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SHORT PROBABILISTIC PROOF OF THE BRASCAMP-LIEB AND BARTHE THEOREMS

JOSEPH LEHEC

ABSTRACT. We give a short proof of the Brascamp-Lieb theorem, which asserts that a certain general form of Young's convolution inequality is saturated by Gaussian functions. The argument is inspired by Borell's stochastic proof of the Prékopa-Leindler inequality and applies also to the reversed Brascamp-Lieb inequality, due to Barthe.

1. INTRODUCTION

A Brascamp-Lieb datum on \mathbb{R}^n is a finite sequence

$$(1) \quad (c_1, B_1), \dots, (c_m, B_m)$$

where c_i is a positive number and $B_i: \mathbb{R}^n \rightarrow \mathbb{R}^{n_i}$ is linear and onto. The Brascamp-Lieb constant associated to this datum is the smallest real number C such that the inequality

$$(2) \quad \int_{\mathbb{R}^n} \prod_{i=1}^m (f_i \circ B_i)^{c_i} dx \leq C \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i dx \right)^{c_i}$$

holds for every set of non-negative integrable functions $f_i: \mathbb{R}^{n_i} \rightarrow \mathbb{R}$. The Brascamp-Lieb theorem [9, 13] asserts that (2) is saturated by Gaussian functions. In other words if (2) holds for every functions f_1, \dots, f_m of the form

$$f_i(x) = e^{-\langle A_i x, x \rangle / 2}$$

where A_i is a symmetric positive definite matrix on \mathbb{R}^{n_i} then (2) holds for every set of functions f_1, \dots, f_m .

The reversed Brascamp-Lieb constant associated to (1) is the smallest constant C_r such that for every non-negative measurable functions f_1, \dots, f_m, f satisfying

$$(3) \quad \prod_{i=1}^m f_i(x_i)^{c_i} \leq f\left(\sum_{i=1}^m c_i B_i^* x_i\right)$$

for every $(x_1, \dots, x_m) \in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m}$ we have

$$(4) \quad \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i dx \right)^{c_i} \leq C_r \int_{\mathbb{R}^n} f dx.$$

It was shown by Barthe [1] that again Gaussian functions saturate the inequality. The original paper of Brascamp and Lieb [9] rely on symmetrization techniques. Barthe's argument uses optimal transport and works for both the direct and the reversed inequality. More recent proofs of the direct inequality [5, 6, 10, 11] all

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rely on semi-group techniques. There are also semi-group proofs of the reversed inequality, at least when the Brascamp-Lieb datum has the following property

$$(5) \quad \begin{aligned} B_i B_i^* &= \text{id}_{\mathbb{R}^{n_i}}, \quad \forall i \leq m, \\ \sum_{i=1}^m c_i B_i^* B_i &= \text{id}_{\mathbb{R}^n}, \end{aligned}$$

called *frame condition* in the sequel. This was achieved by Barthe and Cordero-Erausquin [2] in the rank 1 case (when all dimensions n_i equal 1) and Barthe and Huet [3] in any dimension.

The purpose of the present article is to give a short probabilistic proof of the Brascamp-Lieb and Barthe theorems. Our main tool shall be a representation formula for the quantity

$$\ln \left(\int e^{g(x)} \gamma(dx) \right),$$

where γ is a Gaussian measure. Let us describe it briefly. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, let $(\mathcal{F}_t)_{t \in [0, T]}$ be a filtration and let

$$(W_t)_{t \in [0, T]}$$

be a Brownian motion taking values in \mathbb{R}^n (we fix a finite time horizon T). Assuming that the covariance matrix A of the random vector W_1 has full rank, we let \mathbb{H} be the Cameron-Martin space associated to W ; namely the Hilbert space of absolutely continuous paths $u: [0, T] \rightarrow \mathbb{R}^n$ starting from 0, equipped with the norm

$$\|u\|_{\mathbb{H}} = \left(\int_0^T \langle A^{-1} \dot{u}_s, \dot{u}_s \rangle ds \right)^{1/2}.$$

In the sequel we call *drift* any adapted process U which belongs to \mathbb{H} almost surely. The following formula is due to Boué and Dupuis [8] (see also [7, 12]).

Proposition 1. *Let $g: \mathbb{R}^n \rightarrow \mathbb{R}$ be measurable and bounded from below, then*

$$\log \left(\mathbb{E} e^{g(W_T)} \right) = \sup \left\{ \mathbb{E} \left(g(W_T + U_T) - \frac{1}{2} \|U\|_{\mathbb{H}}^2 \right) \right\}$$

where the supremum is taken over all drifts U .

In [7], Borell rediscovers this formula and shows that it yields the Prékopa-Leindler inequality (a reversed form of Hölder's inequality) very easily. Later on Cordero and Maurey noticed that under the frame condition, both the direct and reversed Brascamp-Lieb inequalities could be recovered this way (this was not published but is explained in [12]). The purpose of this article is, following Borell, Cordero and Maurey, to show that the Brascamp-Lieb and Barthe theorems in full generality are direct consequences of Proposition 1.

2. THE DIRECT INEQUALITY

Replace f_i by $x \mapsto f_i(x/\lambda)$ in inequality (2). The left-hand side of the inequality is multiplied by λ^n and the right-hand side by $\lambda^{\sum_{i=1}^m c_i n_i}$. Therefore, a necessary condition for C to be finite is

$$\sum_{i=1}^m c_i n_i = n.$$

This homogeneity condition will be assumed throughout the rest of the article.

Theorem 2. *Assume that there exists a positive definite matrix A satisfying*

$$(6) \quad A^{-1} = \sum_{i=1}^m c_i B_i^* (B_i A B_i^*)^{-1} B_i.$$

Then the Brascamp-Lieb constant is

$$C = \left(\frac{\det(A)}{\prod_{i=1}^m \det(B_i A B_i^*)^{c_i}} \right)^{1/2},$$

and there is equality in (2) for the following Gaussian functions

$$(7) \quad f_i: x \in \mathbb{R}^{n_i} \mapsto e^{-\langle (B_i A B_i^*)^{-1} x, x \rangle / 2}, \quad i \leq m.$$

Remark. If the frame condition (5) holds then $A = \text{id}_{\mathbb{R}^n}$ satisfies (6) and the Brascamp-Lieb constant is 1.

Proof. Because of (6), if the functions f_i are defined by (7) then

$$\prod_{i=1}^m (f_i(B_i x))^{c_i} = e^{-\langle A^{-1} x, x \rangle / 2}.$$

The equality case follows easily (recall the homogeneity condition $\sum c_i n_i = n$). Let us prove the inequality. Let f_1, \dots, f_m be non-negative integrable functions on $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$, respectively and let

$$f: x \in \mathbb{R}^n \mapsto \prod_{i=1}^m f_i(B_i x)^{c_i}.$$

Fix $\delta > 0$, let $g_i = \log(f_i + \delta)$ for every $i \leq m$ and let

$$g(x) = \sum_{i=1}^m c_i g_i(B_i x).$$

The functions $(g_i)_{i \leq m}, g$ are bounded from below. Fix a time horizon T , let $(W_t)_{t \geq T}$ be a Brownian motion on \mathbb{R}^n , starting from 0 and having covariance A ; and let \mathbb{H} be the associated Cameron-Martin space. By Proposition 1, given $\epsilon > 0$, there exists a drift U such that

$$(8) \quad \begin{aligned} \log\left(\mathbb{E}e^{g(W_T)}\right) &\leq \mathbb{E}\left(g(W_T + U_T) - \frac{1}{2}\|U\|_{\mathbb{H}}^2\right) + \epsilon \\ &= \sum_{i=1}^m c_i \mathbb{E}g_i(B_i W_T + B_i U_T) - \frac{1}{2}\mathbb{E}\|U\|_{\mathbb{H}}^2 + \epsilon. \end{aligned}$$

The process $B_i W$ is a Brownian motion on \mathbb{R}^{n_i} with covariance $B_i A B_i^*$. Set $A_i = B_i A B_i^*$ and let \mathbb{H}_i be the Cameron-Martin space associated to $B_i W$. Equality (6) gives

$$\langle A^{-1} x, x \rangle = \sum_{i=1}^m c_i \langle A_i^{-1} B_i x, B_i x \rangle$$

for every $x \in \mathbb{R}^n$. This implies that

$$\|u\|_{\mathbb{H}}^2 = \sum_{i=1}^m c_i \|B_i u\|_{\mathbb{H}_i}^2$$

for every absolutely continuous path $u: [0, T] \rightarrow \mathbb{R}^n$. So that (8) becomes

$$\log\left(\mathbb{E}e^{g(W_T)}\right) \leq \sum_{i=1}^m c_i \mathbb{E}\left(g_i(B_i W_T + B_i U_T) - \frac{1}{2}\|B_i U\|_{\mathbb{H}_i}^2\right) + \epsilon.$$

By Proposition 1 again we have

$$\mathbb{E}\left(g_i(B_i W_T + B_i U_T) - \frac{1}{2}\|B_i U\|_{\mathbb{H}_i}^2\right) \leq \log\left(\mathbb{E}e^{g_i(B_i W_T)}\right)$$

for every $i \leq m$. We obtain (dropping ϵ which is arbitrary)

$$(9) \quad \log\left(\mathbb{E}e^{g(W_T)}\right) \leq \sum_{i=1}^m c_i \log\left(\mathbb{E}e^{g_i(B_i W_T)}\right).$$

Recall that $f \leq e^g$ and observe that

$$\prod_{i=1}^m \left(\mathbb{E}e^{g_i(B_i W_T)}\right)^{c_i} \leq \prod_{i=1}^m \left(\mathbb{E}f_i(B_i W_T)\right)^{c_i} + O(\delta^c),$$

for some positive constant c . Inequality (9) becomes (dropping the $O(\delta^c)$ term)

$$(10) \quad \mathbb{E}f(W_T) \leq \prod_{i=1}^m \left(\mathbb{E}f_i(B_i W_T)\right)^{c_i}.$$

Since W_T is a centered Gaussian vector with covariance TA

$$\mathbb{E}f(W_T) = \frac{1}{(2\pi T)^{n/2} \det(A)^{1/2}} \int_{\mathbb{R}^n} f(x) e^{-\langle A^{-1}x, x \rangle / 2T} dx,$$

and there is a similar equality for $\mathbb{E}f_i(B_i W_T)$. Then it is easy to see that letting T tend to $+\infty$ in inequality (10) yields the result (recall that $\sum c_i n_i = n$). \square

Example (Optimal constant in Young's inequality). Young's convolution inequality asserts that if $p, q, r \geq 1$ and are linked by the equation

$$(11) \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

then

$$\|F * G\|_r \leq \|F\|_p \|G\|_q,$$

for all $F \in L_p$ and $G \in L_q$. When either p, q or r equals 1 or $+\infty$ the inequality is a consequence of Hölder's inequality and is easily seen to be sharp. On the other hand when p, q, r belong to the open interval $(1, +\infty)$ the best constant C in the inequality

$$\|F * G\|_r \leq C \|F\|_p \|G\|_q,$$

is actually smaller than 1. Let us compute it using the previous theorem. Observe that by duality C is the best constant in the inequality

$$(12) \quad \int_{\mathbb{R}^2} f^{c_1}(x+y) g^{c_2}(y) h^{c_3}(x) dx dy \leq C \left(\int_{\mathbb{R}} f\right)^{c_1} \left(\int_{\mathbb{R}} g\right)^{c_2} \left(\int_{\mathbb{R}} h\right)^{c_3},$$

where

$$c_1 = \frac{1}{p}, \quad c_2 = \frac{1}{q}, \quad c_3 = 1 - \frac{1}{r}.$$

In other words C is the Brascamp-Lieb constant in \mathbb{R}^2 associated to the data

$$(c_1, B_1), (c_2, B_2), (c_3, B_3),$$

where $B_1 = (1, 1)$, $B_2 = (0, 1)$ and $B_3 = (1, 0)$. According to the previous result, we have to find a positive definite matrix A satisfying

$$A^{-1} = \sum_{i=1}^3 c_i B_i^* (B_i A B_i^*)^{-1} B_i.$$

Letting $A = \begin{pmatrix} x & z \\ z & y \end{pmatrix}$, this equation turns out to be equivalent to

$$\begin{aligned} (1 - c_2)xy + yz + c_2z^2 &= 0 \\ (1 - c_3)xy + xz + c_3z^2 &= 0 \\ c_1 + c_2 + c_3 &= 2. \end{aligned}$$

The third equation is just the Young constraint (11). The first two equations admit two families of solutions: either (x, y, z) is a multiple of $(1, 1, -1)$ or (x, y, z) is a multiple of

$$(c_3(1 - c_3), c_2(1 - c_2), -(1 - c_2)(1 - c_3)).$$

The constraint $xy - z^2 > 0$ rules out the first solution. The second solution is fine since c_1, c_2 and c_3 are assumed to belong to the open interval $(0, 1)$. By Theorem 2, the best constant in (12) is

$$C = \left(\frac{\det(A)}{\prod_{i=1}^3 \det(B_i A B_i^*)^{c_i}} \right)^{1/2} = \left(\frac{(1 - c_1)^{1-c_1} (1 - c_2)^{1-c_2} (1 - c_3)^{1-c_3}}{c_1^{c_1} c_2^{c_2} c_3^{c_3}} \right)^{1/2}.$$

In terms of p, q, r we have

$$C = \left(\frac{p^{1/p} q^{1/q} r^{1/r'}}{p^{1/p'} q^{1/q'} r^{1/r}} \right)^{1/2}$$

where p', q', r' are the conjugate exponents of p, q, r , respectively. This is indeed the best constant in Young's inequality, first obtained by Beckner [4].

3. THE REVERSED INEQUALITY

Theorem 3. *Again, assume that there is a matrix A satisfying (6). Then the reversed Brascamp-Lieb constant is*

$$C_r = \left(\frac{\det(A)}{\prod_{i=1}^m \det(B_i A B_i^*)^{c_i}} \right)^{1/2}.$$

There is equality in (4) for the following Gaussian functions

$$f_i: x \in \mathbb{R}^{n_i} \mapsto e^{-\langle B_i A B_i^* x, x \rangle / 2}, \quad i \leq m.$$

$$f: x \in \mathbb{R}^n \mapsto e^{-\langle A x, x \rangle / 2}.$$

Remark. Observe that under condition (6) the Brascamp-Lieb constant and the reversed constant are the same, but the extremizers differ.

We shall use the following elementary lemma.

Lemma 4. *Let A_1, \dots, A_m be positive definite matrices on $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$, respectively and let*

$$A = \left(\sum_{i=1}^m c_i B_i^* A_i^{-1} B_i \right)^{-1}.$$

Then for all $x \in \mathbb{R}^n$

$$\langle Ax, x \rangle = \inf \left\{ \sum_{i=1}^m c_i \langle A_i x_i, x_i \rangle, \sum_{i=1}^m c_i B_i^* x_i = x \right\}.$$

Proof. Let x_1, \dots, x_m and let

$$(13) \quad x = \sum_{i=1}^m c_i B_i^* x_i.$$

Then by the Cauchy-Schwarz inequality (recall that the matrices A_i are positive definite)

$$\begin{aligned} \langle Ax, x \rangle &= \sum_{i=1}^m c_i \langle Ax, B_i^* x_i \rangle = \sum_{i=1}^m c_i \langle B_i Ax, x_i \rangle \\ &\leq \left(\sum_{i=1}^m c_i \langle A_i^{-1} B_i Ax, B_i Ax \rangle \right)^{1/2} \left(\sum_{i=1}^m c_i \langle A_i x_i, x_i \rangle \right)^{1/2} \\ &= \langle Ax, x \rangle^{1/2} \left(\sum_{i=1}^m c_i \langle A_i x_i, x_i \rangle \right)^{1/2}. \end{aligned}$$

Besides, given $x \in \mathbb{R}^n$, set $x_i = A_i^{-1} B_i Ax$ for all $i \leq m$. Then (13) holds and there is equality in the above Cauchy-Schwarz inequality. This concludes the proof. \square

Proof of Theorem 3. The equality case is a straightforward consequence of the hypothesis (6) and Lemma 4, details are left to the reader.

Let us prove the inequality. There is no loss of generality assuming that the functions f_1, \dots, f_m are bounded from above (otherwise replace f_i by $\max(f_i, k)$, let k tend to $+\infty$ and use monotone convergence). Fix $\delta > 0$ and let $g_i = \log(f_i + \delta)$ for every $i \leq m$. By (3) and since the functions f_i are bounded from above, there exist positive constants c, C such that the function

$$g: x \in \mathbb{R}^n \mapsto \log(f(x) + C\delta^c),$$

satisfies

$$(14) \quad \sum_{i=1}^m c_i g_i(x_i) \leq g\left(\sum_{i=1}^m c_i B_i^* x_i\right)$$

for every x_1, \dots, x_m . Observe that the functions $(g_i)_{i \leq m}, g$ are bounded from below. Let $(W_t)_{t \leq T}$ be a Brownian motion on \mathbb{R}^n having covariance matrix A . Set $A_i = B_i A B_i^*$, then $A_i^{-1} B_i W$ is a Brownian motion on \mathbb{R}^{n_i} with covariance matrix

$$(A_i^{-1} B_i) A (A_i^{-1} B_i)^* = A_i^{-1} (B_i A B_i^*) A_i^{-1} = A_i^{-1}.$$

Let \mathbb{H}_i be the associated Cameron-Martin space. By Proposition 1 there exists a (\mathbb{R}^{n_i} -valued) drift U_i such that

$$(15) \quad \log\left(\mathbb{E} e^{g_i(A_i^{-1} B_i W_T)}\right) \leq \mathbb{E}\left(g_i(A_i^{-1} B_i W_T + (U_i)_T) - \frac{1}{2} \|U_i\|_{\mathbb{H}_i}^2\right) + \epsilon.$$

By (14) and (6)

$$\begin{aligned} \sum_{i=1}^m c_i g_i(A_i^{-1} B_i W_T + (U_i)_T) &\leq g\left(\sum_{i=1}^m c_i B_i^*(A_i^{-1} B_i W_T + (U_i)_T)\right) \\ &= g\left(A^{-1} W_T + \sum_{i=1}^m c_i B_i^*(U_i)_T\right). \end{aligned}$$

The Brownian motion $(A^{-1}W)_{t \leq T}$ has covariance matrix $A^{-1}A(A^{-1})^* = A^{-1}$. Let \mathbb{H} be the associated Cameron-Martin space. Lemma 4 shows that

$$\left\langle A\left(\sum_{i=1}^m c_i B_i^* x_i\right), \sum_{i=1}^m c_i B_i^* x_i \right\rangle \leq \sum_{i=1}^m c_i \langle A_i x_i, x_i \rangle$$

for every x_1, \dots, x_m in $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$, respectively. Therefore

$$\left\| \sum_{i=1}^m c_i B_i^* u_i \right\|_{\mathbb{H}}^2 \leq \sum_{i=1}^m c_i \|u_i\|_{\mathbb{H}_i}^2.$$

for every sequence of absolutely continuous paths $(u_i: [0, T] \rightarrow \mathbb{R}^{n_i})_{i \leq m}$. Thus multiplying (15) by c_i and summing over i yields

$$\begin{aligned} \sum_{i=1}^m c_i \log\left(\mathbb{E} e^{g_i(A_i^{-1} B_i W_T)}\right) \\ \leq \mathbb{E} \left[g\left(A^{-1} W_T + \sum_{i=1}^m c_i B_i^*(U_i)_T\right) - \frac{1}{2} \left\| \sum_{i=1}^m c_i B_i^* U_i \right\|_{\mathbb{H}}^2 \right] + \sum_{i=1}^m c_i \epsilon. \end{aligned}$$

Hence, using Proposition 1 again and dropping ϵ again,

$$(16) \quad \sum_{i=1}^m c_i \log\left(\mathbb{E} e^{g_i(A_i^{-1} B_i W_T)}\right)^{c_i} \leq \log\left(\mathbb{E} e^{g(A^{-1} W_T)}\right).$$

Recall that $f_i \leq e^{g_i}$ for every $i \leq m$ and that $e^g = f + C\delta^c$. Since δ is arbitrary, inequality (16) becomes

$$\prod_{i=1}^m \left(\mathbb{E} f_i(A_i^{-1} B_i W_T)\right)^{c_i} \leq \mathbb{E} f(A^{-1} W_T).$$

Again, letting T tend to $+\infty$ in this inequality yields the result. \square

4. THE BRASCAMP-LIEB AND BARTHE THEOREMS

So far we have seen that both the direct inequality and the reversed version are saturated by Gaussian functions when there exists a matrix A such that

$$(17) \quad A^{-1} = \sum_{i=1}^m c_i B_i^* (B_i A B_i^*)^{-1} B_i.$$

In this section, we briefly explain why this yields the Brascamp-Lieb and Barthe theorems.

Applying (2) to Gaussian functions gives

$$(18) \quad \prod_{i=1}^m \det(A_i)^{c_i} \leq C^2 \det\left(\sum_{i=1}^m c_i B_i^* A_i B_i\right),$$

for every sequence A_1, \dots, A_m of positive definite matrices on $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$. Let C_g be the Gaussian Brascamp-Lieb constant; namely the best constant in the previous inequality. We have $C_g \leq C$ and it turns out that applying (4) to Gaussian functions yields $C_g \leq C_r$ (one has to apply Lemma 4 at some point).

It is known since the work of Carlen and Cordero [10] that there is a dual formulation of (2) in terms of relative entropy. In the same way, there is a dual formulation of (18). For every positive matrix A on \mathbb{R}^n , one has

$$\log \det(A) = \inf_{B>0} \{ \operatorname{tr}(AB) - n - \log(\det(B)) \},$$

with equality when $B = A^{-1}$. Using this and the equality $\sum_{i=1}^m c_i n_i = n$, it is easily seen that C_g is also the best constant such that the inequality

$$(19) \quad \det(A) \leq C_g^2 \prod_{i=1}^m \det(B_i A B_i^*)^{c_i}$$

holds for every positive definite matrix A on \mathbb{R}^n .

Example. Assume that $m = n$, that $c_1 = \dots = c_n = 1$ and that $B_i(x) = x_i$ for $i \in [n]$. Inequality (18) trivially holds with constant 1 (and there is equality for every A_1, \dots, A_n). On the other hand (19) becomes

$$\det(A) \leq \prod_{i=1}^n a_{ii},$$

for every positive definite A , with equality when A is diagonal. This is Hadamard's inequality.

Lemma 5. *If A is extremal in (19) then A satisfies (17).*

Proof. Just compute the gradient of the map

$$A > 0 \mapsto \log \det(A) - \sum_{i=1}^m c_i \log \det(B_i A B_i^*). \quad \square$$

Therefore, if the constant C_g is finite and if there is an extremizer A in (19) then A satisfies (17) and together with the results of the previous sections we get the Brascamp-Lieb and Barthe equalities

$$(20) \quad C = C_r = C_g.$$

Although it may happen that $C_g < +\infty$ and no Gaussian extremizer exists, there is a way to bypass this issue. For the Brascamp-Lieb theorem, there is an abstract argument showing that it is enough to prove the equality $C = C_g$ when there is a Gaussian extremizer. This argument relies on:

- (1) A criterion for having a Gaussian extremizer, due to Barthe [1] in the rank 1 case (namely when the dimensions n_i are all equal to 1) and Bennett, Carbery, Christ and Tao [6] in the general case.
- (2) A multiplicativity property of C and C_g due to Carlen, Lieb and Loss [11] in the rank 1 case and obtained in full generality in [6] again.

There is no point repeating this argument here, and we refer to [11, 6] instead. This settles the case of the $C = C_g$ equality. As for the $C = C_r$ equality, we observe that the above argument can be carried out verbatim once the multiplicativity property of the reversed Brascamp-Lieb constant is established. This is the purpose of the rest of the article.

Definition 6. Given a proper subspace E of \mathbb{R}^n , we let for $i \leq m$

$$\begin{aligned} B_{i,E}: E &\rightarrow B_i E & B_{i,E^\perp}: E^\perp &\rightarrow (B_i E)^\perp \\ x &\mapsto B_i x, & x &\mapsto q_i(B_i x), \end{aligned}$$

where q_i is the orthogonal projection onto $(B_i E)^\perp$. Observe that both $B_{i,E}$ and B_{i,E^\perp} are onto. Now we let $C_{r,E}$ be the reversed Brascamp-Lieb constant on E associated to the datum

$$(c_1, B_{1,E}), \dots, (c_m, B_{m,E})$$

and C_{r,E^\perp} be the reversed Brascamp-Lieb constant on E^\perp associated to the datum

$$(c_1, B_{1,E^\perp}), \dots, (c_m, B_{m,E^\perp})$$

Proposition 7. *Let E be a proper subspace of \mathbb{R}^n , and assume that E is critical, in the sense that*

$$\dim(E) = \sum_{i=1}^m c_i \dim(B_i E).$$

Then $C_r = C_{r,E} \times C_{r,E^\perp}$.

Bennett, Carbery, Christ and Tao proved the corresponding property of C and C_g , we adapt their argument to prove the multiplicativity of C_r .

Let us prove the inequality $C_r \leq C_{r,E} \times C_{r,E^\perp}$ first. This does not require E to be critical. Let f_1, \dots, f_m, f be functions on $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}, \mathbb{R}$ respectively, satisfying

$$\prod_{i=1}^m f_i(z_i)^{c_i} \leq f\left(\sum_{i=1}^m c_i B_i^* z_i\right)$$

for all z_1, \dots, z_m . Fix $(x_1, \dots, x_m) \in B_1 E \times \dots \times B_m E$. Since $(B_{i,E^\perp})^* y_i = B_i^* y_i$ for every $y_i \in (B_i E)^\perp$, applying the reversed Brascamp-Lieb inequality on E^\perp to the functions

$$y \in (B_i E)^\perp \mapsto f_i(x_i + y), \quad i \leq m$$

yields

$$\begin{aligned} \prod_{i=1}^m \left(\int_{(B_i E)^\perp} f_i(x_i + y) \, dy \right)^{c_i} &\leq C_{r,E^\perp} \int_{E^\perp} f\left(\sum_{i=1}^m c_i B_i^* x_i + y\right) \, dy \\ &= C_{r,E^\perp} \int_{E^\perp} f\left(\sum_{i=1}^m c_i (B_{i,E})^* x_i + y\right) \, dy. \end{aligned}$$

For the latter equality, observe that $(B_{i,E})^* x_i = p(B_i^* x_i)$ where p is the orthogonal projection with range E and use the translation invariance of the Lebesgue measure. Applying the reversed Brascamp-Lieb inequality (this time on E) and using Fubini's theorem we get

$$\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i \, dx \right)^{c_i} \leq C_{r,E} C_{r,E^\perp} \int_{\mathbb{R}^n} f \, dx,$$

which is the result.

We start the proof of the inequality $C_{r,E} C_{r,E^\perp} \leq C_r$ with a couple of simple observations.

Lemma 8. *Upper semi-continuous functions having compact support saturate the reversed Brascamp-Lieb inequality.*

Proof. The regularity of the Lebesgue measure implies that given a non-negative integrable function f_i on \mathbb{R}^{n_i} and $\epsilon > 0$ there exists a non-negative linear combination of indicators of compact sets g_i satisfying

$$g_i \leq f_i \quad \text{and} \quad \int_{\mathbb{R}^{n_i}} f_i \, dx \leq (1 + \epsilon) \int_{\mathbb{R}^{n_i}} g_i \, dx.$$

The lemma follows easily. \square

The proof of the following lemma is left to the reader.

Lemma 9. *If f_1, \dots, f_m are compactly supported and upper semi-continuous on $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$ respectively, then the function f defined on \mathbb{R}^n by*

$$f(x) = \sup \left\{ \prod_{i=1}^m f_i(x_i)^{c_i}, \sum_{i=1}^m c_i B_i^* x_i = x \right\},$$

is compactly supported and upper semi-continuous as well.

Remark. If the Brascamp-Lieb datum happens to be *degenerate*, in the sense that the map $(x_1, \dots, x_m) \mapsto \sum_{i=1}^m B_i^* x_i$ is not onto, then Brascamp-Lieb constants are easily seen to be $+\infty$. Still the previous lemma remains valid, provided the convention $\sup \emptyset = 0$ is adopted.

Let us prove that $C_{r,E} \times C_{r,E^\perp} \leq C_r$. By Lemma 8, it is enough to prove that the inequality

$$\prod_{i=1}^m \left(\int_{B_i E} f_i \, dx \right)^{c_i} \times \prod_{i=1}^m \left(\int_{(B_i E)^\perp} g_i \, dx \right)^{c_i} \leq C_r \left(\int_E f \, dx \right) \left(\int_{E^\perp} g \, dx \right).$$

holds for every compactly supported upper semi-continuous functions $(f_i)_{i \leq m}$ and $(g_i)_{i \leq m}$, where f and g are defined by

$$\begin{aligned} f: x \in E &\mapsto \sup \left\{ \prod_{i=1}^m f_i(x_i)^{c_i}, \sum_{i=1}^m c_i (B_i E)^* x_i = x \right\} \\ g: y \in E^\perp &\mapsto \sup \left\{ \prod_{i=1}^m g_i(y_i)^{c_i}, \sum_{i=1}^m c_i (B_i E^\perp)^* y_i = y \right\}. \end{aligned}$$

Let $\epsilon > 0$. For $i \leq m$ define a function h_i on \mathbb{R}^{n_i} by

$$h_i(x + y) = f_i(x/\epsilon) g_i(y), \quad \forall x \in B_i E, \forall y \in (B_i E)^\perp,$$

and let

$$h: z \in \mathbb{R}^n \mapsto \sup \left\{ \prod_{i=1}^m h_i(z_i)^{c_i}, \sum_{i=1}^m c_i B_i^* z_i = z \right\}.$$

By definition of the reversed Brascamp-Lieb constant C_r

$$(21) \quad \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} h_i \, dx \right)^{c_i} \leq C_r \int_{\mathbb{R}^n} h \, dx.$$

Using the equality $\sum_{i=1}^m c_i \dim(B_i E) = \dim(E)$ we get

$$\epsilon^{-\dim(E)} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} h_i \, dx \right)^{c_i} = \prod_{i=1}^m \left(\int_{B_i E} f_i \, dx \right)^{c_i} \times \prod_{i=1}^m \left(\int_{(B_i E)^\perp} g_i \, dx \right)^{c_i}.$$

On the other hand, we let the reader check that for every $x \in E, y \in E^\perp$

$$h(\epsilon x + y) \leq f(x)g_\epsilon(y),$$

where

$$g_\epsilon(y) = \sup\{g(y'), |y - y'| \leq K\epsilon\}$$

and K is a constant depending on the diameters of the supports of the functions f_i . Therefore

$$\epsilon^{-\dim E} \int_{\mathbb{R}^n} h \, dx = \int_{E \times E^\perp} h(\epsilon x + y) \, dx dy \leq \left(\int_E f \, dx \right) \left(\int_{E^\perp} g_\epsilon \, dx \right).$$

Inequality (21) becomes

$$\prod_{i=1}^m \left(\int_{B_i E} f_i \, dx \right)^{c_i} \times \prod_{i=1}^m \left(\int_{(B_i E)^\perp} g_i \, dx \right)^{c_i} \leq C_r \left(\int_E f \, dx \right) \left(\int_{E^\perp} g_\epsilon \, dx \right).$$

By Lemma 9, the function g has compact support and is upper semi-continuous. This implies easily that

$$\lim_{\epsilon \rightarrow 0} \int_{E^\perp} g_\epsilon \, dx = \int_{E^\perp} g \, dx,$$

which concludes the proof.

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REFERENCES

- [1] F. Barthe, On a reverse form of the Brascamp-Lieb inequality. *Invent. Math.* **134** (1998), 335–361.
- [2] F. Barthe and D. Cordero-Erausquin, Inverse Brascamp-Lieb inequalities along the heat equation. In *Geometric aspects of functional analysis*, 65–71, Lecture Notes in Math., 1850, Springer, Berlin, 2004.
- [3] F. Barthe and N. Huet, On Gaussian Brunn-Minkowski inequalities, *Studia Math.* **191** (2009), 283–304.
- [4] W. Beckner, Inequalities in Fourier analysis, *Ann. of Math.* **102** (1975), 159–182.
- [5] J. Bennett, N. Bez and A. Carbery, Heat-flow monotonicity related to the Hausdorff-Young inequality, *Bull. Lond. Math. Soc.* **41** (2009), 971–979.
- [6] J. Bennett, A. Carbery, M. Christ and T. Tao, The Brascamp-Lieb inequalities: finiteness, structure and extremals, *Geom. Funct. Anal.* **17** (2008), 1343–1415.
- [7] C. Borell, Diffusion equations and geometric inequalities, *Potential Anal.* **12** (2000), 49–71.

- [8] M. Boué and P. Dupuis, A variational representation for certain functionals of Brownian motion, *Ann. Probab.* **26** (1998), 1641–1659.
- [9] H.J. Brascamp and E.H. Lieb, Best constants in Young’s inequality, its converse, and its generalization to more than three functions, *Advances in Math.* **20** (1976), 151–173.
- [10] E.A. Carlen and D. Cordero-Erausquin, Subadditivity of the entropy and its relation to Brascamp-Lieb type inequalities, *Geom. Funct. Anal.* **19** (2009), 373–405.
- [11] E.A. Carlen, E.H. Lieb and M. Loss, A sharp analog of Young’s inequality on S^N and related entropy inequalities, *J. Geom. Anal.* **14** (2004), 487–520.
- [12] J. Lehec, Representation formula for the entropy and functional inequalities, *Ann. Inst. Henri Poincaré Probab. Stat.* **49** (2013), 885–899.
- [13] E.H. Lieb, Gaussian kernels have only Gaussian maximizers, *Invent. Math.* **102** (1990), 179–208.

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