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# SHORT PROBABILISTIC PROOF OF THE BRASCAMP-LIEB AND BARTHE THEOREMS

JOSEPH LEHEC

ABSTRACT. We give a short proof of the Brascamp-Lieb theorem, which asserts that a certain general form of Young's convolution inequality is saturated by Gaussian functions. The argument is inspired by Borell's stochastic proof of the Prékopa-Leindler inequality and applies also to the reversed Brascamp-Lieb inequality, due to Barthe.

## 1. INTRODUCTION

A Brascamp-Lieb datum on  $\mathbb{R}^n$  is a finite sequence

$$(1) \quad (c_1, B_1), \dots, (c_m, B_m)$$

where  $c_i$  is a positive number and  $B_i: \mathbb{R}^n \rightarrow \mathbb{R}^{n_i}$  is linear and onto. The Brascamp-Lieb constant associated to this datum is the smallest real number  $C$  such that the inequality

$$(2) \quad \int_{\mathbb{R}^n} \prod_{i=1}^m (f_i \circ B_i)^{c_i} dx \leq C \prod_{i=1}^m \left( \int_{\mathbb{R}^{n_i}} f_i dx \right)^{c_i}$$

holds for every set of non-negative integrable functions  $f_i: \mathbb{R}^{n_i} \rightarrow \mathbb{R}$ . The Brascamp-Lieb theorem [9, 13] asserts that (2) is saturated by Gaussian functions. In other words if (2) holds for every functions  $f_1, \dots, f_m$  of the form

$$f_i(x) = e^{-\langle A_i x, x \rangle / 2}$$

where  $A_i$  is a symmetric positive definite matrix on  $\mathbb{R}^{n_i}$  then (2) holds for every set of functions  $f_1, \dots, f_m$ .

The reversed Brascamp-Lieb constant associated to (1) is the smallest constant  $C_r$  such that for every non-negative measurable functions  $f_1, \dots, f_m, f$  satisfying

$$(3) \quad \prod_{i=1}^m f_i(x_i)^{c_i} \leq f\left(\sum_{i=1}^m c_i B_i^* x_i\right)$$

for every  $(x_1, \dots, x_m) \in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m}$  we have

$$(4) \quad \prod_{i=1}^m \left( \int_{\mathbb{R}^{n_i}} f_i dx \right)^{c_i} \leq C_r \int_{\mathbb{R}^n} f dx.$$

It was shown by Barthe [1] that again Gaussian functions saturate the inequality. The original paper of Brascamp and Lieb [9] rely on symmetrization techniques. Barthe's argument uses optimal transport and works for both the direct and the reversed inequality. More recent proofs of the direct inequality [5, 6, 10, 11] all

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rely on semi-group techniques. There are also semi-group proofs of the reversed inequality, at least when the Brascamp-Lieb datum has the following property

$$(5) \quad \begin{aligned} B_i B_i^* &= \text{id}_{\mathbb{R}^{n_i}}, \quad \forall i \leq m, \\ \sum_{i=1}^m c_i B_i^* B_i &= \text{id}_{\mathbb{R}^n}, \end{aligned}$$

called *frame condition* in the sequel. This was achieved by Barthe and Cordero-Erausquin [2] in the rank 1 case (when all dimensions  $n_i$  equal 1) and Barthe and Huet [3] in any dimension.

The purpose of the present article is to give a short probabilistic proof of the Brascamp-Lieb and Barthe theorems. Our main tool shall be a representation formula for the quantity

$$\ln \left( \int e^{g(x)} \gamma(dx) \right),$$

where  $\gamma$  is a Gaussian measure. Let us describe it briefly. Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space, let  $(\mathcal{F}_t)_{t \in [0, T]}$  be a filtration and let

$$(W_t)_{t \in [0, T]}$$

be a Brownian motion taking values in  $\mathbb{R}^n$  (we fix a finite time horizon  $T$ ). Assuming that the covariance matrix  $A$  of the random vector  $W_1$  has full rank, we let  $\mathbb{H}$  be the Cameron-Martin space associated to  $W$ ; namely the Hilbert space of absolutely continuous paths  $u: [0, T] \rightarrow \mathbb{R}^n$  starting from 0, equipped with the norm

$$\|u\|_{\mathbb{H}} = \left( \int_0^T \langle A^{-1} \dot{u}_s, \dot{u}_s \rangle ds \right)^{1/2}.$$

In the sequel we call *drift* any adapted process  $U$  which belongs to  $\mathbb{H}$  almost surely. The following formula is due to Boué and Dupuis [8] (see also [7, 12]).

**Proposition 1.** *Let  $g: \mathbb{R}^n \rightarrow \mathbb{R}$  be measurable and bounded from below, then*

$$\log \left( \mathbb{E} e^{g(W_T)} \right) = \sup \left\{ \mathbb{E} \left( g(W_T + U_T) - \frac{1}{2} \|U\|_{\mathbb{H}}^2 \right) \right\}$$

where the supremum is taken over all drifts  $U$ .

In [7], Borell rediscovers this formula and shows that it yields the Prékopa-Leindler inequality (a reversed form of Hölder's inequality) very easily. Later on Cordero and Maurey noticed that under the frame condition, both the direct and reversed Brascamp-Lieb inequalities could be recovered this way (this was not published but is explained in [12]). The purpose of this article is, following Borell, Cordero and Maurey, to show that the Brascamp-Lieb and Barthe theorems in full generality are direct consequences of Proposition 1.

## 2. THE DIRECT INEQUALITY

Replace  $f_i$  by  $x \mapsto f_i(x/\lambda)$  in inequality (2). The left-hand side of the inequality is multiplied by  $\lambda^n$  and the right-hand side by  $\lambda^{\sum_{i=1}^m c_i n_i}$ . Therefore, a necessary condition for  $C$  to be finite is

$$\sum_{i=1}^m c_i n_i = n.$$

This homogeneity condition will be assumed throughout the rest of the article.

**Theorem 2.** *Assume that there exists a positive definite matrix  $A$  satisfying*

$$(6) \quad A^{-1} = \sum_{i=1}^m c_i B_i^* (B_i A B_i^*)^{-1} B_i.$$

*Then the Brascamp-Lieb constant is*

$$C = \left( \frac{\det(A)}{\prod_{i=1}^m \det(B_i A B_i^*)^{c_i}} \right)^{1/2},$$

*and there is equality in (2) for the following Gaussian functions*

$$(7) \quad f_i: x \in \mathbb{R}^{n_i} \mapsto e^{-\langle (B_i A B_i^*)^{-1} x, x \rangle / 2}, \quad i \leq m.$$

*Remark.* If the frame condition (5) holds then  $A = \text{id}_{\mathbb{R}^n}$  satisfies (6) and the Brascamp-Lieb constant is 1.

*Proof.* Because of (6), if the functions  $f_i$  are defined by (7) then

$$\prod_{i=1}^m (f_i(B_i x))^{c_i} = e^{-\langle A^{-1} x, x \rangle / 2}.$$

The equality case follows easily (recall the homogeneity condition  $\sum c_i n_i = n$ ). Let us prove the inequality. Let  $f_1, \dots, f_m$  be non-negative integrable functions on  $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$ , respectively and let

$$f: x \in \mathbb{R}^n \mapsto \prod_{i=1}^m f_i(B_i x)^{c_i}.$$

Fix  $\delta > 0$ , let  $g_i = \log(f_i + \delta)$  for every  $i \leq m$  and let

$$g(x) = \sum_{i=1}^m c_i g_i(B_i x).$$

The functions  $(g_i)_{i \leq m}, g$  are bounded from below. Fix a time horizon  $T$ , let  $(W_t)_{t \geq T}$  be a Brownian motion on  $\mathbb{R}^n$ , starting from 0 and having covariance  $A$ ; and let  $\mathbb{H}$  be the associated Cameron-Martin space. By Proposition 1, given  $\epsilon > 0$ , there exists a drift  $U$  such that

$$(8) \quad \begin{aligned} \log\left(\mathbb{E}e^{g(W_T)}\right) &\leq \mathbb{E}\left(g(W_T + U_T) - \frac{1}{2}\|U\|_{\mathbb{H}}^2\right) + \epsilon \\ &= \sum_{i=1}^m c_i \mathbb{E}g_i(B_i W_T + B_i U_T) - \frac{1}{2}\mathbb{E}\|U\|_{\mathbb{H}}^2 + \epsilon. \end{aligned}$$

The process  $B_i W$  is a Brownian motion on  $\mathbb{R}^{n_i}$  with covariance  $B_i A B_i^*$ . Set  $A_i = B_i A B_i^*$  and let  $\mathbb{H}_i$  be the Cameron-Martin space associated to  $B_i W$ . Equality (6) gives

$$\langle A^{-1} x, x \rangle = \sum_{i=1}^m c_i \langle A_i^{-1} B_i x, B_i x \rangle$$

for every  $x \in \mathbb{R}^n$ . This implies that

$$\|u\|_{\mathbb{H}}^2 = \sum_{i=1}^m c_i \|B_i u\|_{\mathbb{H}_i}^2$$

for every absolutely continuous path  $u: [0, T] \rightarrow \mathbb{R}^n$ . So that (8) becomes

$$\log\left(\mathbb{E}e^{g(W_T)}\right) \leq \sum_{i=1}^m c_i \mathbb{E}\left(g_i(B_i W_T + B_i U_T) - \frac{1}{2}\|B_i U\|_{\mathbb{H}_i}^2\right) + \epsilon.$$

By Proposition 1 again we have

$$\mathbb{E}\left(g_i(B_i W_T + B_i U_T) - \frac{1}{2}\|B_i U\|_{\mathbb{H}_i}^2\right) \leq \log\left(\mathbb{E}e^{g_i(B_i W_T)}\right)$$

for every  $i \leq m$ . We obtain (dropping  $\epsilon$  which is arbitrary)

$$(9) \quad \log\left(\mathbb{E}e^{g(W_T)}\right) \leq \sum_{i=1}^m c_i \log\left(\mathbb{E}e^{g_i(B_i W_T)}\right).$$

Recall that  $f \leq e^g$  and observe that

$$\prod_{i=1}^m \left(\mathbb{E}e^{g_i(B_i W_T)}\right)^{c_i} \leq \prod_{i=1}^m \left(\mathbb{E}f_i(B_i W_T)\right)^{c_i} + O(\delta^c),$$

for some positive constant  $c$ . Inequality (9) becomes (dropping the  $O(\delta^c)$  term)

$$(10) \quad \mathbb{E}f(W_T) \leq \prod_{i=1}^m \left(\mathbb{E}f_i(B_i W_T)\right)^{c_i}.$$

Since  $W_T$  is a centered Gaussian vector with covariance  $TA$

$$\mathbb{E}f(W_T) = \frac{1}{(2\pi T)^{n/2} \det(A)^{1/2}} \int_{\mathbb{R}^n} f(x) e^{-\langle A^{-1}x, x \rangle / 2T} dx,$$

and there is a similar equality for  $\mathbb{E}f_i(B_i W_T)$ . Then it is easy to see that letting  $T$  tend to  $+\infty$  in inequality (10) yields the result (recall that  $\sum c_i n_i = n$ ).  $\square$

*Example* (Optimal constant in Young's inequality). Young's convolution inequality asserts that if  $p, q, r \geq 1$  and are linked by the equation

$$(11) \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

then

$$\|F * G\|_r \leq \|F\|_p \|G\|_q,$$

for all  $F \in L_p$  and  $G \in L_q$ . When either  $p, q$  or  $r$  equals 1 or  $+\infty$  the inequality is a consequence of Hölder's inequality and is easily seen to be sharp. On the other hand when  $p, q, r$  belong to the open interval  $(1, +\infty)$  the best constant  $C$  in the inequality

$$\|F * G\|_r \leq C \|F\|_p \|G\|_q,$$

is actually smaller than 1. Let us compute it using the previous theorem. Observe that by duality  $C$  is the best constant in the inequality

$$(12) \quad \int_{\mathbb{R}^2} f^{c_1}(x+y) g^{c_2}(y) h^{c_3}(x) dx dy \leq C \left(\int_{\mathbb{R}} f\right)^{c_1} \left(\int_{\mathbb{R}} g\right)^{c_2} \left(\int_{\mathbb{R}} h\right)^{c_3},$$

where

$$c_1 = \frac{1}{p}, \quad c_2 = \frac{1}{q}, \quad c_3 = 1 - \frac{1}{r}.$$

In other words  $C$  is the Brascamp-Lieb constant in  $\mathbb{R}^2$  associated to the data

$$(c_1, B_1), (c_2, B_2), (c_3, B_3),$$

where  $B_1 = (1, 1)$ ,  $B_2 = (0, 1)$  and  $B_3 = (1, 0)$ . According to the previous result, we have to find a positive definite matrix  $A$  satisfying

$$A^{-1} = \sum_{i=1}^3 c_i B_i^* (B_i A B_i^*)^{-1} B_i.$$

Letting  $A = \begin{pmatrix} x & z \\ z & y \end{pmatrix}$ , this equation turns out to be equivalent to

$$\begin{aligned} (1 - c_2)xy + yz + c_2z^2 &= 0 \\ (1 - c_3)xy + xz + c_3z^2 &= 0 \\ c_1 + c_2 + c_3 &= 2. \end{aligned}$$

The third equation is just the Young constraint (11). The first two equations admit two families of solutions: either  $(x, y, z)$  is a multiple of  $(1, 1, -1)$  or  $(x, y, z)$  is a multiple of

$$(c_3(1 - c_3), c_2(1 - c_2), -(1 - c_2)(1 - c_3)).$$

The constraint  $xy - z^2 > 0$  rules out the first solution. The second solution is fine since  $c_1, c_2$  and  $c_3$  are assumed to belong to the open interval  $(0, 1)$ . By Theorem 2, the best constant in (12) is

$$C = \left( \frac{\det(A)}{\prod_{i=1}^3 \det(B_i A B_i^*)^{c_i}} \right)^{1/2} = \left( \frac{(1 - c_1)^{1-c_1} (1 - c_2)^{1-c_2} (1 - c_3)^{1-c_3}}{c_1^{c_1} c_2^{c_2} c_3^{c_3}} \right)^{1/2}.$$

In terms of  $p, q, r$  we have

$$C = \left( \frac{p^{1/p} q^{1/q} r^{1/r'}}{p^{1/p'} q^{1/q'} r^{1/r}} \right)^{1/2}$$

where  $p', q', r'$  are the conjugate exponents of  $p, q, r$ , respectively. This is indeed the best constant in Young's inequality, first obtained by Beckner [4].

### 3. THE REVERSED INEQUALITY

**Theorem 3.** *Again, assume that there is a matrix  $A$  satisfying (6). Then the reversed Brascamp-Lieb constant is*

$$C_r = \left( \frac{\det(A)}{\prod_{i=1}^m \det(B_i A B_i^*)^{c_i}} \right)^{1/2}.$$

There is equality in (4) for the following Gaussian functions

$$f_i: x \in \mathbb{R}^{n_i} \mapsto e^{-\langle B_i A B_i^* x, x \rangle / 2}, \quad i \leq m.$$

$$f: x \in \mathbb{R}^n \mapsto e^{-\langle A x, x \rangle / 2}.$$

*Remark.* Observe that under condition (6) the Brascamp-Lieb constant and the reversed constant are the same, but the extremizers differ.

We shall use the following elementary lemma.

**Lemma 4.** *Let  $A_1, \dots, A_m$  be positive definite matrices on  $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$ , respectively and let*

$$A = \left( \sum_{i=1}^m c_i B_i^* A_i^{-1} B_i \right)^{-1}.$$

Then for all  $x \in \mathbb{R}^n$

$$\langle Ax, x \rangle = \inf \left\{ \sum_{i=1}^m c_i \langle A_i x_i, x_i \rangle, \sum_{i=1}^m c_i B_i^* x_i = x \right\}.$$

*Proof.* Let  $x_1, \dots, x_m$  and let

$$(13) \quad x = \sum_{i=1}^m c_i B_i^* x_i.$$

Then by the Cauchy-Schwarz inequality (recall that the matrices  $A_i$  are positive definite)

$$\begin{aligned} \langle Ax, x \rangle &= \sum_{i=1}^m c_i \langle Ax, B_i^* x_i \rangle = \sum_{i=1}^m c_i \langle B_i Ax, x_i \rangle \\ &\leq \left( \sum_{i=1}^m c_i \langle A_i^{-1} B_i Ax, B_i Ax \rangle \right)^{1/2} \left( \sum_{i=1}^m c_i \langle A_i x_i, x_i \rangle \right)^{1/2} \\ &= \langle Ax, x \rangle^{1/2} \left( \sum_{i=1}^m c_i \langle A_i x_i, x_i \rangle \right)^{1/2}. \end{aligned}$$

Besides, given  $x \in \mathbb{R}^n$ , set  $x_i = A_i^{-1} B_i Ax$  for all  $i \leq m$ . Then (13) holds and there is equality in the above Cauchy-Schwarz inequality. This concludes the proof.  $\square$

*Proof of Theorem 3.* The equality case is a straightforward consequence of the hypothesis (6) and Lemma 4, details are left to the reader.

Let us prove the inequality. There is no loss of generality assuming that the functions  $f_1, \dots, f_m$  are bounded from above (otherwise replace  $f_i$  by  $\max(f_i, k)$ , let  $k$  tend to  $+\infty$  and use monotone convergence). Fix  $\delta > 0$  and let  $g_i = \log(f_i + \delta)$  for every  $i \leq m$ . By (3) and since the functions  $f_i$  are bounded from above, there exist positive constants  $c, C$  such that the function

$$g: x \in \mathbb{R}^n \mapsto \log(f(x) + C\delta^c),$$

satisfies

$$(14) \quad \sum_{i=1}^m c_i g_i(x_i) \leq g\left(\sum_{i=1}^m c_i B_i^* x_i\right)$$

for every  $x_1, \dots, x_m$ . Observe that the functions  $(g_i)_{i \leq m}, g$  are bounded from below. Let  $(W_t)_{t \leq T}$  be a Brownian motion on  $\mathbb{R}^n$  having covariance matrix  $A$ . Set  $A_i = B_i A B_i^*$ , then  $A_i^{-1} B_i W$  is a Brownian motion on  $\mathbb{R}^{n_i}$  with covariance matrix

$$(A_i^{-1} B_i) A (A_i^{-1} B_i)^* = A_i^{-1} (B_i A B_i^*) A_i^{-1} = A_i^{-1}.$$

Let  $\mathbb{H}_i$  be the associated Cameron-Martin space. By Proposition 1 there exists a ( $\mathbb{R}^{n_i}$ -valued) drift  $U_i$  such that

$$(15) \quad \log\left(\mathbb{E} e^{g_i(A_i^{-1} B_i W_T)}\right) \leq \mathbb{E}\left(g_i(A_i^{-1} B_i W_T + (U_i)_T) - \frac{1}{2} \|U_i\|_{\mathbb{H}_i}^2\right) + \epsilon.$$

By (14) and (6)

$$\begin{aligned} \sum_{i=1}^m c_i g_i(A_i^{-1} B_i W_T + (U_i)_T) &\leq g\left(\sum_{i=1}^m c_i B_i^* (A_i^{-1} B_i W_T + (U_i)_T)\right) \\ &= g\left(A^{-1} W_T + \sum_{i=1}^m c_i B_i^* (U_i)_T\right). \end{aligned}$$

The Brownian motion  $(A^{-1}W)_{t \leq T}$  has covariance matrix  $A^{-1}A(A^{-1})^* = A^{-1}$ . Let  $\mathbb{H}$  be the associated Cameron-Martin space. Lemma 4 shows that

$$\left\langle A \left( \sum_{i=1}^m c_i B_i^* x_i \right), \sum_{i=1}^m c_i B_i^* x_i \right\rangle \leq \sum_{i=1}^m c_i \langle A_i x_i, x_i \rangle$$

for every  $x_1, \dots, x_m$  in  $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$ , respectively. Therefore

$$\left\| \sum_{i=1}^m c_i B_i^* u_i \right\|_{\mathbb{H}}^2 \leq \sum_{i=1}^m c_i \|u_i\|_{\mathbb{H}_i}^2.$$

for every sequence of absolutely continuous paths  $(u_i: [0, T] \rightarrow \mathbb{R}^{n_i})_{i \leq m}$ . Thus multiplying (15) by  $c_i$  and summing over  $i$  yields

$$\begin{aligned} \sum_{i=1}^m c_i \log\left(\mathbb{E} e^{g_i(A_i^{-1} B_i W_T)}\right) \\ \leq \mathbb{E} \left[ g\left(A^{-1} W_T + \sum_{i=1}^m c_i B_i^* (U_i)_T\right) - \frac{1}{2} \left\| \sum_{i=1}^m c_i B_i^* U_i \right\|_{\mathbb{H}}^2 \right] + \sum_{i=1}^m c_i \epsilon. \end{aligned}$$

Hence, using Proposition 1 again and dropping  $\epsilon$  again,

$$(16) \quad \sum_{i=1}^m c_i \log\left(\mathbb{E} e^{g_i(A_i^{-1} B_i W_T)}\right)^{c_i} \leq \log\left(\mathbb{E} e^{g(A^{-1} W_T)}\right).$$

Recall that  $f_i \leq e^{g_i}$  for every  $i \leq m$  and that  $e^g = f + C\delta^c$ . Since  $\delta$  is arbitrary, inequality (16) becomes

$$\prod_{i=1}^m \left(\mathbb{E} f_i(A_i^{-1} B_i W_T)\right)^{c_i} \leq \mathbb{E} f(A^{-1} W_T).$$

Again, letting  $T$  tend to  $+\infty$  in this inequality yields the result.  $\square$

#### 4. THE BRASCAMP-LIEB AND BARTHE THEOREMS

So far we have seen that both the direct inequality and the reversed version are saturated by Gaussian functions when there exists a matrix  $A$  such that

$$(17) \quad A^{-1} = \sum_{i=1}^m c_i B_i^* (B_i A B_i^*)^{-1} B_i.$$

In this section, we briefly explain why this yields the Brascamp-Lieb and Barthe theorems.

Applying (2) to Gaussian functions gives

$$(18) \quad \prod_{i=1}^m \det(A_i)^{c_i} \leq C^2 \det\left(\sum_{i=1}^m c_i B_i^* A_i B_i\right),$$



for every sequence  $A_1, \dots, A_m$  of positive definite matrices on  $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$ . Let  $C_g$  be the Gaussian Brascamp-Lieb constant; namely the best constant in the previous inequality. We have  $C_g \leq C$  and it turns out that applying (4) to Gaussian functions yields  $C_g \leq C_r$  (one has to apply Lemma 4 at some point).

It is known since the work of Carlen and Cordero [10] that there is a dual formulation of (2) in terms of relative entropy. In the same way, there is a dual formulation of (18). For every positive matrix  $A$  on  $\mathbb{R}^n$ , one has

$$\log \det(A) = \inf_{B>0} \{ \operatorname{tr}(AB) - n - \log(\det(B)) \},$$

with equality when  $B = A^{-1}$ . Using this and the equality  $\sum_{i=1}^m c_i n_i = n$ , it is easily seen that  $C_g$  is also the best constant such that the inequality

$$(19) \quad \det(A) \leq C_g^2 \prod_{i=1}^m \det(B_i A B_i^*)^{c_i}$$

holds for every positive definite matrix  $A$  on  $\mathbb{R}^n$ .

*Example.* Assume that  $m = n$ , that  $c_1 = \dots = c_n = 1$  and that  $B_i(x) = x_i$  for  $i \in [n]$ . Inequality (18) trivially holds with constant 1 (and there is equality for every  $A_1, \dots, A_n$ ). On the other hand (19) becomes

$$\det(A) \leq \prod_{i=1}^n a_{ii},$$

for every positive definite  $A$ , with equality when  $A$  is diagonal. This is Hadamard's inequality.

**Lemma 5.** *If  $A$  is extremal in (19) then  $A$  satisfies (17).*

*Proof.* Just compute the gradient of the map

$$A > 0 \mapsto \log \det(A) - \sum_{i=1}^m c_i \log \det(B_i A B_i^*). \quad \square$$

Therefore, if the constant  $C_g$  is finite and if there is an extremizer  $A$  in (19) then  $A$  satisfies (17) and together with the results of the previous sections we get the Brascamp-Lieb and Barthe equalities

$$(20) \quad C = C_r = C_g.$$

Although it may happen that  $C_g < +\infty$  and no Gaussian extremizer exists, there is a way to bypass this issue. For the Brascamp-Lieb theorem, there is an abstract argument showing that it is enough to prove the equality  $C = C_g$  when there is a Gaussian extremizer. This argument relies on:

- (1) A criterion for having a Gaussian extremizer, due to Barthe [1] in the rank 1 case (namely when the dimensions  $n_i$  are all equal to 1) and Bennett, Carbery, Christ and Tao [6] in the general case.
- (2) A multiplicativity property of  $C$  and  $C_g$  due to Carlen, Lieb and Loss [11] in the rank 1 case and obtained in full generality in [6] again.

There is no point repeating this argument here, and we refer to [11, 6] instead. This settles the case of the  $C = C_g$  equality. As for the  $C = C_r$  equality, we observe that the above argument can be carried out verbatim once the multiplicativity property of the reversed Brascamp-Lieb constant is established. This is the purpose of the rest of the article.

**Definition 6.** Given a proper subspace  $E$  of  $\mathbb{R}^n$ , we let for  $i \leq m$

$$\begin{aligned} B_{i,E}: E &\rightarrow B_i E & B_{i,E^\perp}: E^\perp &\rightarrow (B_i E)^\perp \\ x &\mapsto B_i x, & x &\mapsto q_i(B_i x), \end{aligned}$$

where  $q_i$  is the orthogonal projection onto  $(B_i E)^\perp$ . Observe that both  $B_{i,E}$  and  $B_{i,E^\perp}$  are onto. Now we let  $C_{r,E}$  be the reversed Brascamp-Lieb constant on  $E$  associated to the datum

$$(c_1, B_{1,E}), \dots, (c_m, B_{m,E})$$

and  $C_{r,E^\perp}$  be the reversed Brascamp-Lieb constant on  $E^\perp$  associated to the datum

$$(c_1, B_{1,E^\perp}), \dots, (c_m, B_{m,E^\perp})$$

**Proposition 7.** *Let  $E$  be a proper subspace of  $\mathbb{R}^n$ , and assume that  $E$  is critical, in the sense that*

$$\dim(E) = \sum_{i=1}^m c_i \dim(B_i E).$$

Then  $C_r = C_{r,E} \times C_{r,E^\perp}$ .

Bennett, Carbery, Christ and Tao proved the corresponding property of  $C$  and  $C_g$ , we adapt their argument to prove the multiplicativity of  $C_r$ .

Let us prove the inequality  $C_r \leq C_{r,E} \times C_{r,E^\perp}$  first. This does not require  $E$  to be critical. Let  $f_1, \dots, f_m, f$  be functions on  $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}, \mathbb{R}$  respectively, satisfying

$$\prod_{i=1}^m f_i(z_i)^{c_i} \leq f\left(\sum_{i=1}^m c_i B_i^* z_i\right)$$

for all  $z_1, \dots, z_m$ . Fix  $(x_1, \dots, x_m) \in B_1 E \times \dots \times B_m E$ . Since  $(B_{i,E^\perp})^* y_i = B_i^* y_i$  for every  $y_i \in (B_i E)^\perp$ , applying the reversed Brascamp-Lieb inequality on  $E^\perp$  to the functions

$$y \in (B_i E)^\perp \mapsto f_i(x_i + y), \quad i \leq m$$

yields

$$\begin{aligned} \prod_{i=1}^m \left( \int_{(B_i E)^\perp} f_i(x_i + y) \, dy \right)^{c_i} &\leq C_{r,E^\perp} \int_{E^\perp} f\left(\sum_{i=1}^m c_i B_i^* x_i + y\right) \, dy \\ &= C_{r,E^\perp} \int_{E^\perp} f\left(\sum_{i=1}^m c_i (B_{i,E})^* x_i + y\right) \, dy. \end{aligned}$$

For the latter equality, observe that  $(B_{i,E})^* x_i = p(B_i^* x_i)$  where  $p$  is the orthogonal projection with range  $E$  and use the translation invariance of the Lebesgue measure. Applying the reversed Brascamp-Lieb inequality (this time on  $E$ ) and using Fubini's theorem we get

$$\prod_{i=1}^m \left( \int_{\mathbb{R}^{n_i}} f_i \, dx \right)^{c_i} \leq C_{r,E} C_{r,E^\perp} \int_{\mathbb{R}^n} f \, dx,$$

which is the result.

We start the proof of the inequality  $C_{r,E} C_{r,E^\perp} \leq C_r$  with a couple of simple observations.

**Lemma 8.** *Upper semi-continuous functions having compact support saturate the reversed Brascamp-Lieb inequality.*

*Proof.* The regularity of the Lebesgue measure implies that given a non-negative integrable function  $f_i$  on  $\mathbb{R}^{n_i}$  and  $\epsilon > 0$  there exists a non-negative linear combination of indicators of compact sets  $g_i$  satisfying

$$g_i \leq f_i \quad \text{and} \quad \int_{\mathbb{R}^{n_i}} f_i \, dx \leq (1 + \epsilon) \int_{\mathbb{R}^{n_i}} g_i \, dx.$$

The lemma follows easily.  $\square$

The proof of the following lemma is left to the reader.

**Lemma 9.** *If  $f_1, \dots, f_m$  are compactly supported and upper semi-continuous on  $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$  respectively, then the function  $f$  defined on  $\mathbb{R}^n$  by*

$$f(x) = \sup \left\{ \prod_{i=1}^m f_i(x_i)^{c_i}, \sum_{i=1}^m c_i B_i^* x_i = x \right\},$$

*is compactly supported and upper semi-continuous as well.*

*Remark.* If the Brascamp-Lieb datum happens to be *degenerate*, in the sense that the map  $(x_1, \dots, x_m) \mapsto \sum_{i=1}^m B_i^* x_i$  is not onto, then Brascamp-Lieb constants are easily seen to be  $+\infty$ . Still the previous lemma remains valid, provided the convention  $\sup \emptyset = 0$  is adopted.

Let us prove that  $C_{r,E} \times C_{r,E^\perp} \leq C_r$ . By Lemma 8, it is enough to prove that the inequality

$$\prod_{i=1}^m \left( \int_{B_i E} f_i \, dx \right)^{c_i} \times \prod_{i=1}^m \left( \int_{(B_i E)^\perp} g_i \, dx \right)^{c_i} \leq C_r \left( \int_E f \, dx \right) \left( \int_{E^\perp} g \, dx \right).$$

holds for every compactly supported upper semi-continuous functions  $(f_i)_{i \leq m}$  and  $(g_i)_{i \leq m}$ , where  $f$  and  $g$  are defined by

$$\begin{aligned} f: x \in E &\mapsto \sup \left\{ \prod_{i=1}^m f_i(x_i)^{c_i}, \sum_{i=1}^m c_i (B_i E)^* x_i = x \right\} \\ g: y \in E^\perp &\mapsto \sup \left\{ \prod_{i=1}^m g_i(y_i)^{c_i}, \sum_{i=1}^m c_i (B_i E^\perp)^* y_i = y \right\}. \end{aligned}$$

Let  $\epsilon > 0$ . For  $i \leq m$  define a function  $h_i$  on  $\mathbb{R}^{n_i}$  by

$$h_i(x + y) = f_i(x/\epsilon) g_i(y), \quad \forall x \in B_i E, \forall y \in (B_i E)^\perp,$$

and let

$$h: z \in \mathbb{R}^n \mapsto \sup \left\{ \prod_{i=1}^m h_i(z_i)^{c_i}, \sum_{i=1}^m c_i B_i^* z_i = z \right\}.$$

By definition of the reversed Brascamp-Lieb constant  $C_r$

$$(21) \quad \prod_{i=1}^m \left( \int_{\mathbb{R}^{n_i}} h_i \, dx \right)^{c_i} \leq C_r \int_{\mathbb{R}^n} h \, dx.$$

Using the equality  $\sum_{i=1}^m c_i \dim(B_i E) = \dim(E)$  we get

$$\epsilon^{-\dim(E)} \prod_{i=1}^m \left( \int_{\mathbb{R}^{n_i}} h_i \, dx \right)^{c_i} = \prod_{i=1}^m \left( \int_{B_i E} f_i \, dx \right)^{c_i} \times \prod_{i=1}^m \left( \int_{(B_i E)^\perp} g_i \, dx \right)^{c_i}.$$

On the other hand, we let the reader check that for every  $x \in E, y \in E^\perp$

$$h(\epsilon x + y) \leq f(x)g_\epsilon(y),$$

where

$$g_\epsilon(y) = \sup\{g(y'), |y - y'| \leq K\epsilon\}$$

and  $K$  is a constant depending on the diameters of the supports of the functions  $f_i$ . Therefore

$$\epsilon^{-\dim E} \int_{\mathbb{R}^n} h \, dx = \int_{E \times E^\perp} h(\epsilon x + y) \, dx dy \leq \left( \int_E f \, dx \right) \left( \int_{E^\perp} g_\epsilon \, dx \right).$$

Inequality (21) becomes

$$\prod_{i=1}^m \left( \int_{B_i E} f_i \, dx \right)^{c_i} \times \prod_{i=1}^m \left( \int_{(B_i E)^\perp} g_i \, dx \right)^{c_i} \leq C_r \left( \int_E f \, dx \right) \left( \int_{E^\perp} g_\epsilon \, dx \right).$$

By Lemma 9, the function  $g$  has compact support and is upper semi-continuous. This implies easily that

$$\lim_{\epsilon \rightarrow 0} \int_{E^\perp} g_\epsilon \, dx = \int_{E^\perp} g \, dx,$$

which concludes the proof.

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