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# Bounding the norm of a log-concave vector via thin-shell estimates

Ronen Eldan and Joseph Lehec

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## Abstract

Chaining techniques show that if  $X$  is an isotropic log-concave random vector in  $\mathbb{R}^n$  and  $\Gamma$  is a standard Gaussian vector then

$$\mathbb{E}\|X\| \leq Cn^{1/4}\mathbb{E}\|\Gamma\|$$

for any norm  $\|\cdot\|$ , where  $C$  is a universal constant. Using a completely different argument we establish a similar inequality relying on the *thin-shell* constant

$$\sigma_n = \sup\left(\sqrt{\text{Var}(|X|)}; X \text{ isotropic and log-concave on } \mathbb{R}^n\right).$$

In particular, we show that if the thin-shell conjecture  $\sigma_n = O(1)$  holds, then  $n^{1/4}$  can be replaced by  $\log(n)$  in the inequality. As a consequence, we obtain certain bounds for the mean-width, the dual mean-width and the isotropic constant of an isotropic convex body. In particular, we give an alternative proof of the fact that a positive answer to the thin-shell conjecture implies a positive answer to the slicing problem, up to a logarithmic factor.

## 1 Introduction

Given a stochastic process  $(X_t)_{t \in T}$ , the question of obtaining bounds for the quantity

$$\mathbb{E}\left(\sup_{t \in T} X_t\right)$$

is a fundamental question in probability theory dating back to Kolmogorov, and the theory behind this type of question has applications in a variety of fields.

The case that  $(X_t)_{t \in T}$  is a Gaussian process is perhaps the most important one. It has been studied intensively over the past 50 years, and numerous bounds on the supremum in terms of the geometry of the set  $T$  have been attained by Dudley, Fernique, Talagrand and many others.

The case of interest in this paper is a certain generalization of the Gaussian process. We consider the supremum of the process

$$(X_t = \langle X, t \rangle)_{t \in T}$$

where  $X$  is a log-concave random vector in  $\mathbb{R}^n$  and  $T \subset \mathbb{R}^n$  is a compact set. Throughout the article  $\langle x, y \rangle$  denotes the inner product of  $x, y \in \mathbb{R}^n$  and  $|x| = \sqrt{\langle x, x \rangle}$  the Euclidean norm of  $x$ . Our aim is to obtain an upper bound on this supremum in terms of the supremum of a corresponding Gaussian process  $Y_t = \langle \Gamma, t \rangle$  where  $\Gamma$  is a gaussian random vector having the same covariance structure as  $X$ .

Before we formulate the results, we begin with some notation. A probability density  $\rho : \mathbb{R}^n \rightarrow [0, \infty)$  is called *log-concave* if it takes the form  $\rho = \exp(-H)$  for a convex function  $H : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ . A probability measure is log-concave if it has a log-concave density and a random vector taking values in  $\mathbb{R}^n$  is said to be log-concave if its law is log-concave. Two canonical examples of log-concave measures are the uniform probability measure on a convex body and the Gaussian measure. It is a well-known fact that any log-concave probability density decays exponentially at infinity, and thus has moments of all orders. A log-concave random vector  $X$  is said to be *isotropic* if its expectation and covariance matrix satisfy

$$\mathbb{E}(X) = 0, \quad \text{cov}(X) = \text{id}.$$

Let  $\sigma_n$  be the so-called *thin-shell* constant:

$$\sigma_n = \sup_X \sqrt{\text{Var}(|X|)} \tag{1}$$

where the supremum runs over all isotropic, log-concave random vectors  $X$  in  $\mathbb{R}^n$ . It is trivial that  $\sigma_n \leq \sqrt{n}$  and it was proven initially by Klartag [K2] that in fact

$$\sigma_n = o(\sqrt{n}).$$

Shortly afterwards, Fleury-Guédon-Paouris [FGP] gave an alternative proof of this fact. Several improvements on the bound have been established since then, and the current best estimate is  $\sigma_n = O(n^{1/3})$  due to Guédon-Milman [Gu-M]. The *thin-shell* conjecture, which asserts that the sequence  $(\sigma_n)_{n \geq 1}$  is bounded, is still open. Another related constant is:

$$\tau_n^2 = \sup_X \sup_{\theta \in \mathbb{S}^{n-1}} \sum_{i,j=1}^n \mathbb{E}(X_i X_j \langle X, \theta \rangle)^2, \tag{2}$$

where the supremum runs over all isotropic log-concave random vectors  $X$  in  $\mathbb{R}^n$ . Although it is not known whether  $\tau_n = O(\sigma_n)$ , we have the following estimate, proven in [E]

$$\tau_n^2 = O\left(\sum_{k=1}^n \frac{\sigma_k^2}{k}\right). \tag{3}$$

The estimate  $\sigma_n = O(n^{1/3})$  thus gives  $\tau_n = O(n^{1/3})$ , whereas the *thin-shell* conjecture yields  $\tau_n = O(\sqrt{\log n})$ .

We denote by  $\Gamma$  the standard Gaussian vector in  $\mathbb{R}^n$  (with identity covariance matrix). We are now ready to formulate our main theorem.

**Theorem 1.** *Let  $X$  be an isotropic log-concave random vector in  $\mathbb{R}^n$  and let  $\|\cdot\|$  be a norm. Then there is a universal constant  $C$  such that*

$$\mathbb{E}\|X\| \leq C\sqrt{\log n} \tau_n \mathbb{E}\|\Gamma\|. \quad (4)$$

*Remark.* It is well known that, as far as  $\mathbb{E}\|X\|$  is concerned, there is no loss of generality assuming additionally that the support of  $X$  is contained in a ball of radius  $C_0\sqrt{n}$  for some sufficiently large constant  $C_0$  (see for instance Lemma 8 below). Then it is easily seen that  $X$  satisfies the following  $\psi_2$  estimate

$$\mathbb{P}(|\langle X, \theta \rangle| \geq t) \leq Ce^{-ct^2/\sqrt{n}}, \quad \forall t \geq 0, \forall \theta \in \mathbb{S}^{n-1},$$

where  $C, c$  are universal constants. Combining this with chaining methods developed by Dudley-Fernique-Talagrand (more precisely, using Theorem 1.2.6. and Theorem 2.1.1. of [T]), one gets the inequality

$$\mathbb{E}\|X\| \leq C'n^{1/4}\mathbb{E}\|\Gamma\|,$$

we refer to [Bou] for more details. This means that using the current best-known bound for the thin-shell constant:  $\sigma_n = O(n^{1/3})$ , the above theorem does not give us anything new. On the other hand, under the thin-shell hypothesis we obtain using (3)

$$\mathbb{E}\|X\| \leq C \log n \mathbb{E}\|\Gamma\|.$$

As an application of Theorem 1, we derive several bounds related to the mean width and dual mean width of isotropic convex bodies and to the so-called *hyperplane conjecture*. We begin with a few definitions. A convex body  $K \subset \mathbb{R}^n$  is a compact convex set whose interior contains the origin. For  $x \in \mathbb{R}^n$ , we define

$$\|x\|_K = \inf\{\lambda; x \in \lambda K\}$$

to be the gauge associated to  $K$  (it is a norm if  $K$  is symmetric about 0). The polar body of  $K$  is denoted by

$$K^\circ = \{y \in \mathbb{R}^n; \langle x, y \rangle \leq 1, \forall x \in K\}.$$

Next we define

$$M(K) = \int_{\mathbb{S}^{n-1}} \|x\|_K \sigma(dx),$$

$$M^*(K) = \int_{\mathbb{S}^{n-1}} \|x\|_{K^\circ} \sigma(dx),$$

where  $\sigma$  is the Haar measure on the sphere, normalized to be a probability measure. These two parameters play an important rôle in the asymptotic theory of convex bodies.

The convex body  $K$  is said to be isotropic if a random vector uniform on  $K$  is isotropic. When  $K$  is isotropic, the *isotropic constant* of  $K$  is then defined to be

$$L_K = |K|^{-1/n},$$

where  $|K|$  denotes the Lebesgue measure of  $K$ . More generally, the isotropic constant of an isotropic log-concave random vector is  $L_X = f(0)^{1/n}$  where  $f$  is the density of  $X$ . The *slicing* or *hyperplane* conjecture asserts that  $L_K \leq C$  for some universal constant  $C$ . The current best estimate is  $L_K \leq Cn^{1/4}$  due to Klartag [K1]. We are ready to formulate our corollary:

**Corollary 2.** *Let  $K$  be an isotropic convex body. Then one has,*

$$(i) \quad M^*(K) \geq c\sqrt{n}/(\sqrt{\log n} \tau_n),$$

$$(ii) \quad L_K \leq C\tau_n(\log n)^{3/2},$$

where  $c, C > 0$  are universal constants.

*Remark.* Part (ii) of the corollary is nothing new. Indeed, in [EK], it is shown that  $L_K \leq C\sigma_n$  for a universal constant  $C > 0$ . Our proof uses different methods and could therefore shed some more light on this relation, which is the reason why we provide it.

Using similar methods, we attain an alternative proof of the following correlation inequality proven initially by Hargé in [H].

**Proposition 3** (Hargé). *Let  $X$  be a random vector on  $\mathbb{R}^n$ . Assume that  $\mathbb{E}(X) = 0$  and that  $X$  is more log-concave than  $\Gamma$ , i.e. the density of  $X$  has the form*

$$x \mapsto \exp(-V(x) - \frac{1}{2}|x|^2)$$

for some convex function  $V: \mathbb{R}^n \rightarrow (-\infty, +\infty]$ . Then for every convex function  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$  we have

$$\mathbb{E}\varphi(X) \leq \mathbb{E}\varphi(\Gamma).$$

The structure of the paper is as follows: in section 2 we recall some properties of a stochastic process constructed in [E], which will serve as one of the central ingredients in the proof of Theorem 1, as well as establish some new facts about this process. In section 3 we prove the main theorem and Proposition 3. Finally, in section 4 we prove Corollary 2.

In this note, the letters  $c, \tilde{c}, c', C, \tilde{C}, C', C''$  will denote positive universal constants, whose value is not necessarily the same in different appearances. Further notation used throughout the text:  $\text{id}$  will denote the identity  $n \times n$  matrix. The Euclidean unit sphere is denoted by  $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$ . The operator norm and the trace of a matrix  $A$  are denoted by  $\|A\|_{op}$  and  $\text{Tr}(A)$ , respectively. For two probability measures  $\mu, \nu$  on  $\mathbb{R}^n$ , we let  $T_2(\mu, \nu)$  be their *transportation cost* for the Euclidean distance squared:

$$T_2(\mu, \nu) = \inf_{\xi} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 \xi(dx, dy)$$

where the infimum is taken over all measures  $\xi$  on  $\mathbb{R}^{2n}$  whose marginals onto the first and last  $n$  coordinates are the measures  $\mu$  and  $\nu$  respectively. Finally, given

a continuous martingale  $(X_t)_{t \geq 0}$ , we denote by  $[X]_t$  its quadratic variation. If  $X$  is  $\mathbb{R}^n$ -valued, then  $[X]_t$  is a non-negative matrix whose  $i, j$  coefficient is the quadratic covariation of the  $i$ -th and  $j$ -th coordinates of  $X$  at time  $t$ .

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## 2 The stochastic construction

We make use of the construction described in [E]. There it is shown that, given a probability measure  $\mu$  having compact support and whose density with respect to the Lebesgue measure is  $f$ , and given a standard Brownian motion  $(W_t)_{t \geq 0}$  on  $\mathbb{R}^n$ ; there exists an adapted random process  $(\mu_t)_{t \geq 0}$  taking values in the space of absolutely continuous probability measures such that  $\mu_0 = \mu$  and such that the density  $f_t$  of  $\mu_t$  satisfies

$$df_t(x) = f_t(x) \langle A_t^{-1/2}(x - a_t), dW_t \rangle, \quad \forall t \geq 0, \quad (5)$$

for every  $x \in \mathbb{R}^n$ , where

$$\begin{aligned} a_t &= \int_{\mathbb{R}^n} x \mu_t(dx), \\ A_t &= \int_{\mathbb{R}^n} (x - a_t) \otimes (x - a_t) \mu_t(dx) \end{aligned}$$

are the barycenter and the covariance matrix of  $\mu_t$ , respectively.

Let us give now the main properties of this process. Some of these properties have already been established in [E], in this case we will only give the general idea of the proof. We refer the reader to [E, Section 2,3] for complete proofs. Firstly, for every test function  $\phi$  the process

$$\left( \int_{\mathbb{R}^n} \phi d\mu_t \right)_{t \geq 0}$$

is a martingale. In particular

$$\mathbb{E} \int_{\mathbb{R}^n} \phi d\mu_t = \int_{\mathbb{R}^n} \phi d\mu, \quad \forall t \geq 0. \quad (6)$$

The Itô differentials of  $a_t$  and  $A_t$  read

$$da_t = A_t^{1/2} dW_t \quad (7)$$

$$dA_t = -A_t dt + \int_{\mathbb{R}^n} (x - a_t) \otimes (x - a_t) \langle A_t^{-1/2}(x - a_t), dW_t \rangle \mu_t(dx). \quad (8)$$

Recall that  $\mu$  is assumed to have compact support, and observe that the support of  $\mu_t$  is included in that of  $\mu$ , almost surely. This shows that the processes

$(a_t)$  and  $(A_t)$  as well as the process involved in the local martingale part of equation (8) are uniformly bounded. In particular the local martingales of the last two equations are actually genuine martingales. Thus we get from (8)

$$\frac{d}{dt}\mathbb{E}\mathrm{Tr}(A_t) = -\mathbb{E}\mathrm{Tr}(A_t).$$

Integrating this differential equation we obtain

$$\mathbb{E}\mathrm{Tr}(A_t) = e^{-t}\mathrm{Tr}(A_0), \quad t \geq 0. \quad (9)$$

Combining this with (7) we obtain

$$\mathbb{E}|a_t|^2 = |a_0|^2 + \int_0^t \mathbb{E}\mathrm{Tr}(A_s) ds = |a_0|^2 + (1 - e^{-t})\mathrm{Tr}(A_0).$$

The process  $(a_t)_{t \geq 0}$  is thus a martingale bounded in  $L^2$ . By Doob's theorem, it converges almost surely and in  $L^2$  to some random vector  $a_\infty$ .

**Proposition 4.** *The random vector  $a_\infty$  has law  $\mu$ .*

*Proof.* Let  $\phi, \psi$  be functions on  $\mathbb{R}^n$  satisfying

$$\phi(x) + \psi(y) \leq |x - y|^2, \quad x, y \in \mathbb{R}^n. \quad (10)$$

Then

$$\phi(a_t) + \int_{\mathbb{R}^n} \psi(y) \mu_t(dy) \leq \int_{\mathbb{R}^n} |a_t - y|^2 dy = \mathrm{Tr}(A_t).$$

Taking expectation and using (6) and (9) we obtain

$$\int_{\mathbb{R}^n} \phi d\nu_t + \int_{\mathbb{R}^n} \psi d\mu \leq \mathrm{Tr}(A_0)e^{-t},$$

where  $\nu_t$  is the law of  $a_t$ . This holds for every pair of functions satisfying the constraint (10). By the Monge-Kantorovich duality (see for instance [V, Theorem 5.10]) we obtain

$$T_2(\nu_t, \mu) \leq e^{-t}\mathrm{Tr}(A_0)$$

where  $T_2$  is the transport cost associated to the Euclidean distance squared, defined in the introduction. Thus  $\nu_t \rightarrow \mu$  in the  $T_2$  sense, which implies that  $a_t \rightarrow \mu$  in law, hence the result.  $\square$

Let us move on to properties of the operator norm of  $A_t$ . We shall use the following lemma which follows for instance from a theorem of Brascamp-Lieb [BL, Theorem 4.1.]. We provide an elementary proof using the Prékopa-Leindler inequality.

**Lemma 5.** *Let  $X$  be a random vector on  $\mathbb{R}^n$  whose density  $\rho$  has the form*

$$\rho(x) = \exp\left(-\frac{1}{2}\langle Bx, x \rangle - V(x)\right)$$

where  $B$  is a positive definite matrix, and  $V: \mathbb{R}^n \rightarrow (-\infty + \infty]$  is a convex function. Then one has,

$$\text{cov}(X) \leq B^{-1}.$$

In other words, if a random vector  $X$  is more log-concave than a Gaussian vector  $Y$ , then  $\text{cov}(X) \leq \text{cov}(Y)$ .

*Proof.* There is no loss of generality assuming that  $B = \text{id}$  (replace  $X$  by  $B^{1/2}X$  otherwise). Let

$$\Lambda: x \mapsto \log \mathbb{E}(e^{\langle x, X \rangle}).$$

Since log-concave vectors have exponential moment  $\Lambda$  is  $\mathcal{C}^\infty$  in a neighborhood of 0 and it is easily seen that

$$\nabla^2 \Lambda(0) = \text{cov}(X). \tag{11}$$

Fix  $a \in \mathbb{R}^n$  and define

$$\begin{aligned} f: x &\mapsto \langle a, x \rangle - \frac{1}{2}|x|^2 - V(x), \\ g: y &\mapsto -\langle a, y \rangle - \frac{1}{2}|y|^2 - V(y), \\ h: z &\mapsto -\frac{1}{2}|z|^2 - V(z). \end{aligned}$$

Using the inequality

$$\frac{1}{2}\langle a, x - y \rangle - \frac{1}{4}|x|^2 - \frac{1}{4}|y|^2 \leq \frac{1}{2}|a|^2 - \frac{1}{8}|x + y|^2,$$

and the convexity of  $V$  we obtain

$$\frac{1}{2}f(x) + \frac{1}{2}g(y) \leq \frac{1}{2}|a|^2 + h\left(\frac{x+y}{2}\right), \quad \forall x, y \in \mathbb{R}^n.$$

Hence by Prékopa-Leindler

$$\left(\int_{\mathbb{R}^n} e^{f(x)} dx\right)^{1/2} \left(\int_{\mathbb{R}^n} e^{g(y)} dy\right)^{1/2} \leq e^{|a|^2/2} \int_{\mathbb{R}^n} e^{h(z)} dz.$$

This can be rewritten as

$$\frac{1}{2}\Lambda(a) + \frac{1}{2}\Lambda(-a) - \Lambda(0) \leq \frac{1}{2}|a|^2.$$

Letting  $a$  tend to 0 we obtain  $\langle \nabla^2 \Lambda(0)a, a \rangle \leq |a|^2$  which, together with (11), yields the result.  $\square$



Recalling (5) and applying Itô's formula to  $\log(f_t)$  yields

$$d\log(f_t)(x) = \langle A_t^{-1/2}(x - a_t), dW_t \rangle - \frac{1}{2} \langle A_t^{-1}(x - a_t), x - a_t \rangle dt$$

This shows that the density of the measure  $\mu_t$  satisfies

$$f_t(x) = f(x) \exp\left(c_t + \langle b_t, x \rangle - \frac{1}{2} \langle B_t x, x \rangle\right) \quad (12)$$

where  $c_t, b_t$  are some random processes, and

$$B_t = \int_0^t A_s^{-1} ds. \quad (13)$$

**Lemma 6.** *If the initial measure  $\mu$  is more-log-concave than the standard Gaussian measure, then almost surely*

$$\|A_t\|_{op} \leq e^{-t}, \quad \forall t \geq 0.$$

*Proof.* Since  $\mu$  is more log-concave than the Gaussian, equation (12) shows that the density  $f_t$  of  $\mu_t$  satisfies

$$f_t(x) = \exp\left(-\frac{1}{2}|x|^2 - \frac{1}{2} \langle B_t x, x \rangle - V_t(x)\right)$$

for some convex function  $V_t$ . By the previous lemma, the covariance matrix  $A_t$  of  $\mu_t$  satisfies

$$A_t \leq (B_t + \text{id})^{-1} \leq \frac{1}{\lambda_t + 1} \text{id},$$

where  $\lambda_t$  is the lowest eigenvalue of  $B_t$ . Therefore

$$\|A_t\|_{op} \leq \frac{1}{\lambda_t + 1}.$$

On the other hand, the equality (13) yields

$$\lambda_t \geq \int_0^t \|A_s\|_{op}^{-1} ds,$$

showing that

$$\int_0^t \|A_s\|_{op}^{-1} ds + 1 \leq \|A_t\|_{op}^{-1}.$$

Integrating this differential inequality yields the result.  $\square$

The following proposition will be crucial for the proof of our main theorem. Its proof is more involved than the proof of previous estimate, and we refer to [E, Section 3].

**Proposition 7.** *If the initial measure  $\mu$  is log-concave then*

$$\mathbb{E}\|A_t\|_{op} \leq C_0 \|A_0\|_{op} \tau_n^2 \log(n) e^{-t}, \quad \forall t \geq 0,$$

where  $C_0$  is a universal constant.

### 3 Proof of the main theorem

We start with an elementary lemma.

**Lemma 8.** *Let  $X$  be a log-concave random vector in  $\mathbb{R}^n$  and let  $\|\cdot\|$  be a norm. Then for any event  $F$*

$$\mathbb{E}(\|X\|; F) \leq C_1 \sqrt{\mathbb{P}(F)} \mathbb{E}(\|X\|),$$

where  $C_1$  is a universal constant. In particular, if  $\mathbb{P}(F) \leq (2C_1)^{-2}$ , one has

$$\mathbb{E}(\|X\|) \leq 2\mathbb{E}(\|X\|; F^c), \quad (14)$$

where  $F^c$  is the complement of  $F$ .

*Proof.* This is an easy consequence of Borell's lemma, which states as follows. There exist universal constants  $C, c > 0$  such that,

$$\mathbb{P}\left(\|X\| > t\mathbb{E}(\|X\|)\right) \leq Ce^{-ct}.$$

By Fubini's theorem and the Cauchy-Schwarz inequality

$$\mathbb{E}(\|X\|; F) = \int_0^\infty \mathbb{P}(\|X\| > t, F) dt \leq \left(\int_0^\infty \sqrt{\mathbb{P}(\|X\| > t)} dt\right) \times \sqrt{\mathbb{P}(F)}.$$

Plugging in Borell's inequality yields the result, with constant  $C_1 = 2C/c$ .  $\square$

The next ingredient we will need is the following proposition, which we learnt from B.Maurey. The authors are not aware of any published similar result.

**Proposition 9.** *Let  $(M_t)_{t \geq 0}$  be a continuous martingale taking values in  $\mathbb{R}^n$ . Assume that  $M_0 = 0$  and that the quadratic variation of  $M$  satisfies*

$$\forall t > 0, \quad [M]_t \leq \text{id},$$

*almost surely. Then  $(M_t)_{t \geq 0}$  converges almost surely, and the limit satisfies the following inequality. Letting  $\Gamma$  be a standard Gaussian vector, we have for every convex function  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$*

$$\mathbb{E}\varphi(M_\infty) \leq \mathbb{E}\varphi(\Gamma).$$

*Proof.* The hypothesis implies that  $M$  is bounded in  $L^2$ , hence convergent by Doob's theorem. Let  $X$  be a standard Gaussian vector on  $\mathbb{R}^n$  independent of  $(M_t)_{t \geq 0}$ . We claim that

$$Y = M_\infty + (\text{id} - [M]_\infty)^{1/2} X$$

is also a standard Gaussian vector. Indeed, for a fixed  $x \in \mathbb{R}^n$  one has

$$\begin{aligned} \mathbb{E}\left(e^{i\langle x, Y \rangle} \mid (M_t)_{t \geq 0}\right) &= \exp\left(i\langle x, M_\infty \rangle + \frac{1}{2}\langle [M]_\infty x, x \rangle - \frac{1}{2}|x|^2\right) \\ &= \exp\left(iL_\infty + \frac{1}{2}[L]_\infty - \frac{1}{2}|x|^2\right), \end{aligned}$$

where  $L$  is the real martingale defined by  $L_t = \langle M_t, x \rangle$ . Itô's formula shows that

$$D_t = \exp\left(iL_t + \frac{1}{2}[L]_t\right)$$

is a local martingale. On the other hand the hypothesis yields

$$|D_t| = \exp\left(\frac{1}{2}\langle [M]_t x, x \rangle\right) \leq \exp\left(\frac{1}{2}|x|^2\right)$$

almost surely. This shows that  $(D_t)_{t \geq 0}$  is a bounded martingale; in particular

$$\mathbb{E}(D_\infty) = \mathbb{E}(D_0) = 1,$$

since  $M_0 = 0$ . Therefore

$$\mathbb{E}(e^{i\langle x, Y \rangle}) = e^{-|x|^2/2},$$

proving the claim. Similarly (just replace  $X$  by  $-X$ )

$$Z = M_\infty - (\text{id} - [M]_\infty)^{1/2} X$$

is also standard Gaussian vector. Now, given a convex function  $\phi$ , we have

$$\mathbb{E}\phi(M_\infty) = \mathbb{E}\phi\left(\frac{Y+Z}{2}\right) \leq \frac{1}{2}\mathbb{E}(\phi(Y) + \phi(Z)) = \mathbb{E}\phi(Y),$$

which is the result.  $\square$

We are now ready to prove the main theorem.

*Proof of Theorem 1.* Let us prove that given a norm  $\|\cdot\|$  and a log-concave vector  $X$  satisfying  $\mathbb{E}(X) = 0$  we have

$$\mathbb{E}\|X\| \leq C\tau_n(\log n)^{1/2} \|\text{cov}(X)\|_{op}^{1/2} \mathbb{E}\|\Gamma\|, \quad (15)$$

for some universal constant  $C$ . If  $X$  is assumed to be isotropic, then  $\text{cov}(X) = \text{id}$  and we end up with the desired inequality (4).

Our first step is to reduce the proof to the case that  $X$  has a compact support. Assume that (15) holds for such vectors, and for  $r > 0$ , let  $Y_r$  be a random vector distributed according to the conditional law of  $X$  given the event  $\{|X| \leq r\}$ . Then  $Y_r$  is a compactly supported log-concave vector, and by our assumption,

$$\mathbb{E}\|Y_r - \mathbb{E}(Y_r)\| \leq C\tau_n(\log n)^{1/2} \|\text{cov}(Y_r)\|_{op}^{1/2} \mathbb{E}\|\Gamma\|. \quad (16)$$

Besides, it is easily seen by dominated convergence that

$$\begin{aligned} \lim_{r \rightarrow +\infty} \mathbb{E}\|Y_r - \mathbb{E}Y_r\| &= \mathbb{E}\|X\|, \\ \lim_{r \rightarrow +\infty} \text{cov}(Y_r) &= \text{cov}(X). \end{aligned}$$

So letting  $r$  tend to  $+\infty$  in (16) yields (15). Therefore, we may continue the proof under the assumption that  $X$  is compactly supported.

We use the stochastic process  $(\mu_t)_{t \geq 0}$  defined in the beginning of the previous section, with the starting law  $\mu$  being the law of  $X$ . Let  $T$  be the following stopping time:

$$T = \inf\left(t \geq 0, \int_0^t A_s ds > C^2 \tau_n^2 \log n \|A_0\|_{op}\right),$$

where  $C$  is a positive constant to be fixed later and with the usual convention that  $\inf(\emptyset) = +\infty$ . Define the stopped process  $a^T$  by

$$(a^T)_t = a_{\min(t, T)}.$$

By the optional stopping theorem, this process is also a martingale and by definition of  $T$  its quadratic variation satisfies

$$[a^T]_t \leq C^2 \tau_n^2 \log n \|A_0\|_{op}, \quad \forall t \geq 0.$$

Also  $(a^T)_0 = a_0 = \mathbb{E}(X) = 0$ . Applying Proposition 9 we get

$$\mathbb{E}\|a_T\| = \mathbb{E}\|(a^T)_\infty\| \leq C \tau_n (\log n)^{1/2} \|A_0\|_{op}^{1/2} \mathbb{E}\|\Gamma\|. \quad (17)$$

On the other hand, using Proposition 7 and Markov inequality we get

$$\mathbb{P}(T < +\infty) = \mathbb{P}\left(\int_0^\infty \|A_s\|_{op} ds > C^2 \tau_n^2 \log n \|A_0\|_{op}\right) \leq \frac{C_0}{C^2}.$$

So  $\mathbb{P}(T < +\infty)$  can be rendered arbitrarily small by choosing  $C$  large enough. By Proposition 4 we have  $a_\infty = X$  in law; in particular  $a_\infty$  is log-concave. If  $\mathbb{P}(T < +\infty)$  is small enough, we get using Lemma 8

$$\begin{aligned} \mathbb{E}\|X\| &= \mathbb{E}\|a_\infty\| \leq 2\mathbb{E}(\|a_\infty\|; T = \infty) \\ &= 2\mathbb{E}(\|a_T\|; T = \infty) \leq 2\mathbb{E}\|a_T\|. \end{aligned}$$

Combining this with (17) and recalling that  $A_0 = \text{cov}(X)$  we obtain the result (15).  $\square$

The proof of Proposition 3 follows the same lines. The main difference is that Proposition 6 is used in lieu of Proposition 7.

*Proof of Proposition 3.* Let  $Y_r$  be a random vector distributed according to the conditional law of  $X$  given  $|X| \leq r$ . Then  $Y_r$  is also more log-concave than  $\Gamma$  and

$$\mathbb{E}\varphi(Y_r) \rightarrow \mathbb{E}\varphi(X)$$

as  $r \rightarrow +\infty$ . So again we can assume that  $X$  is compactly supported, and consider the process  $(\mu_t)_{t \geq 0}$  starting from the law of  $X$ .

By Lemma 6, the process  $(a_t)_{t \geq 0}$  is a martingale whose quadratic variation satisfies

$$[a]_t = \int_0^t A_s ds \leq \text{id}, \quad \forall t \geq 0,$$

almost surely. Since again  $a_0 = \mathbb{E}(X) = 0$ , Proposition 9 yields the result.  $\square$

*Remark.* This proof is essentially due to Maurey; although his (unpublished) argument relied on a different stochastic construction.

## 4 Application to Mean Width and to the Isotropic Constant

In this section, we prove Corollary 2.

Let  $\Gamma$  be a standard Gaussian vector in  $\mathbb{R}^n$  and let  $\Theta$  be a point uniformly distributed in  $\mathbb{S}^{n-1}$ . Integration in polar coordinates shows that for any norm  $\|\cdot\|$ ,

$$\mathbb{E}\|\Gamma\| = c_n \mathbb{E}\|\Theta\|,$$

where

$$c_n = \mathbb{E}|\Gamma| = \sqrt{n} + O(1),$$

since  $\Gamma$  has the thin-shell property. Theorem 1 can thus be restated as follows. If  $Y$  is an isotropic log-concave random vector and  $K$  is a convex body containing 0 in its interior then

$$\mathbb{E}\|Y\|_K \leq C \sqrt{n \log n} \tau_n M(K). \quad (18)$$

Now let  $K$  be an isotropic convex body and let  $X$  be a random vector uniform on  $K$ . Since  $X \in K$  almost surely, we have  $\|X\|_{K^\circ} \geq |X|^2$ , hence

$$\mathbb{E}\|X\|_{K^\circ} \geq \mathbb{E}|X|^2 = n.$$

Applying (18) to  $K^\circ$  and to  $X$  thus gives

$$M^*(K) \geq \frac{c\sqrt{n}}{\sqrt{\log n} \tau_n},$$

which is (i).

In [Bou], Bourgain combined the inequality

$$\mathbb{E}\|X\| \leq Cn^{1/4} \mathbb{E}\|\Gamma\| \quad (19)$$

with a theorem of Pisier to get the estimate

$$L_K \leq Cn^{1/4} \log n.$$

Part (ii) of the corollary is obtained along the same lines, replacing (19) by our main theorem. We sketch the argument for completeness.

Recall that  $K$  is assumed to be isotropic and that  $X$  is uniform on  $K$ . Let  $T$  be a positive linear map of determinant 1. Then by the arithmetic-geometric inequality

$$\mathbb{E}\|X\|_{(TK)^\circ} \geq \mathbb{E}\langle X, TX \rangle = \text{Tr}(T) \geq n.$$

Applying (18) to the random vector  $X$  and the convex body  $(TK)^\circ$  we get

$$M^*(TK) \geq \frac{c\sqrt{n}}{\sqrt{\log n} \tau_n}. \quad (20)$$

Now we claim that given a convex body  $K$  containing 0 in its interior, there exists a positive linear map  $T$  of determinant 1 such that

$$M^*(TK) \leq C|K|^{1/n}\sqrt{n}\log n. \quad (21)$$

Taking this for granted and combining it with (20) we obtain

$$|K|^{-1/n} \leq C'(\log n)^{3/2}\tau_n.$$

which is part (ii) of the corollary.

It remains to prove the claim (21). Clearly

$$M^*(K) \leq M^*(K - K),$$

and by the Rogers-Shephard inequality (see [RS])

$$|K - K| \leq 4^n|K|.$$

This shows that it is enough to prove the claim when  $K$  is symmetric about the origin. Now if  $K$  is a symmetric convex body in  $\mathbb{R}^n$ , Pisier's Rademacher-projection estimate together with a result of Figiel and Tomczak-Jaegermann (see e.g. [P, Theorem 2.5 and Theorem 3.11]) guarantee the existence of  $T$  such that

$$M(TK)M^*(TK) \leq C\log(n),$$

where  $C$  is a universal constant. On the other hand, using Jensen's inequality and integrating in polar coordinate we get

$$\begin{aligned} M(TK) &= \int_{\mathbb{S}^{n-1}} \|\theta\|_{TK} \sigma(d\theta) \geq \left( \int_{\mathbb{S}^{n-1}} \|\theta\|_{TK}^{-n} \sigma(d\theta) \right)^{-1/n} \\ &= \left( \frac{|B_2^n|}{|TK|} \right)^{1/n} \geq \frac{c}{\sqrt{n}|K|^{1/n}}, \end{aligned}$$

finishing the proof of (21).

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