varepsilon}{(X^n)_{n-j}} - 1\right)(X^n)_l, & \text{if } l > n-j \end{cases}.$$

The following properties are straightforward:

$$0 \le Z_1 \le \dots \le Z_{n-j} \le \varepsilon \ \text{ and } \ |Z_{n-j+1}|, \dots, |Z_n| \le Z_{n-j}, \tag{5.4}$$

$$(X^n)_1 = \varepsilon - Z_1, \dots, (X^n)_{n-j} = \varepsilon - Z_{n-j},$$

$$(X^n)_{n-j+1} = \frac{\varepsilon - Z_{n-j}}{Z_{n-j}} Z_{n-j+1}, \dots, (X^n)_n = \frac{\varepsilon - Z_{n-j}}{Z_{n-j}} Z_n,$$

$$\left|\frac{dX^n}{dZ}\right| = \left(\frac{\varepsilon - Z_{n-j}}{Z_{n-j}}\right)^j.$$

Thus

$$\overline{I} \le \frac{C}{\varepsilon^{n+sp}} \int_{\mathbb{R}^n} dU \int_0^\varepsilon dZ_{n-j} \int_{|\hat{Z}_{n-j}| \le Z_{n-j}} d\hat{Z}_{n-j} \left(\frac{\varepsilon - Z_{n-j}}{Z_{n-j}}\right)^j |f(U) - f(U-Z)|^p. \tag{5.5}$$

Since for any Z satisfying (5.4) we have

$$\frac{1}{\varepsilon^{n+sp}} \left(\frac{\varepsilon - Z_{n-j}}{Z_{n-j}}\right)^j \le \frac{1}{(Z_{n-j})^{n+sp}} = \frac{1}{|Z|^{n+sp}},$$

(5.5) implies that

$$\overline{I} \le C \int_{\mathbb{R}^n} dU \int_{|Z|\le\varepsilon} dZ\, \frac{|f(U) - f(U-Z)|^p}{|Z|^{n+sp}} = C \iint_{|U-W|<\varepsilon} dU\, dW\, \frac{|f(U) - f(W)|^p}{|U-W|^{n+sp}} \to 0 \text{ as } \varepsilon \to 0.$$

Step 3. Estimate of D

With D_L as in (4.11), we have

$$D \le \sum_{\substack{L\in\mathbb{Z}^n \\ |L|\le 1}} D_L.$$

Step 3.1. Estimate of D_0
Recall that

$$D_0 = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} dU \int_{Q_\varepsilon} dX^n \int_{Q_\varepsilon} dY^n \, \frac{|f_U(U+X^n) - f_U(U+Y^n)|^p}{|X^n - Y^n|^{n+sp}}$$
$$= \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} dU \int_{Q_\varepsilon} dX^n \int_{Q_\varepsilon} dY^n \, \frac{|f(U+X^j) - f(U+Y^j)|^p}{|X^n - Y^n|^{n+sp}}.$$

If we take the partition (5.1) of Q_ε into account, we find that

$$D_0 = \sum_{q \in \{-1,1\}^{n-j}} \sum_{\sigma \in S_{n-j,n}} \sum_{r \in \{-1,1\}^{n-j}} \sum_{\tau \in S_{n-j,n}} D_{0,q,\sigma,r,\tau}, \tag{5.6}$$

where

$$D_{0,q,\sigma,r,\tau} = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} dU \int_{Q_{\varepsilon,q,\sigma}} dX^n \int_{Q_{\varepsilon,r,\tau}} dY^n \, \frac{|f(U+X^j) - f(U+Y^j)|^p}{|X^n - Y^n|^{n+sp}}. \tag{5.7}$$

We next consider a convenient parametrization of $Q_{\varepsilon,q,\sigma}$, given by

$$\begin{cases} (X^n)_{\sigma(1)} & = \varepsilon q_1 t_1, \ 0 \le t_1 \le 1 \\ (X^n)_{\sigma(2)} & = \varepsilon q_2 t_1 t_2, \ 0 \le t_2 \le 1 \\ \quad \vdots & \\ (X^n)_{\sigma(n-j)} & = \varepsilon q_{n-j} t_1 t_2 \dots t_{n-j}, \ 0 \le t_{n-j} \le 1 \\ (X^n)_l & = \varepsilon t_1 t_2 \dots t_{n-j} \omega_l, \ |\omega_l| \le 1, \ \forall\, l \ne \sigma(1), \dots, \sigma(n-j) \end{cases}. \tag{5.8}$$

We note that

$$(X^j)_l = \begin{cases} \varepsilon q_{\sigma^{-1}(l)}, & \text{if } l = \sigma(i) \text{ for some } i \\ \varepsilon \omega_l, & \text{else} \end{cases}. \tag{5.9}$$

In particular, X^j depends only on the ω_l's, not on the t_i's; this will be used to give a meaning to (5.10) below.

We consider a similar parametrization of $C_{\varepsilon,r,\tau}$, the t's being replaced by u's and the ω's by λ's. We use the following compact notations:

$$\omega = (\omega_l)_{l \notin \sigma([\![1,n-j]\!])} \text{ and } t = (t_i)_{i \in [\![1,n-j]\!]} \ (\lambda, u \text{ are defined similarly}).$$

Note that X^j depends only on ω, σ and q; similarly, Y^j is expressed in terms of λ, τ and r.
With the convention that $0 \le t \le 1$ stands for $0 \le t_i \le 1$ for each i, we find that

$$D_{0,q,\sigma,r,\tau} = \frac{1}{\varepsilon^{sp}} \int_{\mathbb{R}^n} dU \int_{|\omega| \le 1} d\omega \int_{|\lambda| \le 1} d\lambda \, k(\omega,\lambda) |f(U+X^j) - f(U+Y^j)|^p, \tag{5.10}$$

where

$$k(\omega,\lambda) = \int_{0 \le t \le 1} dt \int_{0 \le u \le 1} du \, t_1^{n-1} \dots t_{n-j}^j u_1^{n-1} \dots u_{n-j}^j \, \frac{\varepsilon^{n+sp}}{|X^n - Y^n|^{n+sp}}.$$

We rely on the following generalization of Lemma 4.2.

Lemma 5.2. Let $0 < s < 1, sp < j+1 \le n$. Then

$$k(\omega,\lambda) \le \frac{C\varepsilon^{j+sp}}{|X^j - Y^j|^{j+sp}}. \tag{5.11}$$

The case $j = n-1$ corresponds to Lemma 4.2.

Proof. We note that inequality (5.11) makes sense, since X^j (respectively Y^j) depends only on ω (respectively λ).
On the other hand, the formula that gives $k(\omega,\lambda)$ does not depend on ε; neither does the r.h.s. of (5.11). Thus, when we estimate $k(\omega,\lambda)$, we may assume that $\varepsilon = 1$.
We proceed by induction on n: assuming that (5.11) holds for all integers $m \le n-1$ and all $j \le m-1$, we prove it for n and each $j \le n-1$. Note that the case $n = 1$ (and $j = 0$) is covered by Lemma 4.2.
Since $X^n = t_1 X^{n-1}$ and $Y^n = u_1 Y^{n-1}$, we have

$$k(\omega,\sigma) = \int\limits_{0\le t\le 1} dt \int\limits_{0\le u\le 1} du\; t_1^{n-1}\dots t_{n-j}^{j} u_1^{n-1}\dots u_{n-j}^{j} \frac{1}{|t_1 X^{n-1} - u_1 Y^{n-1}|^{n+sp}}.$$

Using the fact that $|X^{n-1}| = |Y^{n-1}| = 1$ and Lemma 4.2, we find that

$$\int_0^1 t_1^{n-1} dt_1 \int_0^1 u_1^{n-1} du_1 \frac{1}{|t_1 X^{n-1} - u_1 Y^{n-1}|^{n+sp}} \le \frac{C}{|X^{n-1} - Y^{n-1}|^{n+sp-1}}. \tag{5.12}$$

If $j = n-1$, then (5.12) is the desired inequality. Assume $j < n-1$. Then (5.12) implies that

$$k(\omega,\sigma) \le C \int\limits_{0\le \hat t_1\le 1} d\hat t_1 \int\limits_{0\le \hat u_1\le 1} d\hat u_1\; t_2^{n-2}\dots t_{n-j}^{j} u_2^{n-2}\dots u_{n-j}^{j} \frac{1}{|X^{n-1} - Y^{n-1}|^{n+sp-1}}. \tag{5.13}$$

Next we note that one of the three cases occurs.

Case 1. $\sigma(1) = \tau(1), q_1 = r_1$.
Case 2. $\sigma(1) = \tau(1), q_1 \ne r_1$.
Case 3. $\sigma(1) \ne \tau(1)$.

We will estimate the right-hand side of (5.13) in each of these cases.

Case 1. Assume e.g. $\sigma(1) = \tau(1) = 1, q_1 = r_1 = 1$. In this case, the first coordinate of X^{n-1} or Y^{n-1} is 1, so that

$$|X^{n-1} - Y^{n-1}| = |\widehat{X^{n-1}}_1 - \widehat{Y^{n-1}}_1|.$$

The vectors $\widehat{X^{n-1}}_1$ and $\widehat{Y^{n-1}}_1$ belong to $\mathbb{R}^{n-1}$ and are obtained from ω and λ via (5.8), with an obvious shift in the indices of the coordinates and with n replaced by $n-1$.
Thus, in this case, (5.11) follows from (5.13) and the fact that the conclusion of the lemma holds for $n-1$ and j.

Case 2. In this case, we have $|X^{n-1} - Y^{n-1}| = 2$ and $|X^j - Y^j| = 2$. Inequality (5.11) follows easily from (5.13).

Case 3. With no loss of generality, we may assume $\sigma(1) = 1, \tau(1) = 2, q_1 = 1, r_1 = 1$. Thus

$$X^{n-1} = e_1 + t_2 v,\ v \perp e_1,\ |v| = 1 \tag{5.14}$$

and

$$Y^{n-1} = e_2 + u_2 w,\ w \perp e_2,\ |w| = 1. \tag{5.15}$$

We rely on the fact that (5.14)-(5.15) imply

$$|(e_1+t_2v)-(e_2+u_2w)| \geq C|(e_1+t_2v)-(e_2+w)|,\ 0\leq t_2,u_2\leq 1. \tag{5.16}$$

The proof of this inequality is postponed (see Lemma 5.3 below).
Using (5.16), we obtain

$$\begin{aligned} M :&= \int_0^1 t_2^{n-2}dt_2 \int_0^1 u_2^{n-2}du_2\ \frac{1}{|X^{n-1}-Y^{n-1}|^{n+sp-1}} \\ &\leq C\int_0^1 t_2^{n-2}dt_2 \int_0^1 u_2^{n-2}du_2\ \frac{1}{|(e_1+t_2v)-(e_2+w)|^{n+sp-1}} \\ &\leq C\int_0^1 t_2^{n-2}dt_2\ \frac{1}{|(e_1+t_2v)-(e_2+w)|^{n+sp-1}} \leq \frac{C}{|(e_1+v)-(e_2+w)|^{n+sp-2}}. \end{aligned} \tag{5.17}$$

The last inequality in (5.17) is a consequence of (4.17).[1]

Since $e_1+v=X^{n-2}$ and $e_2+w=Y^{n-2}$, we find that, with $\overline{t}=(t_3,\dots,t_{n-j})$ and $\overline{u}=(u_3,\dots,u_{n-j})$, we have

$$k(\omega,\sigma)\leq C\int_{0\leq\overline{t}\leq 1} dt \prod_{i=3}^{n-j} t_i^{n-i} \int_{0\leq\overline{u}\leq 1} du \prod_{l=3}^{n-j} u_l^{n-l}\ \frac{1}{|X^{n-2}-Y^{n-2}|^{n+sp-2}}.$$

If $j=n-2$, then we are done. Otherwise, we continue as in the estimate of (5.13), distinguishing at each step the three cases mentioned before (and using again the induction assumption when encountering Case 1). At the end of this process, we are led to

$$k(\omega,\lambda)\leq \frac{C}{|X^j-Y^j|^{j+sp}},\ \forall\, j\in[\![0,n-1]\!],$$

assuming the same inequality valid up to $n-1$.

The proof of Lemma 5.2 is complete. □

As promised, we now established (5.16).

Lemma 5.3. If $0\leq t,u\leq 1$ are real numbers, and if $v\perp e_1$, $|v|=1$, $w\perp e_2$, $|w|=1$, then

$$|(e_1+tv)-(e_2+uw)|\geq C|(e_1+tv)-(e_2+w)|. \tag{5.18}$$

Proof of Lemma 5.3. Assume first that $u\leq\dfrac{1}{2}$. Then

$$|e_1+tv-(e_2+uw)|\geq|\langle e_1+tv-(e_2+uw),e_1\rangle|=|1-uw_1|\geq\frac{1}{2}$$

and $|e_1+tv-(e_2+w)|\leq 4$, so that (5.18) is clear in this case.

We next consider the case $u\geq\dfrac{1}{2}$. Consider the following compact subset of ∂Q_1:

$$\mathscr{K}=\{X\in\mathbb{R}^n;|X|=1\}\setminus\{X\in\mathbb{R}^n;X_2=1,|\widehat{X}_2|<1\}.$$

[1] We are in position to apply (4.17) since $sp<n-1$.

Let P be the radial projection centered at e_2 of $Q_1 \setminus \{e_2\}$ onto $\mathscr{K}$.[2] Then

$$P \text{ is Lipschitz in } Q_1 \setminus Q_{1/2}(e_2), \tag{5.19}$$

$$P(e_2+uw) = e_2+w \text{ and } P(e_1+tv) = e_1+tv, \tag{5.20}$$

$$e_1+tv, e_2+uw \in Q_1 \setminus Q_{1/2}(e_2) \text{ if } u \geq 1/2. \tag{5.21}$$

Inequality (5.18), which is equivalent to

$$|P(e_1+tv)-P(e_2+uw)| \leq \frac{1}{C}|(e_1+tv)-(e_2+uw)|,$$

is then a consequence of (5.19) - (5.21).

The proof of Lemma 5.3 is complete. □

Step 3.1 continued. Recall that we want to establish an estimate of the form $D_0 \leq a(\varepsilon)+b(\varepsilon)$.

For this purpose, we start by establishing (5.25), which is the analog of (4.23) adapted to the case of a general j.

By Lemma 5.2 and (5.10), we find that

$$D_{0,q,\sigma,r,\tau} \leq C\varepsilon^j \int_{\mathbb{R}^n} dU \int_{|\omega|\leq 1} d\omega \int_{|\lambda|\leq 1} d\lambda \, \frac{|f(U+X^j)-f(U+Y^j)|^p}{|X^j-Y^j|^{j+sp}} := \overline{D}_{0,q,\sigma,r,\tau} := \overline{D}_0. \tag{5.22}$$

Estimate (5.22) leads to the following:

$$\begin{aligned} \overline{D}_0 &\leq \frac{C}{\varepsilon^{sp}} \int_{\mathbb{R}^n} dU \int_{\substack{|\omega_l|\leq 2, \forall\, l\in(\sigma,\tau)_1 \\ \omega_l = q_{\sigma^{-1}(l)} - r_{\tau^{-1}(l)}, \forall\, l\in(\sigma,\tau)_2}} \bigotimes_{l\in(\sigma,\tau)_1} d\omega_l \, \frac{|f(U+\varepsilon\omega)-f(U)|^p}{|\omega|^{j+sp}} \\ &= C\varepsilon^{n(\sigma,\tau)} \int_{\mathbb{R}^n} dU \int_{\substack{|\omega_l|\leq 2\varepsilon, \forall\, l\in(\sigma,\tau)_1 \\ \omega_l = \varepsilon(q_{\sigma^{-1}(l)} - r_{\tau^{-1}(l)}), \forall\, l\in(\sigma,\tau)_2}} \bigotimes_{l\in(\sigma,\tau)_1} d\omega_l \, \frac{|f(U+\omega)-f(U)|^p}{|\omega|^{j+sp}}. \end{aligned} \tag{5.23}$$

Here,

$$(\sigma,\tau)_2 = \sigma(\{1,\dots,n-j\}) \cap \tau(\{1,\dots,n-j\}) \subset \{1,\dots,n\};$$

$$(\sigma,\tau)_1 = \{1,\dots,n\} \setminus (\sigma,\tau)_2;$$

$$n(\sigma,\tau) = j-n+\#(\sigma,\tau)_2.$$

Indeed, inequality (5.23) is easily proved by noting that

$$(X^j-Y^j)_l = \begin{cases} \varepsilon(q_i-r_m), & \text{if } l=\sigma(i)=\tau(m)\in(\sigma,\tau)_2 \\ \varepsilon(q_i-\lambda_m), & \text{if } l=\sigma(i)\in\sigma(\{1,\dots,n-j\})\setminus(\sigma,\tau)_2 \\ \varepsilon(\omega_l-r_m), & \text{if } l=\tau(m)\in\tau(\{1,\dots,n-j\})\setminus(\sigma,\tau)_2 \\ \varepsilon(\omega_l-\lambda_l), & \text{if } l\notin\sigma(\{1,\dots,n-j\})\cup\tau(\{1,\dots,n-j\}) \end{cases}.$$

For further use, let us prove that

$$n-2j \leq \#(\sigma,\tau)_2 \leq n-j. \tag{5.24}$$

To see this, we note that on the one hand we have

$$\#\sigma(\{1,\dots,n-j\})\cup\tau(\{1,\dots,n-j\}) = 2n-2j-\#(\sigma,\tau)_2 \leq n.$$

[2] P is given by the formula $P(X) = e_2+\tau(X-e_2)$, where τ is the only number ≥ 1 such that $|e_2+\tau(X-e_2)|=1$.

On the other hand, clearly $\#(\sigma,\tau)_2 \le n-j$.

If we insert (5.23) into (5.22) and next take the sum over q,σ,r,τ and use (5.24), we obtain the following analog of (4.23):

$$\begin{aligned} D_0 &\le C \sum_{k=\min(0,n-2j)}^{n-j} \sum_{\substack{A\subset\{1,\dots,n\}\\ \#A=k}} \frac{1}{\varepsilon^{n-j-k}} \int_{\mathbb{R}^n} dU \int_{\substack{\omega_l\in\{0,\pm 2\varepsilon\},\,\forall\, l\in A\\ |\omega_l|\le 2\varepsilon,\,\forall\, l\notin A}} \bigotimes_{l\notin A} d\omega_l \, \frac{|f(U+\omega)-f(U)|^p}{|\omega|^{j+sp}} \\ &:= C\sum_{k,A} D_{0,k,A}. \end{aligned} \tag{5.25}$$

We complete the proof of Step 3.1 by estimating each $D_{0,k,A}$. By symmetry, it suffices to estimate the integrals

$$I_{l,m} = \frac{1}{\varepsilon^{n-j-(l+m)}} \int_{\mathbb{R}^n} dU \int_{\substack{\omega_1=\dots=\omega_l=2\varepsilon\\ \omega_{l+1}=\dots=\omega_{l+m}=0\\ |\omega_k|\le 2\varepsilon,\,\forall\, k>l+m}} d\omega_{l+m+1}\dots d\omega_n \, \frac{|f(U+\omega)-f(U)|^p}{|\omega|^{j+sp}},$$

with $\max\{0,n-2j\} \le l+m \le n-j$.

Case 1. $l=0, m>0$

In this case, we have

$$I_{0,m} = \frac{1}{\varepsilon^{n-j-m}} \int_{\mathbb{R}^n} dU \int_{\substack{\omega_1=\dots=\omega_m=0\\ |\omega_k|\le 2\varepsilon,\,\forall\, k>m}} d\omega_{m+1}\dots d\omega_n \, \frac{|f(U+\omega)-f(U)|^p}{|\omega|^{j+sp}}. \tag{5.26}$$

Case 1.1. $m=n-j$

By Lemma 4.6, we have

$$\begin{aligned} I_{0,n-j} &= \int_{\mathbb{R}^n} dU \int_{\substack{\omega'\in\mathbb{R}^j\\ |\omega'|\le 2\varepsilon}} d\omega' \, \frac{|f(U+(0,\dots 0,\omega'))-f(U)|^p}{|\omega'|^{j+sp}} \le C\int_{\mathbb{R}^n} dU \int_{\substack{\omega\in\mathbb{R}^n\\ |\omega|\le 4\varepsilon}} d\omega \, \frac{|f(U+\omega)-f(U)|^p}{|\omega|^{n+sp}} \\ &= C \iint_{|U-V|<4\varepsilon} dU\,dV \, \frac{|f(U)-f(V)|^p}{|U-V|^{n+sp}} \to 0 \text{ as } \varepsilon\to 0. \end{aligned}$$

Case 1.2. $m<n-j$

Using again Lemma 4.6, we find

$$\begin{aligned} \int_0^1 \frac{I_{0,m}}{\varepsilon} d\varepsilon &= \int_0^1 \frac{d\varepsilon}{\varepsilon^{n+1-j-m}} \int_{\mathbb{R}^n} dU \int_{\substack{\omega'\in\mathbb{R}^{n-m}\\ |\omega'|\le 2\varepsilon}} d\omega' \, \frac{|f(U+(0,\dots,0,\omega'))-f(U)|^p}{|\omega'|^{j+sp}} \\ &\le C\int_{\mathbb{R}^n} dU \int_{\substack{\omega'\in\mathbb{R}^{n-m}\\ |\omega'|\le 2}} d\omega' \, \frac{|f(U+(0,\dots,0,\omega'))-f(U)|^p}{|\omega'|^{n-m+sp}} \\ &\le C\int_{\mathbb{R}^n} dU \int_{\substack{\omega\in\mathbb{R}^n\\ |\omega'|\le 4}} d\omega \, \frac{|f(U+\omega)-f(U)|^p}{|\omega|^{n+sp}} = C\iint_{|U-V|<4} dU\,dV \, \frac{|f(U)-f(V)|^p}{|U-V|^{n+sp}} < \infty. \end{aligned}$$

Case 2. $l>0$
In this case we have $|\omega|=2\varepsilon$. We set $V=U+2\varepsilon e_1+\cdots+2\varepsilon e_l$ and $\omega'=(\omega_k)_{k\in[\![l+m+1,n]\!]}$. Since

$$\begin{aligned}|f(U+\omega)-f(U)|^p \le C(&|f(U+2\varepsilon e_1)-f(U)|^p+|f(U+2\varepsilon e_1+2\varepsilon e_2)-f(U+2\varepsilon e_1)|^p\\&+\cdots+|f(V)-f(U+2\varepsilon e_1+\cdots+2\varepsilon e_{l-1})|^p\\&+|f(V+(0,\dots,0,\omega'))-f(V)|^p),\end{aligned}$$

we find that

$$\begin{aligned}I_{l,m}\le&\frac{C}{\varepsilon^{sp}}\sum_{j=1}^{l}\int_{\mathbb{R}^n}dU\ |f(U+2\varepsilon e_j)-f(U)|^p\\&+\frac{C}{\varepsilon^{n-(l+m)+sp}}\int_{\mathbb{R}^n}dU\int_{\substack{\omega'\in\mathbb{R}^{n-(l+m)}\\|\omega'|\le 2\varepsilon}}d\omega'\ |f(U+(0,\dots,0,\omega'))-f(U)|^p:=C\left(\sum_{j=1}^{l}P_j+P_0\right).\end{aligned}$$

Estimate of P_0. We have

$$\begin{aligned}\int_0^1\frac{P_0}{\varepsilon}d\varepsilon&=\int_0^1\frac{d\varepsilon}{\varepsilon^{n+1-(l+m)+sp}}\int_{\mathbb{R}^n}dU\int_{\substack{\omega'\in\mathbb{R}^{n-(l+m)}\\|\omega'|\le 2\varepsilon}}d\omega'\ |f(U+(0,\dots,0,\omega'))-f(U)|^p\\&\le C\int_{\mathbb{R}^n}dU\int_{\substack{\omega'\in\mathbb{R}^{n-(l+m)}\\|\omega'|\le 2}}d\omega'\ \frac{|f(U+(0,\dots,0,\omega'))-f(U)|^p}{|\omega'|^{n-(l+m)+sp}}\\&\le C\int_{\mathbb{R}^n}dU\int_{\substack{\omega\in\mathbb{R}^n\\|\omega|\le 4}}d\omega\ \frac{|f(U+\omega)-f(U)|^p}{|\omega|^{n+sp}}=C\iint_{|U-V|<4}dUdV\ \frac{|f(U)-f(V)|^p}{|U-V|^{n+sp}}<\infty;\end{aligned}$$

here, we used Lemma 4.6.

Estimate of P_1. (The estimates of $P_2,\dots,P_l$ are similar.) By Lemma 4.6, we have

$$\begin{aligned}\int_0^1\frac{P_1}{\varepsilon}d\varepsilon&=\int_0^1\frac{d\varepsilon}{\varepsilon^{1+sp}}\int_{\mathbb{R}^n}dU\,|f(U+2\varepsilon e_1)-f(U)|^p\\&\le C\int_{\mathbb{R}^n}dU\int_{\substack{\omega\in\mathbb{R}^n\\|\omega|\le 4}}d\omega\ \frac{|f(U+\omega)-f(U)|^p}{|\omega|^{n+sp}}=C\iint_{|U-V|<4}dUdV\ \frac{|f(U)-f(V)|^p}{|U-V|^{n+sp}}<\infty.\end{aligned}$$

Case 3. $l=m=0$
In this case, the inequality

$$\varepsilon^{n-j}|\omega|^{j+sp}\ge 2^{j-n}|\omega|^{n+sp}\ \text{ if }|\omega|\le 2\varepsilon$$

yields

$$\begin{aligned}I_{0,0}&=\frac{1}{\varepsilon^{n-j}}\int_{\mathbb{R}^n}dU\int_{|\omega|\le 2\varepsilon}d\omega\ \frac{|f(U+\omega)-f(U)|^p}{|\omega|^{j+sp}}\le C\int_{\mathbb{R}^n}dU\int_{|\omega|\le 2\varepsilon}d\omega\ \frac{|f(U+\omega)-f(U)|^p}{|\omega|^{n+sp}}\\&=C\iint_{|U-V|\le 2\varepsilon}dUdV\ \frac{|f(U)-f(V)|^p}{|U-V|^{n+sp}}\to 0\ \text{ as }\varepsilon\to 0.\end{aligned}$$

Step 3.1 is complete.

Step 3.2. Estimate of $D_L, L \in \mathbb{Z}^n, |L| = 1$

The proof is essentially the same as for D_0. One has to use instead of Lemma 5.2 its following straightforward variant (which generalizes Lemma 4.7).

Lemma 5.4. Assume that $sp < j+1$. Let $L \in \mathbb{Z}^n$, with $|L| = 1$. Set

$$k(\omega,\lambda) = \int\limits_{0\le t\le 1} dt \int\limits_{0\le u\le 1} du\; t_1^{n-1}\dots t_{n-j}^{j} u_1^{n-1}\dots u_{n-j}^{j}\; \frac{\varepsilon^{n+sp}}{|X^n - (2\varepsilon L + Y^n)|^{n+sp}}.$$

Then

$$k(\omega,\lambda) \le \frac{C\varepsilon^{j+sp}}{|X^j - (2\varepsilon L + Y^j)|^{j+sp}}.$$

Using this lemma, we estimate D_L as in Step 3.2 in Section 4; the details are left to the reader. The proof of Lemma 5.1 is complete. □

Remark 5.5. By Steps 3.1 and 3.2, we have an estimate of the form

$$\sum_{\substack{L\in\mathbb{Z}^n\\ |L|\le 1}} \overline{D}_L \le a(\varepsilon) + b(\varepsilon).$$

Here, $\overline{D}_0$ is as in (5.22), and the quantities $\overline{D}_L$ are defined similarly (this is implicit in Step 3.2). The numbers $a(\varepsilon)$ and $b(\varepsilon)$ satisfy(4.2). If we take a closer look to the averaged estimates leading to the existence of $b(\varepsilon)$ (more specifically, to the estimates of P_0, of P_1, and of $I_{0,m}$ with $m < n-j$), we see that, for a *fixed* ε, there exists some $C(\varepsilon)$ such that

$$\sum_{\substack{L\in\mathbb{Z}^n\\ |L|\le 1}} \overline{D}_L \le C(\varepsilon)|f|^p_{W^{s,p}(\mathbb{R}^n)}. \tag{5.27}$$

In order to justify the above, it suffices to examine e.g. the case of $I_{0,m}$, the other cases being similar. By (5.26) and Lemma 4.6, we have

$$I_{0,m} = \frac{1}{\varepsilon^{n-j-m}} \int\limits_{\mathbb{R}^n} dU \int\limits_{\substack{\omega_1=\dots=\omega_m=0\\ |\omega_k|\le 2\varepsilon,\,\forall\, k>m}} d\omega_{m+1}\dots d\omega_n\; \frac{|f(U+\omega)-f(U)|^p}{|\omega|^{j+sp}} \le \frac{C}{\varepsilon^{n-j-m}}|f|^p_{W^{s,p}(\mathbb{R}^n)}.$$

The conclusion of this remark will be needed in order to complete the proof of Lemma 6.1 below.

We end this section with the

Proof of Theorem 5. Let $g \in W^{s,p}(\mathbb{R}^n;\mathbb{R}^m)$ be an extension of f, not necessarily F-valued. We apply Lemma 5.1 to g. Let $g_k = g_{T_k,\varepsilon_k}$ and let $\mathscr{C}(k)$ be the mesh of size $2\varepsilon_k$ having T_k as one of its centers. We take $f_k = g_{k|\mathscr{C}^k}$, where $\mathscr{C}^k$ is the union of cubes in $\mathscr{C}(k)$ which are contained in ω. Clearly, for large k the maps f_k have all the desired properties. □

Short proof of Theorem 5 when $1 \le sp < n$. We consider the mappings $f \overset{F_\varepsilon}{\longmapsto} F_\varepsilon(f)$, where

$$F_\varepsilon(f) : Q_\varepsilon \times \mathbb{R}^n \to \mathbb{R}^m,\; F_\varepsilon(f)(T,X) = f_{T,\varepsilon}(X).$$

Here, $f_{T,\varepsilon}$ is the piecewise j-homogeneous extension associated to T and ε as in this section.

Step 1. Estimate for $s = 0$
By estimate (4.5) (which holds for an arbitrary j), for $1 \le q < \infty$ we have

$$\|F_\varepsilon(f)\|_{L^q(Q_\varepsilon;L^q(\mathbb{R}^n))} \le C\,\varepsilon^{n/q}\|f\|_{L^q(\mathbb{R}^n)}, \text{ with } C \text{ independent of } \varepsilon. \tag{5.28}$$

Step 2. Estimate for $s = 1$
Let $1 \le r < j+1$. We claim that

$$\|F_\varepsilon(f)\|_{L^r(Q_\varepsilon;W^{1,r}(\mathbb{R}^n))} \le C\,\varepsilon^{n/r}\|f\|_{W^{1,r}(\mathbb{R}^n)}, \text{ with } C \text{ independent of } \varepsilon \text{ or } f. \tag{5.29}$$

In view of Step 1, in order to obtain (5.29) it suffices to establish, with $C = C(n,j,r)$, the estimate

$$\int_{Q_\varepsilon} dT \int_{\mathbb{R}^n} dX\,|\nabla f_T(X)|^r \le C\varepsilon^n \int_{\mathbb{R}^n} dX\,|\nabla f(X)|^r. \tag{5.30}$$

We next observe that it suffices to prove (5.30) when $f \in C_c^\infty$. Indeed, assuming for the moment that (5.30) holds for such f, Step 1 combined with (5.30) for $f \in C_c^\infty$ and with a standard limiting argument implies that (5.30) holds for every $f \in W^{1,r}$.

We finally turn to the proof of (5.30) when $f \in C_c^\infty$. We use the same notation as at the beginning of this section: we set $U = T + 2\varepsilon K$, with $K \in \mathbb{Z}^n$, and we let X^n be a point in Q_ε, whose projection on the j-skeleton of Q_ε is denoted X^j. Set $g_U(X^j) = f(U + X^j)$. Then for a.e. $X^n \in Q_\varepsilon$ we have

$$\nabla f_T(U + X^n) = \nabla g_U(X^j). \tag{5.31}$$

We claim that (5.31) holds also in the sense of distributions. Indeed, let $\mathscr{C}_{\ell,V,\varepsilon}$ denote the ℓ-skeleton obtained from the mesh of cubes of radius ε having V as one of its centers. With this notation, the map f_T is locally Lipschitz in $\mathbb{R}^n \setminus \mathscr{C}_{n-j-1,W,\varepsilon}$, where $W = T + (\varepsilon,\dots,\varepsilon)$. [The skeleton $\mathscr{E} = \mathscr{C}_{n-j-1,W,\varepsilon}$ is the "dual skeleton" of $\mathscr{C}_{j,T,\varepsilon}$.] This observation leads to the validity of (5.31) in the sense of distributions in $\mathbb{R}^n \setminus \mathscr{E}$. On the other hand, as we will see in a moment, we have

$$|\nabla g_U(X^j)| \le C(f,\varepsilon)\frac{1}{\operatorname{dist}(U + X^n, \mathscr{E})}. \tag{5.32}$$

In view of (5.32) and the fact that f is compactly supported, we have

$$\nabla f_T \in L^1(\mathbb{R}^n \setminus \mathscr{E}). \tag{5.33}$$

[Here, we also use the fact that $j \ge 1$ and thus $\mathscr{E}$ is a union of m-planes, with $m = n - j - 1 \le n - 2$.]

In order to obtain (5.31), it then suffices to invoke (5.33) and Lemma 2.1. [Note that this lemma applies to our situation since $j \ge 1$.]

In view of the above, it suffices to prove that

$$\int_{Q_\varepsilon} dT \sum_{K\in\mathbb{Z}^n} \int_{Q_\varepsilon} dX^n\,|\nabla g_{T+2\varepsilon K}(T + 2\varepsilon K + X^j)|^r \le C\varepsilon^n \int_{\mathbb{R}^n} dX\,|\nabla f(X)|^r \tag{5.34}$$

and to obtain, on the way, the estimate (5.32). Splitting, in (5.34), the integral in X^n as a sum over $q \in \{-1,1\}^{n-j}$ and over $\sigma \in S_{n-j,n}$, it suffices, by symmetry, to consider the case where X^n belongs to $Q_{\varepsilon,q,\sigma}$, with

$$q_i = 1,\ \sigma(i) = i,\ \forall\, i \in [\![1, n-j]\!].$$

With q and σ as above, every $X^n \in Q_{\varepsilon,q,\sigma}$ satisfies

$$\varepsilon \ge (X^n)_1 \ge \cdots = (X^n)_{n-j} \ge \max\{|(X^n)_i|;\, n-j+1 \le i \le n\} \tag{5.35}$$

and

$$(X^j)_1 = \cdots = (X^j)_{n-j} = \varepsilon,\ (X^j)_i = \varepsilon \frac{(X^n)_i}{(X^n)_{n-j}},\ \forall\, i \in [\![n-j+1, n]\!]. \tag{5.36}$$

By (5.31), (5.35) and (5.36), for a.e. $X^n \in Q_{\varepsilon,q,\sigma}$ we have

$$|\nabla f_T(U+X^n)| \le C\,\varepsilon \sum_{i=n-j}^{n} \frac{|(X^n)_i|}{[(X^n)_{n-j}]^2} |\nabla f(U+X^j)| \le C \frac{\varepsilon}{(X^n)_{n-j}} |\nabla f(U+X^j)|. \tag{5.37}$$

In particular, (5.37) and the fact that

$$\operatorname{dist}(U+X^n, \mathscr{E}) = (X^n)_{n-j},\ \forall X^n \in Q_{\varepsilon,q,\sigma},$$

lead to (5.32).

In view of (5.37), in order to prove (5.34) it suffices to prove that

$$I = \int_{Q_\varepsilon} dT \sum_{K\in\mathbb{Z}^n} \frac{\varepsilon^r}{[(X^n)_{n-j}]^r} \int_{Q_{\varepsilon,q,\sigma}} dX^n\, |\nabla f(K + 2\varepsilon T + X^j)|^r \le C \int_{\mathbb{R}^n} dX\, |\nabla f(X)|^r. \tag{5.38}$$

We let $X' = (X_1, \ldots, X_{n-j})$ and $Z'' = (Z_{n-j+1}, \ldots, Z_n)$, where

$$Z_i = \varepsilon \frac{(X^n)_i}{(X^n)_{n-j}} \in [-\varepsilon, \varepsilon],\ \forall\, i \in [\![n-j+1, n]\!].$$

We set

$$W = (T_1 + 2\varepsilon K_1, \ldots, T_{n-j} + 2\varepsilon K_{n-j}, T_{n-j+1} + 2\varepsilon K_{n-j+1} + Z_{n-j+1}, \ldots, T_n + 2\varepsilon K_n + Z_n).$$

Then with the change of variables

$$Q_{\varepsilon,q,\sigma} \ni X \mapsto (X', Z''),$$

we have

$$I \le \int_{Q_\varepsilon} dT \sum_{K\in\mathbb{Z}^n} \int_{0<(X^n)_{n-j}\le\cdots\le(X^n)_1\le\varepsilon} dX' \int_{|Z''|<\varepsilon} dZ''\, \varepsilon^{r-j} [(X^n)_{n-j}]^{j-r} |\nabla f(W)|^r. \tag{5.39}$$

If we calculate, in (5.39), the integral with respect to X' and use the assumption $r < j+1$, we find (after summation in K) that

$$\begin{aligned} I &\le C\,\varepsilon^{n-j} \int_{\mathbb{R}^n} dX \int_{|Z''|<\varepsilon} dZ''\, |\nabla f(X_1, \ldots, X_{n-j}, X_{n-j+1} + Z_{n-j+1}, \ldots, X_n + Z_n)|^r \\ &= C\varepsilon^n \int_{\mathbb{R}^n} dX\, |\nabla f(X)|^r. \end{aligned}$$

Step 2 is now completed.

Step 3. Estimate for $0 < s < 1$ (provided $sp \ge 1$ and $sp < j+1$)

Let $0 < s < 1$, $1 \le p < \infty$ and $j \in [\![1, n-1]\!]$ be such that $sp < j+1$. Pick $1 < q < \infty$ and $1 < r < j+1$ such that

$$\frac{1}{p} = \frac{s}{r} + \frac{1-s}{q}. \tag{5.40}$$

This is always possible. Indeed, since $sp < j+1$ we may pick some r such that

$$\max\left\{\frac{1}{j+1}, \frac{1}{sp} - \frac{1}{s} + 1\right\} < \frac{1}{r} < \frac{1}{sp},$$

and for any such r the couple (q,r), with q determined by (5.40), has all the required properties.

We next recall three classical interpolation results. Given two Banach spaces X and Y, we use the standard notation $[X,Y]_{s,p}$; see e.g. [26, Section 1.5]. First, when (5.40) holds we have [26, Section 2.4.2, Theorem 1 (a), eq. (2), p. 185]

$$[W^{1,r}, L^q]_{s,p} = W^{s,p}. \tag{5.41}$$

Next, if X and Y are Banach spaces and s, p, q, r are as above, then [26, Section 1.18.4, Theorem, eq. (3), p. 128]

$$[L^r(\Omega;X), L^q(\Omega;Y)]_{s,p} = L^p(\Omega;[X,Y]_{s,p}). \tag{5.42}$$

By (5.41) and (5.42),

$$\text{with } r,q \text{ as in (5.40), we have } [L^r(Q_\varepsilon;W^{1,r}(\mathbb{R}^n)), L^q(Q_\varepsilon;L^q(\mathbb{R}^n))]_{s,p} = L^p(Q_\varepsilon;W^{s,p}(\mathbb{R}^n)). \tag{5.43}$$

Final classical result. Let s, p, q, r, X and Y be as above. Let F be a linear continuous operator from X into $L^r(\Omega;X)$ and from Y into $L^q(\Omega;Y)$. Then F is linear continuous from $[X,Y]_{s,p}$ into $L^p(\Omega;[X,Y]_{s,p})$ and satisfies the norm inequality

$$\|F\|_{\mathscr{L}([X,Y]_{s,p};L^p(\Omega;[X,Y]_{s,p}))} \le \|F\|^s_{\mathscr{L}(X;L^r(\Omega;X))} \|F\|^{1-s}_{\mathscr{L}(Y;L^q(\Omega;Y))}. \tag{5.44}$$

By (5.28), (5.29) and (5.44), we find that

$$\|F_\varepsilon(f)\|_{L^p(Q_\varepsilon;W^{s,p}(\mathbb{R}^n))} \le C\,\varepsilon^{n/p}\|f\|_{W^{s,p}(\mathbb{R}^n)}, \text{ with } C \text{ independent of } \varepsilon. \tag{5.45}$$

[In principle, the constant C in (5.45) may depend on ε, since we apply the interpolation result (5.43) in an ε-dependent domain. The fact that C does not depend on ε is obtained by a straightforward scaling argument: we consider, instead of F_ε, the map

$$G_\varepsilon(f): Q_1 \times \mathbb{R}^n \to \mathbb{R}^m,\ G_\varepsilon(f)(T,X) = f_{\varepsilon T,\varepsilon}(X).$$

We obtain (5.45) by applying (5.44) to $G_\varepsilon(f)$ in Q_1. Details are left to the reader.]

A clear consequence of (5.45) is

$$\frac{1}{\varepsilon^n}\int_{Q_\varepsilon} \|f_{T,\varepsilon} - f\|^p_{W^{s,p}(\mathbb{R}^n)}\,dT \le C\|f\|^p_{W^{s,p}(\mathbb{R}^n)}. \tag{5.46}$$

In order to complete the proof of Theorem 5, it suffices to obtain (5.47) below.

Step 4. We have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\int_{Q_\varepsilon} \|f_{T,\varepsilon} - f\|^p_{W^{s,p}(\mathbb{R}^n)}\,dT = 0,\ \forall f \in W^{s,p}(\mathbb{R}^n). \tag{5.47}$$

Equation (5.47) is a version of (5.46) and is obtained as follows. We let q, r be as in Step 3.

In view of (5.46), it suffices to prove (5.47) when $f \in C^\infty_c$. For such f, we have $f_{T,\varepsilon} \to f$ uniformly in T when $\varepsilon \to 0$; this leads easily to

$$f_{T,\varepsilon} \to f \text{ in } L^q \text{ uniformly in } T \text{ as } \varepsilon \to 0. \tag{5.48}$$

We obtain (5.47) via (5.48), (5.29) and (5.44). □

6 Restrictions of Sobolev maps to good complexes

Sections 6 to 8 are devoted to the proof of Theorem 6.

The current section is partly inspired by [12, Appendix B, Appendix E]. The results we prove here are fractional Sobolev versions of the following Fubini type result: if $f \in L^1(\mathbb{R}^2)$, then for a.e. $y \in \mathbb{R}$ we have $f(\cdot, y) \in L^1(\mathbb{R})$.

As elsewhere in this paper, we let $0 < s < 1$ and $1 \le p < \infty$, and we let $f \in W^{s,p}(\mathbb{R}^n; \mathbb{R}^m)$.

We use notation consistent with Section 5, but we emphasize dependence of meshes on T by writing, instead of $\mathscr{C}_j$, $\mathscr{C}_{j,T,\varepsilon}$ or (when ε is fixed) $\mathscr{C}_{j,T}$. A generic point of $\mathscr{C}_{j,T}$ is denoted X^j_*, $Y^j_*, \dots$ (instead of $U + X^j$ or $U + Y^j$). Also in order to be consistent with Section 5, the projection of X^n_* on $\mathscr{C}_{j,T}$ is denoted X^j_*. Similarly, if $j \ge 1$, then the projection of X^j_* onto $\mathscr{C}_{j-1,T}$ is denoted X^{j-1}_*; this projection is defined $\mathscr{H}^j$-a.e. on $\mathscr{C}_{j,T}$.

Given a (say Borel and everywhere defined) map $f : \mathbb{R}^n \to \mathbb{R}^m$, an integer $j \in [\![1, n-1]\!]$ and a point $T \in \mathbb{R}^n$, we define the norm

$$\|f\|^p_{W^{s,p}(\mathscr{C}_{j,T})} = \int_{\mathscr{C}_{j,T}} dX^j_* \, |f(X^j_*)|^p + \iint\limits_{\substack{\mathscr{C}_{j,T} \times \mathscr{C}_{j,T} \\ |X^j_* - Y^j_*| < 2\varepsilon}} dX^j_* dY^j_* \, \frac{|f(X^j_*) - f(Y^j_*)|^p}{|X^j_* - Y^j_*|^{j+sp}} = \|f\|^p_{L^p(\mathscr{C}_{j,T})} + |f|^p_{W^{s,p}(\mathscr{C}_{j,T})}.$$

The above definition extends to $j = 0$ by replacing the integrals by sums.

We will prove later in this section two results on slicing, in which ε is fixed.

Lemma 6.1. We have

$$\int_{Q_\varepsilon} \|f\|^p_{W^{s,p}(\mathscr{C}_{j,T})} \, dT \le C(\varepsilon) \|f\|^p_{W^{s,p}}, \ \forall \, j \in [\![0, n-1]\!]. \tag{6.1}$$

Lemma 6.2. We have

$$\int_{Q_\varepsilon} dT \int_{\mathscr{C}_{j,T}} dX^j_* \, \frac{|f(X^j_*) - f(X^{j-1}_*)|^p}{|X^j_* - X^{j-1}_*|^{sp}} \le C(\varepsilon) \|f\|^p_{W^{s,p}}, \ \forall \, j \in [\![1, n-1]\!]. \tag{6.2}$$

For $j \in [\![1, n-1]\!]$, we define an ad hoc space $\mathscr{W}^{s,p}_j = \mathscr{W}^{s,p}_{j,T,\varepsilon}$ as follows: $\mathscr{W}^{s,p}_j$ consists of the functions $g : \mathscr{C}_{j,T} \to \mathbb{R}^m$ such that

$$\|g\|_{W^{s,p}(\mathscr{C}_{\ell,T})} < \infty, \ \forall \, \ell \in [\![1, j]\!] \tag{6.3}$$

and

$$\int_{\mathscr{C}_{\ell,T}} dX^\ell_* \, \frac{|f(X^\ell_*) - f(X^{\ell-1}_*)|^p}{|X^\ell_* - X^{\ell-1}_*|^{sp}} < \infty, \ \forall \, \ell \in [\![1, j]\!]. \tag{6.4}$$

The definition of $\mathscr{W}^{s,p}_j$ is inspired by Hang and Lin [16, Section 3]. Clearly, if $f \in \mathscr{W}^{s,p}_{j,T,\varepsilon}$ and $\ell \le j$, then the restriction of f to an ℓ-dimensional cube C of the mesh $\mathscr{C}_{\ell,T,\varepsilon}$ belongs to $W^{s,p}(C)$. As we will see in the next section, the space $\mathscr{W}^{s,p}_j$ has additional properties that will be useful in the proof of Lemma 7.1.

Let s, p be such that $1 \le sp < n$. Let j be an integer such that $sp < j + 1 \le n$. For such j, we consider $f_{T,\varepsilon}$ as in Section 5. Combining Lemmas 6.1 and 6.2 with the fact that, by the proof of Lemma 5.1, there exists a sequence $\varepsilon_k \to 0$ such that

$$\frac{1}{(\varepsilon_k)^n} \int_{Q_{\varepsilon_k}} \|f - f_{T,\varepsilon_k}\|^p_{W^{s,p}} \, dT \to 0 \text{ as } k \to \infty,$$

we obtain the following

Corollary 6.3. Let s, p, j be such that $1 \le sp < j+1 \le n$. Let $f \in W^{s,p}(\mathbb{R}^n;\mathbb{R}^m)$ be a Borel function. Then there exist sequences $\varepsilon_k \to 0$ and $\{T_k\} \subset \mathbb{R}^n$ such that:
1. The restriction f^k of f to $\mathscr{C}_{j,T_k,\varepsilon_k}$ belongs to $\mathscr{W}^{s,p}_{j,T_k,\varepsilon_k}$, $\forall\, k$.
2. If f_k is the j-homogeneous extension of f^k, then $f_k \to f$ in $W^{s,p}$ as $k \to \infty$.

The remaining part of this section is devoted to the proofs of Lemmas 6.1 and 6.2.

A word about the proofs. Many of the calculations we need in Sections 6–8 are quite close to the ones in Section 5. For such calculations, we point to the analog formulas in Section 5 and omit part of details.

We will use the same notation as in Section 5, and more specifically as in Step 3.1 in the proof of Lemma 5.1; see on the one hand (5.6) and (5.7), and on the other hand (5.8) and the derivation of (5.10) starting from (5.8).

Proof of Lemma 6.1. **Step 1.** Averaged estimate of $\|f\|^p_{L^p(\mathscr{C}_{j,T})}$
We establish here the identity

$$\int\limits_{Q_\varepsilon} \|f\|^p_{L^p(\mathscr{C}_{j,T})}\, dT = C(n,j)\,\varepsilon^j\, \|f\|^p_{L^p(\mathbb{R}^n)}. \tag{6.5}$$

Indeed, arguing as in the proof of (5.6) and (5.10) and with X^j as in (5.9), we have

$$\int\limits_{Q_\varepsilon} \|f\|^p_{L^p(\mathscr{C}_{j,T})}\, dT = 2^{j-n} \sum_{q\in\{-1,1\}^{n-j}} \sum_{\sigma\in S_{n-j,n}} \varepsilon^j \int\limits_{|\omega|\le 1} d\omega \int\limits_{\mathbb{R}^n} dU\, |f(U+X^j)|^p. \tag{6.6}$$

[The constant 2^{j-n} comes from the fact that on the right-hand side of (6.6) the integral over a j-dimensional cube C of $\mathscr{C}_{j,T}$ is counted 2^{n-j} times.]

In order to obtain (6.5), it suffices to observe that the last integral in (6.6) does not depend on ω.

Step 2. Averaged estimate of $|f|^p_{W^{s,p}(\mathscr{C}_{j,T})}$
We have

$$|f|^p_{W^{s,p}(\mathscr{C}_{j,T})} = \iint\limits_{\substack{(X^j_*,Y^j_*)\in\mathscr{C}_{j,T}\times\mathscr{C}_{j,T}\\ |X^j_*-Y^j_*|<2\varepsilon}} dX^j_*\, dY^j_*\, \frac{|f(X^j_*)-f(Y^j_*)|^p}{|X^j_*-Y^j_*|^{j+sp}} = I_1(T)+I_2(T), \tag{6.7}$$

where

$$I_1(T) = \iint\limits_{|X^j_*-Y^j_*|<\varepsilon} \ldots,\quad I_2(T) = \iint\limits_{\varepsilon\le|X^j_*-Y^j_*|<2\varepsilon} \ldots$$

We first note that

$$I_2(T) \le C \int\limits_{\mathscr{C}_{j,T}} dX^j_* \int\limits_{\varepsilon\le|X^j_*-Y^j_*|<2\varepsilon} dY^j_*\, \frac{1}{|Y^j_*-X^j_*|^{j+sp}}\, |f(X^j_*)|^p \le C \int\limits_{\mathscr{C}_{j,T}} dX^j_*\, |f(X^j_*)|^p, \tag{6.8}$$

since

$$\int\limits_{\varepsilon\le|X^j_*-Y^j_*|<2\varepsilon} dY^j_*\, \frac{1}{|Y^j_*-X^j_*|^{j+sp}} \le C < \infty,\ \forall\, T,\ \forall\, X^j_*.$$

By (6.8) and Step 1, we have

$$\int_{Q_\varepsilon} I_2(T)\,dT \le C\|f\|^p_{L^p(\mathbb{R}^n)}. \tag{6.9}$$

We next note that (with notation as in (5.9) and (5.10))

$$I_1(T) \le C\,\varepsilon^{2j} \mathop{\sum\nolimits^{\circ}}_{} \int_{|\omega|\le 1} d\omega \int_{|\lambda|\le 1} d\lambda \frac{|f(T+2\varepsilon K+X^j)-f(T+2\varepsilon K+2\varepsilon L+Y^j)|^p}{|X^j-(2\varepsilon L+Y^j)|^{j+sp}}, \tag{6.10}$$

where

$$\mathop{\sum\nolimits^{\circ}}_{} = \sum_{\substack{L\in\mathbb{Z}^n\\ |L|\le 1}} \sum_{K\in\mathbb{Z}^n} \sum_{q,r\in\{-1,1\}^{n-j}} \sum_{\sigma,\tau\in S_{n-j,n}}.$$

Integrating (6.10), we find that

$$\int_{Q_\varepsilon} I_1(T)\,dT \le C\,\varepsilon^{2j} \sum_{\circ} \int_{|\omega|\le 1} d\omega \int_{|\lambda|\le 1} d\lambda \int_{\mathbb{R}^n} dU \frac{|f(U+X^j)-f(U+2\varepsilon L+Y^j)|^p}{|X^j-(2\varepsilon L+Y^j)|^{j+sp}}, \tag{6.11}$$

with

$$\sum_{\circ} = \sum_{\substack{L\in\mathbb{Z}^n\\ |L|\le 1}} \sum_{q,r\in\{-1,1\}^{n-j}} \sum_{\sigma,\tau\in S_{n-j,n}}.$$

By (6.11) and estimate (5.27) in Remark 5.5, we have

$$\int_{Q_\varepsilon} I_1(T)\,dT \le C(\varepsilon)|f|^p_{W^{s,p}(\mathbb{R}^n)}. \tag{6.12}$$

We complete the proof of Lemma 6.1 using (6.7), (6.9) and (6.12). □

Proof of Lemma 6.2. **Step 1.** A dimensional reduction
Assume for the moment that we proved the following estimate (with X^{n-1}_* the projection of X^n_* onto $\mathscr{C}_{n-1,T,\varepsilon}$):

$$I = \int_{Q_\varepsilon} dT \int_{\mathbb{R}^n} dX^n_* \frac{|f(X^n_*)-f(X^{n-1}_*)|^p}{|X^n_*-X^{n-1}_*|^{sp}} \le C(n,\varepsilon) \iint_{|X^n_*-Y^n_*|<\varepsilon} dX^n_* dY^n_* \frac{|f(X^n_*)-f(Y^n_*)|^p}{|X^n_*-Y^n_*|^{n+sp}}. \tag{6.13}$$

Then we claim that the conclusion of the lemma holds. Indeed, if $j \in [\![1,n-1]\!]$ then (6.13) applied with $n=j$ and with $\mathbb{R}^n$ replaced by the intersection of $\mathscr{C}_{j,T}$ with the j-dimensional plane

$$\{(x_1,\dots,x_n);\, x_l = T_l + 2\varepsilon K_l, \forall\, l\in I\},\ \text{with } \#I = n-j \text{ and } K_l\in\mathbb{Z}$$

leads (after the use of the Fubini theorem in the variables T_l with $l\notin I$ and summation in I) to

$$\int_{Q_\varepsilon} dT \int_{\mathscr{C}_{j,T}} dX^j_* \frac{|f(X^j_*)-f(X^{j-1}_*)|^p}{|X^j_*-X^{j-1}_*|^{sp}} \le C(n,j,\varepsilon) \int_{Q_\varepsilon} dT \iint_{\substack{(X^j_*,Y^j_*)\in\mathscr{C}_{j,T}\times\mathscr{C}_{j,T}\\ |X^j_*-Y^j_*|<\varepsilon}} dX^j_* dY^j_* \frac{|f(X^j_*)-f(Y^j_*)|^p}{|X^j_*-Y^j_*|^{j+sp}}. \tag{6.14}$$

We then obtain the conclusion of Lemma 6.2 using (6.14) and Lemma 6.1.

Step 2. Proof of (6.13)
We follow Step 2 in the proof of Lemma 4.1 in Section 4. Following the calculation (4.9), the left-hand side I of (6.13) satisfies

$$I = \int_{\mathbb{R}^n} dX \int_{|Y|<\varepsilon} dY \, \frac{(\varepsilon - |Y|)^{n-1}}{|Y|^{n+sp-1}} |f(X) - f(X-Y)|^p. \tag{6.15}$$

We obtain (6.13) by noting that

$$\frac{(\varepsilon - |Y|)^{n-1}}{|Y|^{n+sp-1}} \le C(\varepsilon) \frac{1}{|Y|^{n+sp}} \text{ if } |Y| < \varepsilon. \qquad \square$$

7 Approximation of maps defined on good skeletons

Throughout the next two sections, we take $0 < s < 1$, $1 \le p < \infty$, $j \in [\![1, n-1]\!]$ and we use the same notation as in Sections 5 and 6. We consider a fixed finite submesh $\mathscr{C}$ of $\mathscr{C}_n$ and a map $g : \mathscr{C}_j \cap \mathscr{C} \to \mathbb{R}^m$. For such maps, we define the norm

$$\|g\|_{L^p}^p = \|g\|_{L^p(\mathscr{C}_j \cap \mathscr{C})}^p = \int_{\mathscr{C}_j \cap \mathscr{C}} dX_*^j \, |g(X_*^j)|^p$$

and the semi-norm

$$|g|_{W^{s,p}}^p = |g|_{W^{s,p}(\mathscr{C}_j \cap \mathscr{C})}^p = \int_{\mathscr{C}_j \cap \mathscr{C}} dX_*^j \int_{\mathscr{C}_j \cap \mathscr{C}} dY_*^j \, \frac{|g(X_*^j) - g(Y_*^j)|^p}{|X_*^j - Y_*^j|^{j+sp}}.$$

With the natural definition, we also consider the space $\mathscr{W}_j^{s,p} = \mathscr{W}_j^{s,p}(\mathscr{C}_j \cap \mathscr{C})$.

In this section, we adapt to the fractional Sobolev case some approximation techniques of maps defined on skeletons devised by Hang and Lin [16, Section 3]. The main result is the following

Lemma 7.1. Let $0 < s < 1$, $1 \le p < \infty$ and $j \in \mathbb{N}$ be such that $1 \le j \le sp < n$. Let N be a compact manifold without boundary embedded in $\mathbb{R}^m$. Let $g \in \mathscr{W}_j^{s,p}(\mathscr{C}_j \cap \mathscr{C}; N)$. Then there exists a sequence $\{g^k\} \subset \mathrm{Lip}(\mathscr{C}_j \cap \mathscr{C}; N)$ such that $g^k \to g$ in $W^{s,p}(\mathscr{C}_j \cap \mathscr{C})$.

Two difficulties arise in the proof of Lemma 7.1. The first one is to show that $\mathbb{R}^m$-valued maps g in $\mathscr{W}_j^{s,p}$ can be approximated by Lipschitz maps. This is already a non trivial task. An additional difficulty occurs when g is N-valued. In this case, we have to prove approximation with N-valued Lipschitz maps.

It will be convenient to start by reducing Lemma 7.1 to a slightly easier to prove statement.

Lemma 7.2. Let $0 < s < 1$, $1 \le p < \infty$ and $j \in \mathbb{N}$ be such that $1 \le j \le sp < n$. Let N be a compact manifold without boundary embedded in $\mathbb{R}^m$. Let $\delta > 0$ be sufficiently small and define

$$M = \{x \in \mathbb{R}^m ; \mathrm{dist}(x, N) \le \delta\}. \tag{7.1}$$

Let $g \in \mathscr{W}_j^{s,p}(\mathscr{C}_j \cap \mathscr{C}; N)$. Then there exists a sequence $\{G^k\} \subset \mathrm{Lip}(\mathscr{C}_j \cap \mathscr{C}; M)$ such that $G^k \to g$ in $W^{s,p}(\mathscr{C}_j \cap \mathscr{C})$.

Lemma 7.2 implies Lemma 7.1. Let $\Pi : M \to N$ denote the nearest point projection. Let $g^k = \Pi(G^k)$. We note that $g = \Pi(g)$, and that g^k is clearly Lipschitz. In order to conclude, it suffices to invoke the continuity of the map

$$W^{s,p}(\mathscr{C}_j \cap \mathscr{C}; M) \ni G \mapsto \Pi(G) \in W^{s,p}(\mathscr{C}_j \cap \mathscr{C}; N).$$

This is standard for maps in smooth domains; see e.g. [5, Proof of (5.43), p. 56] for a slightly more general continuity result. The argument in [5] adapts readily to maps defined on $\mathscr{C}_j \cap \mathscr{C}$. $\square$

We next turn our attention to the proof of Lemma 7.2. Since $\mathscr{C}$ and j are fixed, we will simplify the notation and omit "$\mathscr{C}_j \cap \mathscr{C}$" in the norms and function spaces. With no loss of generality, we may assume that $\varepsilon = 1$. For the convenience of the reader, we start by stating the main technical ingredients required in the proof of Lemma 7.2. Before proceeding, let us define "a cube in $\mathscr{C}_\ell$" (or "an ℓ-dimensional cube in $\mathscr{C}_\ell$") by backward induction as follows. A cube in $\mathscr{C}_n$ is any cube of the mesh $\mathscr{C}_n$. A cube in $\mathscr{C}_{n-1}$ is any of the 2^n faces of a cube in $\mathscr{C}_n$. For $\ell \le n-2$, a cube in $\mathscr{C}_\ell$ is any of the $2^{\ell+1}$ faces of any cube in $\mathscr{C}_{\ell+1}$.

Let $g : \mathscr{C}_j \cap \mathscr{C} \to \mathbb{R}^m$. For $\mathfrak{C}$ a j-dimensional cube in $\mathscr{C}_j \cap \mathscr{C}$, we let $0_{\mathfrak{C}}$ be its center. Clearly, if $X^j_* \in \mathfrak{C}$, then the projection X^{j-1}_* of X^j_* on $\mathscr{C}_{j-1} \cap \mathscr{C}$ is

$$X^{j-1}_* = 0_{\mathfrak{C}} + \frac{X^j_* - 0_{\mathfrak{C}}}{|X^j_* - 0_{\mathfrak{C}}|}.$$

We now define a convenient approximation g_μ of g. For $0 < \mu < 1$ and $X^j \in \mathfrak{C}$, we set

$$g_\mu(X^j_*) = \begin{cases} g(X^{j-1}_*), & \text{if } |X^j_* - 0_{\mathfrak{C}}| \ge 1-\mu \\ g\left(0_{\mathfrak{C}} + \dfrac{X^j_* - 0_{\mathfrak{C}}}{1-\mu}\right), & \text{if } |X^j_* - 0_{\mathfrak{C}}| < 1-\mu \end{cases}.$$

This definition is inspired by the "filling a hole" technique of Brezis and Li [10]. See also [16, Lemma 3.1] and, in the context of fractional spaces, [12, Appendix D].

We have the following result, whose proof is postponed to the end of this section.

Lemma 7.3. Let $g \in \mathscr{W}^{s,p}_j$. Then $g_\mu \to g$ in $W^{s,p}$ as $\mu \to 0$.

[Here, we do not require $j \le sp$.]

Let $\rho \in C^\infty_c(Q)$ (with Q the unit cube in $\mathbb{R}^j$) be a standard mollifier and set

$$\rho_t(x) = \frac{1}{t^j}\,\rho(x/t), \ \forall\, t > 0, \ \forall\, x \in \mathbb{R}^j.$$

Fix some function $\eta \in C^\infty_c([0,1);[0,1])$. We let $\mathfrak{C}$ and $0_{\mathfrak{C}}$ be as above. Given $g : \mathscr{C}_j \cap \mathscr{C} \to \mathbb{R}^m$, we define, with a slight abuse of notation and after identifying the j-plane containing $\mathfrak{C}$ with $\mathbb{R}^j$,

$$g * \rho_t(X^j_*) = \int_{\mathfrak{C}} dY^j_*\, g(Y^j_*)\rho_t(X^j_* - Y^j_*) \ \text{ for } X^j_* \in \mathfrak{C} \text{ such that } |X^j_* - 0_{\mathfrak{C}}| < 1 - t. \tag{7.2}$$

We note that for small t the quantity

$$g^t(X^j_*) = \eta(|X^j_* - 0_{\mathfrak{C}}|)\, g * \rho_t(X^j_*)$$

is well-defined in $\mathscr{C}_j \cap \mathscr{C}$. We also let

$$g^0(X^j_*) = \eta(|X^j_* - 0_{\mathfrak{C}}|)\, g(X^j_*).$$

We now state a standard result on the approximation by smoothing in fractional Sobolev spaces, whose straightforward proof is left to the reader.

Lemma 7.4. Let $g \in W^{s,p}$. Then $g^t \to g^0$ in $W^{s,p}$ as $t \to 0$.

[Here, we do not require $j \le sp$.]

We next present another auxiliary result, which is a rather easy consequence of Lemma 7.10 (which is fully proved below) and whose proof (granted Lemma 7.10) is left to the reader. Given $f : \mathscr{C}_{j-1} \cap \mathscr{C}$, we consider its homogeneous extension g to $\mathscr{C}_j \cap \mathscr{C}$. Let η be as above. We assume in addition that $\eta = 1$ near the origin. This implies that the map

$$\mathscr{C}_j \cap \mathscr{C} \ni X^j_* \mapsto h(X^j_*) = \Big(1 - \eta(|X^j_* - 0_{\mathfrak{C}}|)\Big)\, g(X^j_*)$$

is well-defined in each point.

Lemma 7.5. The mapping $f \mapsto h$ is continuous from $W^{s,p}(\mathscr{C}_{j-1} \cap \mathscr{C})$ into $W^{s,p}(\mathscr{C}_j \cap \mathscr{C})$.

[Here, we do not require $j \leq sp$.]

The final auxiliary result is deeper, and was essentially observed by Schoen and Uhlenbeck [25]. For the fractional version we present below, see [14, Example 2, p. 210, and eqn (7), p. 206]. The argument in [14] (where maps are defined in domains) adapts readily to the case of maps defined on skeletons.

Lemma 7.6. Let $0 < s < 1$ and $1 \leq p < \infty$ be such that $sp < n$. Let $j \in \mathbb{N}$ be such that $1 \leq j \leq sp$. Let $g \in W^{s,p}(\mathscr{C}_j \cap \mathscr{C}; N)$. Let $0 < t < 1$ and let $\delta > 0$ be arbitrarily small (but fixed). Let M be as in (7.1) and $g * \rho_t$ be as in (7.2). Then, for sufficiently small t, we have

$$g * \rho_t(X^j_*) \in M, \ \forall X^j_* \in \mathscr{C}_j \cap \mathscr{C} \text{ such that } \operatorname{dist}(X^j_*, \mathscr{C}_{j-1} \cap \mathscr{C}) > t. \tag{7.3}$$

[Here, we do require $j \leq sp$.]

Proof of Lemma 7.2 using Lemmas 7.3–7.6. The proof relies on two ingredients: approximation of maps as in Lemma 7.3 and induction on j.

Step 1. Proof of the lemma for $j = 1$
By Lemma 7.3, it suffices to prove the lemma when g is replaced by g_μ. Since $j = 1$ and thus $\mathscr{C}_0 \cap \mathscr{C}$ is a finite collection of points, this simply means that we may assume that g is constant near each point in $\mathscr{C}_0 \cap \mathscr{C}$: there exists some $\mu > 0$ such that

$$g(X^1_*) = g(X^0_*) \text{ if } \operatorname{dist}(X^1_*, \mathscr{C}_0 \cap \mathscr{C}) \leq \mu. \tag{7.4}$$

Let now $\eta \in C^\infty([0,1];[0,1])$ be such that

$$\eta(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 - \mu/2 \\ 0, & \text{if } x > 1 - \mu/3 \end{cases}.$$

When $0 < t < \mu/3$, the map

$$X^1_* \mapsto G^t(X^1_*) = \eta(|X^1_* - 0_{\mathfrak{C}}|)\, g * \rho_t(X^1_*) + \left(1 - \eta(|X^1_* - 0_{\mathfrak{C}}|)\right) g(X^0_*)$$

is well-defined everywhere, and is clearly Lipschitz. Moreover, by Lemma 7.4 and the choice of η, we have

$$G^t \to g \text{ in } W^{s,p} \text{ as } t \to 0.$$

It remains to prove that, for small t, we have

$$G^t(X^1_*) \in M, \ \forall X^1_* \in \mathscr{C}_1 \cap \mathscr{C}. \tag{7.5}$$

By Lemma 7.6, property (7.5) holds when $|X^1_* - 0_{\mathfrak{C}}| \leq 1 - \mu/2$. Clearly, (7.5) holds also when $|X^1_* - 0_{\mathfrak{C}}| \geq 1 - \mu/3$. Finally, when $1 - \mu/2 < |X^1_* - 0_{\mathfrak{C}}| < 1 - \mu/3$ and $t < \mu/3$, we have

$$G^t(X^1_*) = g * \rho_t(X^1_*)$$

(this is easily checked using (7.4)). We complete this case using again Lemma 7.6.

Step 2. Proof of the lemma for $j \geq 2$
Let f be the restriction of g to $\mathscr{C}_{j-1} \cap \mathscr{C}$. By Lemma 7.3, we may assume that there exists some $\mu \in (0,1)$ such that

$$g(X^j_*) = f(X^{j-1}_*), \ \forall \mathfrak{C} \subset \mathscr{C}_j \cap \mathscr{C}, \forall X^j_* \in \mathfrak{C} \text{ such that } |X^j_* - 0_{\mathfrak{C}}| > 1 - \mu. \tag{7.6}$$

We argue by induction on j. By the induction hypothesis and the reduction of Lemma 7.1 to Lemma 7.2, the map f (which clearly belongs to $\mathscr{W}^{s,p}_{j-1}(\mathscr{C}_{j-1}\cap\mathscr{C};N)$) is the limit in $W^{s,p}$ of a sequence $\{F^k\}\subset \mathrm{Lip}(\mathscr{C}_{j-1}\cap\mathscr{C};N)$. With η as in Step 1 and $0<t<\mu/3$, we define the Lipschitz maps

$$X^j_*\mapsto G^{k,t}(X^j_*)=\eta(|X^j_*-0_{\mathfrak{C}}|)\,g*\rho_t(X^j_*)+\Big(1-\eta(|X^j_*-0_{\mathfrak{C}}|)\Big)\,F^k(X^{j-1}_*).$$

By Lemmas 7.4 and 7.5, we have

$$\lim_{k\to\infty}\lim_{t\searrow 0}G^{k,t}=g \text{ in } W^{s,p}.$$

In order to complete Step 2 it remains to prove that, for large k and sufficiently small t (possibly depending on k) we have

$$G^{k,t}(X^j_*)\in M,\ \forall\, X^j_*\in\mathscr{C}_j\cap\mathscr{C}. \tag{7.7}$$

As in Step 1, (7.7) holds when $|X^j_*-0_{\mathfrak{C}}|\le 1-\mu/2$ or $|X^j_*-0_{\mathfrak{C}}|\ge 1-\mu/3$. When $1-\mu/2<|X^j_*-0_{\mathfrak{C}}|<1-\mu/3$, we argue as follows. Since $j\le sp$, we have $j-1<sp$. By the Sobolev embeddings, f and F^k are continuous and we have $F^k\to f$ uniformly. Let k_0 be such that

$$\|F^k-f\|_{L^\infty}\le\delta/2,\ \forall\, k\ge k_0. \tag{7.8}$$

By (7.6) and the continuity of f, for every fixed k we have

$$\begin{aligned}&\lim_{t\searrow 0}G^{k,t}(X^j_*)=\eta(|X^j_*-0_{\mathfrak{C}}|)\,f(X^{j-1}_*)+\Big(1-\eta(|X^j_*-0_{\mathfrak{C}}|)\Big)\,F^k(X^{j-1}_*)\\ &\text{uniformly in the set}\bigcup_{\mathfrak{C}\subset\mathscr{C}_j\cap\mathscr{C}}\{X^j_*\in\mathfrak{C};\,1-\mu/2<|X^j_*-0_{\mathfrak{C}}|<1-\mu/3\}.\end{aligned} \tag{7.9}$$

We complete the proof of (7.7) using (7.8) and (7.9). □

In order to complete the proof of Lemma 7.2, it remains to proceed to the

Proof of Lemma 7.3. We may assume that $\varepsilon=1$ and that $T=0$. We set

$$\Omega=\mathscr{C}_j\cap\mathscr{C},\ E=E_\mu=\{X^j_*\in\Omega;\,|X^j_*-X^{j-1}_*|>\mu\},\ F=F_\mu=\{X^j_*\in\Omega;\,|X^j_*-X^{j-1}_*|<\mu\}. \tag{7.10}$$

If $\mathfrak{C}$ is a cube in $\mathscr{C}_j$ and $0<\mu<\mu_0<1$, then we define

$$\mathfrak{C}_\mu=\{X^j_*\in\mathfrak{C};\,|X^j_*-X^{j-1}_*|>\mu\},\ \mathfrak{C}_{\mu,\mu_0}=\{X^j_*\in\mathfrak{C};\,\mu<|X^j_*-X^{j-1}_*|<\mu_0\},\ \mathfrak{C}^c_\mu=\mathfrak{C}\setminus\mathfrak{C}_\mu. \tag{7.11}$$

If $\mathfrak{C}'$ is another cube in $\mathscr{C}_j$, we define similarly $\mathfrak{C}'_\mu$, etc.
We clearly have

$$g_\mu\to g \text{ in } L^p \text{ as } \mu\to 0. \tag{7.12}$$

[For a more general property, see (7.15) below.]
It thus remains to prove that

$$I=\iint\limits_{\Omega\times\Omega}dX^j_*dY^j_*\,\frac{|[g_\mu(X^j_*)-g(X^j_*)]-[g_\mu(Y^j_*)-g(Y^j_*)]|^p}{|X^j_*-Y^j_*|^{j+sp}}\to 0 \text{ as } \mu\to 0. \tag{7.13}$$

We split

$$I=I_{E,E}+2I_{E,F}+I_{F,F},\ \text{where } I_{A,B}=I_{A,B,\mu}=\iint\limits_{A\times B}\cdots$$

We have to prove that $I_{E,E} \to 0$, $I_{E,F} \to 0$ and $I_{F,F} \to 0$ as $\mu \to 0$.

Step 1. For every cube $\mathfrak{C}$ in $\mathscr{C}_j \cap \mathscr{C}$ we have $I_{\mathfrak{C},\mathfrak{C}} \to 0$ as $\mu \to 0$
Indeed, we may assume that $\mathfrak{C}$ is open, and then we identify $\mathfrak{C}$ with the unit cube $Q_1 \subset \mathbb{R}^j$. We let $\mathfrak{C}^* = Q_2$ denote the double of $\mathfrak{C}$, and set

$$h(X_*^j) = \begin{cases} g(X_*^j), & \text{if } X_*^j \in \mathfrak{C} \\ g(X_*^{j-1}), & \text{if } X_*^j \in \mathfrak{C}^* \setminus \mathfrak{C} \end{cases}.$$

Lemma 7.7. We have $h \in W^{s,p}(\mathfrak{C}^*)$.

Proof of Lemma 7.7. Clearly, since $g \in \mathscr{W}_j^{s,p}$, we have $h \in W^{s,p}(\mathfrak{C})$ and $h \in W^{s,p}(\mathfrak{C}^* \setminus \overline{\mathfrak{C}})$. It thus suffices to prove that $h \in W^{s,p}$ near each point of $\partial\mathfrak{C}$. After a bi-Lipschitz change of variables, and taking the definition of $\mathscr{W}_j^{s,p}$ into account, we are then reduced to the following lemma, established in [12, Appendix B, Lemma B.1]. □

Lemma 7.8. Let $0 < s < 1$, $1 \le p < \infty$, $u \in W^{s,p}((0,1)^j)$ and $v \in W^{s,p}((0,1)^{j-1})$ be such that

$$\int\limits_{(0,1)^j} dX_1 \dots dX_j \frac{|u(X_1,\dots,X_j) - v(X_1,\dots,X_{j-1})|^p}{X_j^{sp}} < \infty.$$

Define

$$w(X_1,\dots,X_j) = \begin{cases} u(X_1,\dots,X_j), & \text{if } (X_1,\dots,X_j) \in (0,1)^j \\ v(X_1,\dots,X_{j-1}), & \text{if } (X_1,\dots,X_j) \in (0,1)^{j-1} \times (-1,0] \end{cases}.$$

Then $w \in W^{s,p}((0,1)^{j-1} \times (-1,1))$.

[In the statement of Lemma B.1 in [12] it is assumed that $1 < p < \infty$, but the argument there still holds for $p = 1$.]

Step 1 completed. We may extend h to a map, still denoted h, in $W^{s,p}(\mathbb{R}^j)$. Define

$$h^t(X_*^j) = h(X_*^j/t), \ \forall X_*^j \in \mathbb{R}^j, \ \forall t > 0. \tag{7.14}$$

Note that

$$\text{if } h \in W^{\sigma,p} \text{ for some } \sigma \ge 0, \text{ then the mapping } t \mapsto h^t \in W^{\sigma,p} \text{ defined by (7.14) is continuous.} \tag{7.15}$$

Using (7.15) with $\sigma = s$, we obtain that

$$I_{\mathfrak{C},\mathfrak{C}} = \iint\limits_{\mathfrak{C}\times\mathfrak{C}} dX_*^j dY_*^j \frac{\left|\left[h^{1-\mu}(X_*^j) - h^1(X_*^j)\right] - \left[h^{1-\mu}(Y_*^j) - h^1(Y_*^j)\right]\right|^p}{|X_*^j - Y_*^j|^{j+sp}} \le \left|h^{1-\mu} - h^1\right|^p_{W^{s,p}(\mathbb{R}^j)} \to 0 \text{ as } \mu \to 0.$$

Step 2. For every cube $\mathfrak{C}$ in $\mathscr{C}_j \cap \mathscr{C}$ and for every fixed $\mu_0 \in (0,1)$ we have $I_{\mathfrak{C},E_{\mu_0}} \to 0$ as $\mu \to 0$
Indeed, by Step 1 it suffices to prove that for every cube $\mathfrak{C}' \ne \mathfrak{C}$ in $\mathscr{C}_j$ we have, with $\mathfrak{C}'_{\mu_0}$ as in (7.11),

$$I_{\mathfrak{C},\mathfrak{C}'_{\mu_0}} \to 0 \text{ as } \mu \to 0. \tag{7.16}$$

We note that

$$|X_*^j - Y_*^j| \ge \mu_0, \ \forall X_*^j \in \mathfrak{C}, \ \forall Y_*^j \in \mathfrak{C}'_{\mu_0}. \tag{7.17}$$

By (7.17), we have

$$I_{\mathfrak{C},\mathfrak{C}'_{\mu_0}} \le C(n,p,\mu_0) \left(\int\limits_{\mathfrak{C}} dX_*^j \, |g_\mu(X_*^j) - g(X_*^j)|^p + \int\limits_{\mathfrak{C}'_{\mu_0}} dY_*^j \, |g_\mu(Y_*^j) - g(Y_*^j)|^p \right). \tag{7.18}$$

We obtain (7.16) using (7.18) and (7.12).

Step 3. We have $I_{E,E} \to 0$ as $\mu \to 0$
By Steps 1 and 2, Step 3 amounts to the following. Let $\xi > 0$ be fixed arbitrarily small. Let $\mathfrak{C} \neq \mathfrak{C}'$ be two cubes in $\mathscr{C}_j$. Then there exists some $0 < \mu_0 < 1$ such that

$$I_{\mathfrak{C}_{\mu,\mu_0},\mathfrak{C}'_{\mu,\mu_0}} < \xi \text{ for every } 0 < \mu < \mu_0. \tag{7.19}$$

In order to establish (7.19), we start from

$$I_{A,B} \leq 2^{p-1}\left(\iint\limits_{A\times B} dX_*^j dY_*^j \frac{|g_\mu(X_*^j) - g_\mu(Y_*^j)|^p}{|X_*^j - Y_*^j|^{j+sp}} + \iint\limits_{A\times B} dX_*^j dY_*^j \frac{|g(X_*^j) - g(Y_*^j)|^p}{|X_*^j - Y_*^j|^{j+sp}} \right). \tag{7.20}$$

We next establish the following estimate.

Lemma 7.9. Let $\mathfrak{C} \neq \mathfrak{C}'$ be two cubes in $\mathscr{C}_j \cap \mathscr{C}$. Then, for $0 < \mu < 1/2$ and for $X_*^j \in \mathfrak{C}$ and $Y_*^j \in \mathfrak{C}'$ such that

$$|X_*^j - 0_{\mathfrak{C}}| < 1-\mu \text{ and } |Y_*^j - 0_{\mathfrak{C}'}| < 1-\mu, \tag{7.21}$$

we have

$$\left| \left[0_{\mathfrak{C}} + \frac{X_*^j - 0_{\mathfrak{C}}}{1-\mu} \right] - \left[0_{\mathfrak{C}'} + \frac{Y_*^j - 0_{\mathfrak{C}'}}{1-\mu} \right] \right| \leq C\,|X_*^j - Y_*^j|. \tag{7.22}$$

Proof of Lemma 7.9. Recall that we assume that $\varepsilon = 1$ and $T = 0$. Write

$$0_{\mathfrak{C}} = (C_1, \ldots, C_n),\ 0_{\mathfrak{C}'} = (C'_1, \ldots, C'_n). \tag{7.23}$$

One may check the following properties of the C_i's:
a) Each C_i is an integer.
b) Exactly $n-j$ C_i's are odd.
c) The open cube $\mathfrak{C}$ is given by the following system of equations and inequalities:

$$X_i = C_i, \text{ if } C_i \text{ is odd, } |X_i - C_i| < 1, \text{ if } C_i \text{ is even.}$$

d) Thus every point X_*^j as in (7.21) is of the form

$$X_*^j = (C_1 + x_1, \ldots, C_n + x_n), \text{ with } x_i = 0 \text{ if } C_i \text{ is odd and } |x_i| < 1-\mu \text{ if } C_i \text{ is even.}$$

Similarly if we write $Y_*^j = (C'_1 + y_1, \ldots, C'_n + y_n)$.
Estimate (7.22) will follows from the following estimate, valid for each coordinate:

$$\left| \left[C_i + \frac{x_i}{1-\mu} \right] - \left[C'_i + \frac{y_i}{1-\mu} \right] \right| \leq C\,|[C_i + x_i] - [C'_i + y_i]| \text{ if } 0 < \mu < 1/2,\ |x_i| < 1-\mu,\ |y_i| < 1-\mu. \tag{7.24}$$

In order to establish the validity of (7.24), we consider the following cases.

Case 1. $|C_i - C'_i| \geq 3$
Then we have

$$|[C_i + x_i] - [C'_i + y_i]| \geq |C_i - C'_i| - 2 \text{ and } \left| \left[C_i + \frac{x_i}{1-\mu} \right] - \left[C'_i + \frac{y_i}{1-\mu} \right] \right| \leq |C_i - C'_i| + 2,$$

and thus (7.24) holds with $C = 5$.

Case 2. $|C_i - C'_i| = 2$ and C_i is odd
Then $x_i = y_i = 0$ and thus (7.24) holds with $C = 1$.

The same argument applies to the next case.

Case 3. $C_i = C_i'$ and C_i is odd

Case 4. $|C_i - C_i'| = 2$ and C_i is even
We may assume that $C_i = 2$, $C_i' = 0$, and we have to prove that

$$\left|2 + \frac{x-y}{1-\mu}\right| \le C\,|2 + (x-y)| \text{ when } |x| < 1-\mu \text{ and } |y| < 1-\mu,$$

which amounts to

$$2 + \frac{x-y}{1-\mu} \le C\,[2 + (x-y)] \text{ when } |x| < 1-\mu \text{ and } |y| < 1-\mu.$$

The above inequality holds with $C = 2$ (provided $0 < \mu < 1/2$).

Case 5. $C_i = C_i'$ and C_i is even
Then (7.24) holds with $C = 2$ (provided $0 < \mu < 1/2$).

Case 6. $|C_i - C_i'| = 1$
We may assume that $C_i = 1$ and $C_i' = 0$. As above, for $0 < \mu < 1/2$ estimate (7.24) follows from

$$\left|1 - \frac{y}{1-\mu}\right| \le 2\,|1-y| \text{ when } |y| < 1-\mu. \qquad \square$$

Step 3 completed. We estimate $I_{\mathfrak{C}_{\mu,\mu_0},\mathfrak{C}'_{\mu,\mu_0}}$ using (7.20) with $A = \mathfrak{C}_{\mu,\mu_0}$ and $B = \mathfrak{C}'_{\mu,\mu_0}$. After the changes of variables

$$\mathfrak{C}_{\mu,\mu_0} \ni X_*^j \mapsto 0_{\mathfrak{C}} + \frac{X_*^j - 0_{\mathfrak{C}}}{1-\mu} \in \mathfrak{C}^c_{(\mu_0-\mu)/(1-\mu)},\ \mathfrak{C}'_{\mu,\mu_0} \ni Y_*^j \mapsto 0_{\mathfrak{C}'} + \frac{Y_*^j - 0_{\mathfrak{C}'}}{1-\mu} \in \mathfrak{C}'^c_{(\mu_0-\mu)/(1-\mu)} \tag{7.25}$$

in the first double integral in (7.20), Lemma 7.9 implies that for $0 < \mu_0 < 1/2$ we have

$$\begin{aligned} I_{\mathfrak{C}_{\mu,\mu_0},\mathfrak{C}'_{\mu,\mu_0}} \le & C(n,p) \iint\limits_{\mathfrak{C}^c_{(\mu_0-\mu)/(1-\mu)} \times \mathfrak{C}'^c_{(\mu_0-\mu)/(1-\mu)}} dX_*^j dY_*^j \frac{|g(X_*^j) - g(Y_*^j)|^p}{|X_*^j - Y_*^j|^{j+sp}} \\ & + C(n,p) \iint\limits_{\mathfrak{C}_{\mu,\mu_0} \times \mathfrak{C}'_{\mu,\mu_0}} dX_*^j dY_*^j \frac{|g(X_*^j) - g(Y_*^j)|^p}{|X_*^j - Y_*^j|^{j+sp}} \\ \le & C(n,p) \iint\limits_{\mathfrak{C}^c_{2\mu_0} \times \mathfrak{C}'^c_{2\mu_0}} dX_*^j dY_*^j \frac{|g(X_*^j) - g(Y_*^j)|^p}{|X_*^j - Y_*^j|^{j+sp}}. \end{aligned} \tag{7.26}$$

We complete Step 3 by noting that the last double integral in (7.26) goes to 0 as $\mu_0 \to 0$ (since $g \in W^{s,p}(\mathscr{C}_j \cap \mathscr{C})$).

Step 4. We have $I_{F,F} \to 0$ as $\mu \to 0$
In view of Step 1, Step 4 is an immediate consequence of the fact that the restriction of g to $\mathscr{C}_{j-1} \cap \mathscr{C}$ belongs to $W^{s,p}$ and of the following

Lemma 7.10. Let $0 < s < 1$, $1 \le p < \infty$, $j \ge 1$ and $h \in W^{s,p}(\mathscr{C}_{j-1} \cap \mathscr{C})$. Then, with $C = C(j,s,p,\mathscr{C})$, we have

$$\sum_{\mathfrak{C} \ne \mathfrak{C}'} \iint\limits_{\mathfrak{C}^c_{1/2} \times \mathfrak{C}'^c_{1/2}} dX_*^j dY_*^j \frac{|h(X_*^{j-1}) - h(Y_*^{j-1})|^p}{|X_*^j - Y_*^j|^{j+sp}} \le C\,|h|^p_{W^{s,p}(\mathscr{C}_{j-1} \cap \mathscr{C})}. \tag{7.27}$$

Proof of Lemma 7.10. Estimate (7.27) is a special case of the following more general inequality, valid for nonegative measurable f:

$$\iint\limits_{\mathfrak{C}^c_{1/2}\times\mathfrak{C}'^c_{1/2}} dX^j_* dY^j_* \frac{f(X^{j-1}_*,Y^{j-1}_*)}{|X^j_*-Y^j_*|^{j+sp}} \le C \iint\limits_{(\overline{\mathfrak{C}}\cap\mathscr{C}_{j-1})\times(\overline{\mathfrak{C}'}\cap\mathscr{C}_{j-1})} dX^{j-1}_* dY^{j-1}_* \frac{f(X^{j-1}_*,Y^{j-1}_*)}{|X^{j-1}_*-Y^{j-1}_*|^{j-1+sp}}. \tag{7.28}$$

If we express the left-hand side of (7.28) using polar coordinates on $\overline{\mathfrak{C}}\cap\mathscr{C}_{j-1}$ (respectively on $\overline{\mathfrak{C}'}\cap\mathscr{C}_{j-1}$), then (7.28) amounts to the following

$$\int\limits_{1/2}^{1}\int\limits_{1/2}^{1} dt d\tau \frac{1}{|[(1-t)0_{\mathfrak{C}}+tX^{j-1}_*]-[(1-\tau)0_{\mathfrak{C}'}+\tau Y^{j-1}_*]|^a} \le C \frac{1}{|X^{j-1}_*-Y^{j-1}_*|^{a-1}}, \tag{7.29}$$

which is valid whenever $a>1$, $\mathfrak{C}\neq\mathfrak{C}'$, $X^{j-1}_*\in\overline{\mathfrak{C}}\cap\mathscr{C}_{j-1}$ and $Y^{j-1}_*\in\overline{\mathfrak{C}'}\cap\mathscr{C}_{j-1}$.

Clearly, estimate (7.29) holds when $\overline{\mathfrak{C}}\cap\overline{\mathfrak{C}'}=\emptyset$ (since both sides of (7.29) are bounded from above and below by finite positive constants).

We may thus assume that

$$\overline{\mathfrak{C}}\cap\overline{\mathfrak{C}'}\neq\emptyset. \tag{7.30}$$

In this case, the idea is to mimic the proof of the estimate (4.14).

Step 1 in the proof of (7.29). We claim that, assuming (7.30), there exists some $C=C(n,j)$ such that for $X^{j-1}_*\in\overline{\mathfrak{C}}\cap\mathscr{C}_{j-1}$ and $Y^{j-1}_*\in\overline{\mathfrak{C}'}\cap\mathscr{C}_{j-1}$ and $1/2\le t,\tau\le 1$ we have

$$|[(1-t)0_{\mathfrak{C}}+tX^{j-1}_*]-[(1-\tau)0_{\mathfrak{C}'}+\tau Y^{j-1}_*]| \ge C\,|[(1-t)0_{\mathfrak{C}}+tX^{j-1}_*]-Y^{j-1}_*|. \tag{7.31}$$

The proof of (7.31) relies on the following geometrically clear inequality, whose proof is postponed.

Lemma 7.11. Assume that (7.30) holds. Then there exists some $C=C(n,j)$ such that if $X^j_*\in\overline{\mathfrak{C}}$ and $Y^j_*\in\overline{\mathfrak{C}'}$, then there exists some $Z^j_*\in\overline{\mathfrak{C}}\cap\mathfrak{C}'$ such that

$$|X^j_*-Z^j_*|+|Y^j_*-Z^j_*|\le C\,|X^j_*-Y^j_*|. \tag{7.32}$$

Equivalently, if $P:\overline{\mathfrak{C}}\cup\overline{\mathfrak{C}'}\to\mathbb{R}^\ell$ is L-Lipschitz on $\overline{\mathfrak{C}}$ and on $\overline{\mathfrak{C}'}$, then P is CL-Lipschitz on $\overline{\mathfrak{C}}\cup\overline{\mathfrak{C}'}$.

Assuming Lemma 7.11 established, we proceed as in the proof of Lemma 5.3: we let

$$P(X^j_*)=X^j_*,\ \forall X^j_*\in\overline{\mathfrak{C}}, P(Y^j_*)=Y^{j-1}_*,\ \forall Y^j_*\in\overline{\mathfrak{C}'}^c_{1/2}.$$

We extend P from $\overline{\mathfrak{C}'}^c_{1/2}$ to $\overline{\mathfrak{C}'}$ without increasing its Lipschitz constant (which is independent of $\mathfrak{C}'$). For this P, estimate (7.31) reads

$$|P(X^j_*)-P(Y^j_*)|\le\frac{1}{C}|X^j_*-Y^j_*|,\ \forall X^j_*\in\mathfrak{C}^c_{1/2},\ \forall Y^j_*\in\mathfrak{C}'^c_{1/2},$$

which follows from Lemma 7.11.

Step 2 in the proof of (7.29). In view of (7.31), we have reduced (7.29) to

$$\int\limits_{1/2}^{1} dt \frac{1}{|[(1-t)0_{\mathfrak{C}}+tX^{j-1}_*]-Y^{j-1}_*|^a} \le C\frac{1}{|X^{j-1}_*-Y^{j-1}_*|^{a-1}}. \tag{7.33}$$

Combining (7.30) with the fact that $\mathfrak{C}\cap\mathfrak{C}'=\emptyset$, we find that

$$1\le|Y^{j-1}_*-0_{\mathfrak{C}}|\le 2 \text{ and } |X^{j-1}_*-Y^{j-1}_*|\le 2. \tag{7.34}$$

Using (7.34), we obtain that

$$|[(1-t)0_{\mathfrak{C}}+tX_*^{j-1}]-Y_*^{j-1}| = |[0_{\mathfrak{C}}-Y_*^{j-1}]-t[0_{\mathfrak{C}}-X_*^{j-1}]| \geq 1-t \text{ when } 1/2 \leq t \leq 1-|X_*^{j-1}-Y_*^{j-1}|/10 \quad (7.35)$$

and

$$\begin{aligned} |[(1-t)0_{\mathfrak{C}}+tX_*^{j-1}]-Y_*^{j-1}| &= |t[X_*^{j-1}-Y_*^{j-1}]+(1-t)[0_{\mathfrak{C}}-Y_*^{j-1}]| \\ &\geq t|X_*^{j-1}-Y_*^{j-1}|-2(1-t) \\ &\geq C|X_*^{j-1}-Y_*^{j-1}| \text{ when } 1-|X_*^{j-1}-Y_*^{j-1}|/10 \leq t \leq 1. \end{aligned} \quad (7.36)$$

Estimate (7.33) follows from (7.35) and (7.36). □

In order to complete Step 4, it remains to proceed to the

Proof of Lemma 7.11. Let $\mathfrak{E} = \overline{\mathfrak{C}} \cap \overline{\mathfrak{C}'}$, and let ℓ be the Hausdorff dimension of $\mathfrak{E}$. Let us note that $\mathfrak{E}$ is a cube in $\mathscr{C}_\ell$. After translation and permutation of the coordinates, we may identify $\mathfrak{E}$ with a cube in $\mathbb{R}^\ell$, and then we may write

$$\overline{\mathfrak{C}} = \mathfrak{D} \times \mathfrak{E},\ \overline{\mathfrak{C}'} = \mathfrak{D}' \times \mathfrak{E}$$

with $\mathfrak{D}$, $\mathfrak{D}'$ closed cubes in $\mathscr{C}_{j-\ell}(\mathbb{R}^{n-\ell})$ such that

$$\mathfrak{D} \cap \mathfrak{D}' = \{Z'\} \text{ for some point } Z'.$$

We will split a point $X_*^j \in \overline{\mathfrak{C}}$ as

$$X_*^j = (X', X''), \text{ with } X' \in \mathfrak{D},\ X'' \in \mathfrak{E}; \text{ similarly for a point } Y_*^j \in \overline{\mathfrak{C}'}.$$

Assume that we have established the estimate

$$|X'-Z'|+|Y'-Z'| \leq C|X'-Y'|,\ \forall X' \in \mathfrak{D},\ \forall Y' \in \mathfrak{D}'. \quad (7.37)$$

Then clearly $(Z', X'') \in \overline{\mathfrak{C}} \cap \overline{\mathfrak{C}'}$ and

$$|X_*^j-(Z',X'')|+|Y_*^j-(Z',X'')| \leq C|X_*^j-Y_*^j|,$$

i.e. (7.32) holds.

It thus remains to prove (7.37). This is obtained by contradiction. Assume that there are sequences $\{X_*^{',k}\} \subset \mathfrak{D} \setminus \{Z'\}$ and $\{Y_*^{',k}\} \subset \mathfrak{D}' \setminus \{Z'\}$ such that

$$|X^{',k}-Z'|+|Y^{',k}-Z'| \geq k|X^{',k}-Y^{',k}|. \quad (7.38)$$

By symmetry and after passing to a subsequence, we may assume that

$$X^{',k}-Z' = \delta_k W^k,\ Y^{',k}-Z' = \lambda_k T^k,\ |W^k| = 1,\ W^k \to W,\ |T^k| = 1,\ T^k \to T,\ 0 \leq \lambda_k \leq \delta_k,\ \lambda_k/\delta_k \to \mu.$$

Using (7.38), we obtain that $W = T$ (and $\mu = 1$). However, this cannot happen. Indeed, since $X^{',k} \in \mathfrak{D}$, we have $Z'+W^k \in \mathfrak{D}$ (check it on a picture). Thus $Z'+W \in \mathfrak{D}$. Similarly, $Z'+T \in \mathfrak{D}'$. Since $W = T$, we obtain that $\mathfrak{D} \cap \mathfrak{D}'$ contains $Z'+W$, a contradiction. □

Step 4 is complete.

Step 5. We have $I_{E,F} \to 0$ as $\mu \to 0$

In view of Steps 1 and 2, it suffices to establish the following. Let $\xi > 0$ be fixed arbitrarily small. Let $\mathfrak{C} \neq \mathfrak{C}'$ be two cubes in $\mathscr{C}_j$. Then there exists some $0 < \mu_0 < 1$ such that

$$I_{\mathfrak{C}_{\mu,\mu_0},\mathfrak{C}'^c_\mu} = \iint\limits_{\mathfrak{C}_{\mu,\mu_0}\times\mathfrak{C}'^c_\mu} dX^j_* dY^j_* \frac{|g_\mu(X^j_*) - g(Y^{j-1}_*)|^p}{|X^j_* - Y^j_*|^{j+sp}} < 2\xi \text{ for every } 0 < \mu < \mu_0. \tag{7.39}$$

By Step 4, we have

$$\iint\limits_{\mathfrak{C}_{\mu,\mu_0}\times\mathfrak{C}'^c_\mu} dX^j_* dY^j_* \frac{|g(X^{j-1}_*) - g(Y^{j-1}_*)|^p}{|X^j_* - Y^j_*|^{j+sp}} \to 0 \text{ uniformly in } 0 < \mu < \mu_0 \text{ as } \mu_0 \to 0. \tag{7.40}$$

Using (7.40), (7.39) amounts to the existence of some μ_0 such that

$$J = \iint\limits_{\mathfrak{C}_{\mu,\mu_0}\times\mathfrak{C}'^c_\mu} dX^j_* dY^j_* \frac{|g_\mu(X^j_*) - g(X^{j-1}_*)|^p}{|X^j_* - Y^j_*|^{j+sp}} < \xi \text{ for every } 0 < \mu < \mu_0. \tag{7.41}$$

The key ingredient in the proof of (7.41) is the following

Lemma 7.12. Let $a > j$. Then for $\mathfrak{C} \neq \mathfrak{C}'$ cubes in $\mathscr{C} \cap \mathscr{C}_j$ and for $X^j_* \in \mathfrak{C}^c_{1/2}$ we have

$$\int\limits_{\mathfrak{C}'^c_{1/2}} dY^j_* \frac{1}{|X^j_* - Y^j_*|^a} \leq C(n,j,a,\mathscr{C}) \frac{1}{|X^j_* - X^{j-1}_*|^{a-j}}. \tag{7.42}$$

Proof of Lemma 7.12. When $\overline{\mathfrak{C}} \cap \overline{\mathfrak{C}'} = \emptyset$, the left-hand side of (7.42) is bounded from above by a positive constant, and the right-hand side of (7.42) is bounded from below by a positive constant, so that the conclusion is clear. We may thus assume that $\overline{\mathfrak{C}} \cap \overline{\mathfrak{C}'} \neq \emptyset$. We are then in position to apply estimate (7.31) (with the roles of X^j_* and Y^j_* reversed) and infer that

$$|X^{j-1}_* - Y^j_*| \leq C|X^j_* - Y^j_*|, \ \forall X^j_* \in \mathfrak{C}^c_{1/2}, \ \forall Y^j_* \in \mathfrak{C}'^c_{1/2}. \tag{7.43}$$

On the other hand, since $\mathfrak{C} \cap \mathfrak{C}' = \emptyset$, we clearly have

$$|X^j - X^{j-1}_*| \leq |X^j_* - Y^j_*|, \ \forall X^j_* \in \mathfrak{C}, \ \forall Y^j_* \in \mathfrak{C}'. \tag{7.44}$$

By (7.43) and (7.44), we have

$$|X^j_* - Y^j_*| \geq C(|X^j - X^{j-1}_*| + |X^{j-1}_* - Y^j_*|), \ \forall X^j_* \in \mathfrak{C}^c_{1/2}, \ \forall Y^j_* \in \mathfrak{C}'^c_{1/2}. \tag{7.45}$$

If Z is the orthogonal projection of X^{j-1}_* on the j-dimensional affine plane Π spanned by $\mathfrak{C}'$, then (7.45) leads to

$$|X^j_* - Y^j_*| \geq C(|X^j - X^{j-1}_*| + |Z - Y^j_*|), \ \forall X^j_* \in \mathfrak{C}^c_{1/2}, \ \forall Y^j_* \in \mathfrak{C}'^c_{1/2}. \tag{7.46}$$

Using (7.46), we find that

$$\int\limits_{\mathfrak{C}'^c_{1/2}} dY^j_* \frac{1}{|X^j_* - Y^j_*|^a} \leq C \int\limits_{\Pi} dY^j_* \frac{1}{[|X^j - X^{j-1}_*| + |Z - Y^j_*|]^a} = C \frac{1}{|X^j - X^{j-1}_*|^{a-j}}. \qquad \square$$

Step 5 completed. Using Lemma 7.12 and a change of variables as in (7.25), we obtain that the left-hand side J of (7.41) satisfies, when $0 < \mu_0 < 1/2$,

$$J \leq C \int\limits_{\mathfrak{C}_{\mu,\mu_0}} dX^j_* \frac{|g_\mu(X^j_*) - g(X^{j-1}_*)|^p}{|X^j_* - X^{j-1}_*|^{sp}} = C(1-\mu)^{j-sp} \int\limits_{\mathfrak{C}^c_{(\mu_0-\mu)/(1-\mu)}} dX^j_* \frac{|g(X^j_*) - g(X^{j-1}_*)|^p}{[|X^j_* - X^{j-1}_*| + \mu/(1-\mu)]^{sp}}$$

$$\leq C \int\limits_{\mathfrak{C}^c_{2\mu_0}} dX^j_* \frac{|g(X^j_*) - g(X^{j-1}_*)|^p}{|X^j_* - X^{j-1}_*|^{sp}} \to 0 \text{ as } \mu_0 \to 0,$$

(here, we use the fact that $g \in \mathscr{W}^{s,p}_j$) and thus (7.41) holds.

Step 5 and the proof of Lemma 7.3 are complete. $\square$

8 Continuity of the map $g \mapsto h$. Proof of Theorem 6

Let $\mathscr{C}$ be a finite submesh of $\mathscr{C}_n$ and let $j \in [\![0, n-1]\!]$. Let $g : \mathscr{C}_j \cap \mathscr{C} \to \mathbb{R}^m$ and let h be its j-piecewise homogeneous extension to $\mathscr{C}$.

The main result in this section is the following

Lemma 8.1. Let $0 < s < 1$, $1 \le p < \infty$ be such that $sp < j+1$. Then we have

$$\|h\|^p_{L^p(\mathscr{C})} \le C \|g\|^p_{L^p(\mathscr{C}_j \cap \mathscr{C})} \tag{8.1}$$

and

$$|h|^p_{W^{s,p}(\mathscr{C})} \le C |g|^p_{W^{s,p}(\mathscr{C}_j \cap \mathscr{C})}. \tag{8.2}$$

[Here, $\| \ \|_{L^p(\mathscr{C})}$ and $| \ |_{W^{s,p}(\mathscr{C})}$ are naturally defined, and we allow constants depending on $\mathscr{C}$.]

Equivalently, the map $W^{s,p}(\mathscr{C}_j \cap \mathscr{C}) \ni g \mapsto h \in W^{s,p}(\mathscr{C})$ is continuous.

Proof. We may assume that $T = 0$. We use the notation in Section 5. In particular, ω and λ will be points in $\mathbb{R}^j$.

Step 1. Estimate of $\|h\|^p_{L^p(\mathscr{C})}$
As in Step 3.1 in the proof of Lemma 5.1 (more precisely, by mimicking the derivation of (5.10) with the help of (5.8)), we have

$$\|h\|^p_{L^p(\mathscr{C})} = \mathop{\sum\nolimits^{\circ}}_{\substack{\sigma \in S_{n-j,n} \\ q \in \{-1,1\}^{n-j} \\ L \in \mathbb{Z}^n}} \int_{|\omega| \le 1} d\omega \, k^\sharp(\omega) |g(2\varepsilon L + X^j)|^p,$$

where $\mathop{\sum\nolimits^{\circ}}$ denotes a sum taken only over the L's such that $Q_\varepsilon + 2\varepsilon L \in \mathscr{C}$ and

$$k^\sharp(\omega) = \varepsilon^n \int_{0 \le t \le 1} dt \, t_1^{n-1} \dots t_{n-j}^j = C(n,j) \varepsilon^n.$$

Thus

$$\|h\|^p_{L^p(\mathscr{C})} = C(n,j) \varepsilon^n \mathop{\sum\nolimits^{\circ}}_{\substack{\sigma \in S_{n-j,n} \\ q \in \{-1,1\}^{n-j} \\ L \in \mathbb{Z}^n}} \int_{|\omega| \le 1} d\omega \, |g(2\varepsilon L + X^j)|^p \le C(n,j) \varepsilon^{n-j} \int_{\mathscr{C}_j \cap \mathscr{C}} |g|^p.$$

Step 2. Estimate of $|h|^p_{W^{s,p}(\mathscr{C})}$
In the spirit of Step 1 above, we have

$$|h|^p_{W^{s,p}(\mathscr{C})} = \mathop{\sum\nolimits^{*}}_{\substack{\sigma, \tau \in S_{n-j,n} \\ q, r \in \{-1,1\}^{n-j} \\ L, M \in \mathbb{Z}^n}} \int_{|\omega| \le 1} d\omega \int_{|\lambda| \le 1} d\lambda \, k^\flat_{L,M}(\omega, \lambda) |g(2\varepsilon L + X^j) - g(2\varepsilon M + Y^j)|^p,$$

where $\mathop{\sum\nolimits^{*}}$ denotes a sum taken only over the L's and M's such that $Q_\varepsilon + 2\varepsilon L \in \mathscr{C}$ and $Q_\varepsilon + 2\varepsilon M \in \mathscr{C}$, and we set

$$k^\flat_{L,M}(\omega, \lambda) = \int_{0 \le t \le 1} dt \int_{0 \le u \le 1} du \, t_1^{n-1} \dots t_{n-j}^j u_1^{n-1} \dots u_{n-j}^j \frac{\varepsilon^{2n}}{|(2\varepsilon L + X^n) - (2\varepsilon M + Y^n)|^{n+sp}}.$$

We rely on the following variant of Lemma 5.4.

Lemma 8.2. Assume that $sp < j+1$. Then

$$k^{\flat}_{L,M}(\omega,\lambda) \le \frac{C(\mathscr{C})}{|(2\varepsilon L + X^j) - (2\varepsilon M + Y^j)|^{j+sp}}. \tag{8.3}$$

Proof. When $L = M$, the conclusion is given by Lemma 5.2. When $|L-M| = 1$, this is Lemma 5.4. Finaly, when $|L-M| \ge 2$, both sides of (8.3) are bounded from above and from below, with finite positive bounds depending on $\mathscr{C}$ (and thus on ε) but independent of L, M, X^n and Y^n. □

Step 2 completed. Using Lemma 8.2, we find that

$$\begin{aligned}|h|^p_{W^{s,p}(\mathscr{C})} \le & C(\mathscr{C}) \mathop{\sum^{\circ}}_{\substack{\sigma,\tau\in S_{n-j,n}\\ q,r\in\{-1,1\}^{n-j}\\ L,M\in\mathbb{Z}^n}} \int_{|\omega|\le 1} d\omega \int_{|\lambda|\le 1} d\lambda \, \frac{|g(2\varepsilon L + X^j) - g(2\varepsilon M + Y^j)|^p}{|(2\varepsilon L + X^j) - (2\varepsilon M + Y^j)|^{j+sp}}\\ \le & C(\mathscr{C})\, |g|^p_{W^{s,p}(\mathscr{C}_j\cap\mathscr{C})}.\end{aligned}$$

□

We end with the

Proof of Theorem 6. Theorem 6 is a straightforward consequence of Corollary 6.3, Lemma 7.1 and Lemma 8.1. □

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