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# PLANAR MARKOVIAN HOLONOMY FIELDS. A FIRST STEP TO THE CHARACTERIZATION OF MARKOVIAN HOLONOMY FIELDS

*by*

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**Abstract.** — We study planar random holonomy fields which are processes indexed by paths on the plane which behave well under the concatenation and orientation-reversing operations on paths. We define the Planar Markovian Holonomy Fields as planar random holonomy fields which satisfy some independence and invariance by area-preserving homeomorphisms properties.

We use the theory of braids in the framework of classical probabilities: for finite and infinite random sequences the notion of invariance by braids is defined and we prove a new version of the de-Finetti's Theorem.

This allows us to construct a family of Planar Markovian Holonomy Fields, the Yang-Mills fields, and we prove that any regular Planar Markovian Holonomy Field is a planar Yang-Mills field. This family of planar Yang-Mills fields can be partitioned into three categories according to the degree of symmetry: we study some equivalent conditions in order to classify them.

Finally, we recall the notion of Markovian Holonomy Fields and construct a bridge between the planar and non-planar theories. Using the results previously proved in the article, we compute, for any Markovian Holonomy Field, the “law” of any family of contractible loops drawn on a surface.

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## 1. Introduction

Yang-Mills theory is a theory of random connections on a principal bundle, the law of which satisfies some local symmetry: the gauge symmetry. It was introduced in the work of Yang and Mills, in 1954, in [31]. Since then, mathematicians have tried to formulate a proper quantum Yang-Mills theory. The construction on a four dimensional manifold for any compact Lie group is still a challenge: we will focus in this article on the 2-dimensional quantum Yang-Mills theory. On a formal level, a Yang-Mills measure is a measure on the space of connections which looks like:

$$e^{S_{YM}(A)} DA,$$

where  $S_{YM}(A)$  is the Yang-Mills action of the connection  $A$ , which is roughly the  $L_2$  norm of the curvature, and  $DA$  is a translation invariant measure on the space of connections. Yet, many problems arise with this formulation. The first of them is that the space of connections can not be endowed with a translation invariant measure. The second is that the action  $S_{YM}$  is invariant under gauge transformations: we are thus “integrating a periodic function under a uniform measure”. It took some time to understand which space could be endowed by a well-defined measure.

One possibility to handle these difficulties in a probabilistic way is to consider holonomies of the random connections along some finite set of paths: thus, after the works of Gross [14], [15], Driver [12], [13], and Sengupta [24], [25] who constructed the Yang-Mills field for a small class of paths but for any surface, it was well understood that the Yang-Mills measure was a process indexed by some nice paths. Their construction uses the fact that the holonomy process under the Yang-Mills measure should satisfy a stochastic differential equation directed by a Brownian white-noise curvature. The Yang-Mills measure has to be constructed on the multiplicative functions from the set of paths to a Lie group, that is the set of functions which have a good behavior under concatenation and orientation-inversion of paths. This idea was already present in the precursory work of Albeverio, Høegh-Krohn and Holden ([1], [3], [4], [2]).

In [19], [20] and [21], Lévy gave a new construction. This construction allows us first to consider any compact Lie groups, any surfaces, and any rectifiable paths.

Besides, it allows us to generalize the definition of Yang-Mills measure to the setting where, in some sense, the curvature of the random connection is a conditioned Lévy noise. The idea was to establish the rigorous discrete construction, as proposed by E. Witten in [29] and [30] and to show that one could take a continuous limit.

The discrete construction was defined by considering a perturbation of a uniform measure, the Ashtekar-Lewandowski measure, by a density. The continuous limit was established using the general Theorem 3.3.1 in [21]. This theorem must be understood as a two-dimensional Kolmogorov's continuity theorem and one should consider it as one of the most important theorem in the theory of two-dimensional holonomy fields. In the article [10] in preparation, G. Cébron, A. Dahlqvist and the author show how to use this theorem in order to construct generalizations of the Master Field constructed in [5] and [18].

In the seminal book [21], Lévy defined also what was a Markovian Holonomy Field. This is the axiomatic point of view on Yang-Mills measures, seen as families of measures, indexed by surfaces which have a good behavior under chirurgical operations on surfaces and are invariant under area-preserving homeomorphisms. The importance of this notion is that Yang-Mills measures are Markovian Holonomy Fields. It is still unknown if any regular Markovian Holonomy Field is a Yang-Mills measure. This paper is a first step in order to prove so.

The axiomatic formulation of the Markovian Holonomy Fields allows us to understand Lévy processes as one-dimensional Planar Markovian Holonomy Fields.

**1.1. Lévy processes and Planar Markovian Holonomy Fields.** — Let  $G$  be a compact Lie group. There exist two notions of Lévy processes depending on the definitions of the increments  $Y_t Y_s^{-1}$  or  $Y_s^{-1} Y_t$ . We will fix the following convention: in this article, a Lévy process on  $G$  is a càdlàg process with independent and stationary increments which begins at the neutral element. In fact one can use a weaker definition and forgot about the càdlàg property and define a Lévy process as a continuous in probability family of random variables  $(Y_t)_{t \in \mathbb{R}^+}$  such that:

- $Y_t Y_s^{-1}$  has same law as  $Y_{t-s}$  for any  $t > s$ ,
- $Y_t Y_s^{-1}$  is independent of  $\sigma(Y_u, u < s)$ ,
- $Y_0 = e$  a.s.

Let  $Y$  be a Lévy process on  $G$ . For any smooth density  $vol$  on  $\mathbb{R}$  with compact support, one can define a measure  $\mathbb{E}_{vol}$  on  $G^{\mathbb{R}}$  such that, under  $\mathbb{E}_{vol}$ , the canonical projection process  $(X_t)_{t \in \mathbb{R}}$  has the law of  $(Y_{vol(\cdot - \infty, t]})_{t \in \mathbb{R}}$ . The family  $(\mathbb{E}_{vol})_{vol}$  satisfies three properties:

- Area-preserving increasing homeomorphism invariance** : Let  $\psi$  be an increasing homeomorphism of  $\mathbb{R}$ . Let  $vol$  and  $vol'$  be two smooth densities on  $\mathbb{R}$  with compact support. Let us suppose that  $\psi$  sends  $vol$  on  $vol'$ . The mapping  $\psi$  induces a measurable mapping from  $G^{\mathbb{R}}$  to itself we will denote

also by  $\psi$  and which is defined by:

$$\psi((x_t)_{t \in \mathbb{R}}) = (x_{\psi(t)})_{t \in \mathbb{R}}.$$

It is then easy to see that  $\mathbb{E}_{vol} = \mathbb{E}_{vol'} \circ \psi^{-1}$ . For example, for any real  $t \in \mathbb{R}$  and any bounded function  $f$  on  $G$ :

$$\mathbb{E}_{vol'} [f(X_{\psi(t)})] = \mathbb{E} [f(Y_{vol'([- \infty, \psi(t)])})] = \mathbb{E} [f(Y_{vol([- \infty, t])})] = \mathbb{E}_{vol} [f(X_t)].$$

**-Independence :** Let  $vol$  be a smooth density on  $\mathbb{R}$  with compact support. Let  $[s_0, t_0]$  and  $[s_1, t_1]$  be two disjoint intervals. Under  $\mathbb{E}_{vol}$ ,  $\sigma((X_t X_s^{-1}), s_0 \leq s < t \leq t_0)$  is independent of  $\sigma((X_t X_s^{-1}), s_1 \leq s < t \leq t_1)$ .

**-Locality property :** Let  $vol$  and  $vol'$  be two smooth densities on  $\mathbb{R}$  with compact support. Let  $t_0$  be a real such that  $vol|_{[- \infty, t_0]} = vol'|_{[- \infty, t_0]}$ . The law of  $(X_t)_{t \leq t_0}$  is the same under  $\mathbb{E}_{vol}$  as  $\mathbb{E}_{vol'}$ .

Besides, for any family of measures  $(\mathbb{E}_{vol})_{vol}$  on  $G^{\mathbb{R}}$  which satisfies the three axioms, there exists a Lévy process  $(Y_t)_{t \geq \mathbb{R}^+}$  such that, for any smooth density  $vol$  on  $\mathbb{R}$ , the canonical projection process  $(X_t)_{t \in \mathbb{R}}$  has the law of  $(Y_{vol([- \infty, t])})_{t \in \mathbb{R}}$ .

With these axioms in mind, looking at the definitions in Section 4.1 of Planar Markovian Holonomy Field, the reader can now understand why we can consider Lévy processes as one dimensional Planar Markovian Holonomy Fields. The surprising fact that we will prove in this paper is that the family of regular two-dimensional Planar Markovian Holonomy Fields is not bigger than the set of one-dimensional Planar Markovian Holonomy Fields.

**1.2. Braids.** — The most innovative idea of this paper is to introduce for the very first time the braid group in the study of Yang-Mills theory. This is also one of the main ingredient in the upcoming article [10].

The braid group is an object which possesses different facets: a combinatorial, a geometric and an algebraic one. One can introduce the braid group using geometric braids: this construction allows us to have a graphical and combinatorial framework to work with. Since it is the most intuitive construction, we quickly present it so that the reader will be familiar with these objects.

**Proposition 1.** — *For any  $n \geq 2$ , let the configuration space  $C_n(\mathbb{R}^2)$  of  $n$  indistinguishable points in the plane be  $((\mathbb{R}^2)^n \setminus \Delta) / \mathfrak{S}_n$  where  $\Delta$  is the union of the hyperplanes  $\{\mathbf{x} \in (\mathbb{R}^2)^n, \mathbf{x}_i = \mathbf{x}_j\}$ . The fundamental group of the configuration space  $C_n(\mathbb{R}^2)$  is the braid group with  $n$  strands  $\mathcal{B}_n$ :*

$$\mathcal{B}_n = \pi_1(C_n(\mathbb{R}^2)).$$

Every continuous loop  $\gamma$  in  $C_n(\mathbb{R}^2)$  parametrized by  $[0, 1]$  and based at  $((1, 0), \dots, (n, 0))$  can be seen as  $n$  continuous functions  $\gamma_j \in \mathcal{C}([0, 1], \mathbb{R}^2)$  such

that, if we set  $\sigma : j \mapsto \gamma_j(1)$  for any  $j \in \{1, \dots, n\}$ , the following conditions hold:

- 1-  $\forall j \in \{1, \dots, n\}, \gamma_j(0) = (j, 0)$ ,
- 2-  $\sigma \in \mathfrak{S}_n$ ,
- 3-  $\forall t \in [0, 1], \forall j \neq j', \gamma_j(t) \neq \gamma_{j'}(t)$ .

Indeed, the function  $\gamma_j$  is given by the image of  $\gamma$  by the natural projection  $\pi_j : (\mathbb{R}^2)^n \rightarrow \mathbb{R}^2$ . We call  $\gamma$  a geometric braid since if we draw the  $(\gamma_j)_{j=1}^n$  in  $\mathbb{R}^3$ , we obtain a physical braid. One can look at the following picture to have an illustration of this fact.

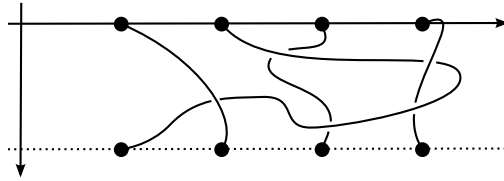


FIGURE 1. A physical braid  $\beta$ .

With this point of view, the composition of two braids is just obtained by gluing two geometric braids, taking then the equivalence class by isotopy of the new braid as shown in Figure 2. In this paper, we will take the convention that, in order to compute  $\beta_1\beta_2$ , one has to put the braid  $\beta_2$  above the braid  $\beta_1$ .

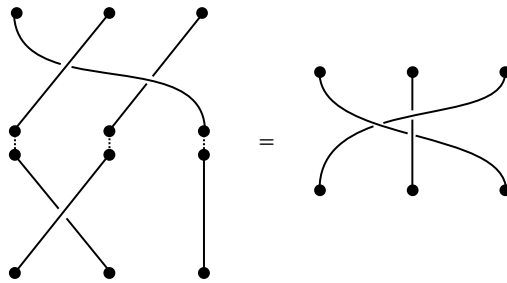


FIGURE 2. The multiplication of two braids.

As we see in Figure 3, one can represent a braid by a two dimensional diagram (or, to be correct, classes of equivalence of two-dimensional diagrams) that we call  $n$ -diagrams. This representation can remind the reader the representation of any permutation by a diagram, yet, this representation differs from the fact that one remembers which string is above an other at each crossing. It is a well-known result that any  $n$ -diagram represents a unique braid with  $n$ -strands. Thus, to construct a braid, we only have to construct a  $n$ -diagram. Besides, every computation can be done with the  $n$ -diagrams.



The Section 3 is devoted to the theory of planar graphs and the notion of  $\mathbb{G} - \mathbb{G}'$  piecewise diffeomorphisms. One of the main result is Corollary 1 which states that one can send, by a  $\mathbb{G} - \mathbb{G}'$  diffeomorphism, any generic planar graph in the  $\mathbb{N}^2$ -graph.

Using the previous sections, we can define four different notions of Planar Markovian Holonomy Fields. Under some regularity condition, it will be proved in the paper that the four notions are essentially equivalent. These objects are processes, indexed by paths drawn on the plane, which are gauge-invariant, invariant under area-preserving homeomorphisms, which satisfy a weak independence property and a locality property. In the remaining of the section, we show how to add a drift to a Planar Markovian Holonomy Field by using the index field. We also consider the questions of restriction and extension of the structure group for Planar Markovian Holonomy Fields.

The equivalence between weak discrete and weak continuous Planar Markovian Holonomy Field is then proved in Section 5, using a Theorem of Moser and Dacorogna proved in [11].

The loop paradigm explained in Section 2 would not be powerful if the group of reduced loops was not introduced in Section 6. In this section, we define the group of reduced group as Lévy did in [21]. Then, we obtain a generalization of Lévy's work in the planar case: this allows us to exhibit general families of loops which generate the group of reduced loops of any planar graph. The proof differs greatly from the proof in the original Lévy's work as we use a splitting/recurrence argument.

Two sections are devoted to the link between braids and probabilities: Section 7 and Section 9. If one is reading the article only for the interactions between the braid group and probabilities, one should go directly to these sections. In the first one, we give an algebraic definition of the braid group. This algebraic definition allows us to construct natural actions of  $\mathcal{B}_n$  on the free group of rank  $n$  and on  $G^n$ . After an application of Artin's theorem to our results on the group of reduced loops, we define the notion of invariance by braid for finite sequences of random variables. Two examples of such braid-invariant finite sequences are then given. Section 9, is devoted to the geometric point of view on braids and to a de-Finetti-Ryll-Nardzewski's theorem for random infinite sequences which are braid-invariant. Then we show that, under an assumption of independence of the diagonal-conjugacy classes, one can characterize the braidable sequences which are sequences of i.i.d. random variables. The end of the section consists in an application of these results to processes.

In Sections 8, 10 and 11 the main results in the theory of Planar Markovian Holonomy Fields are proved. The results of Section 7 on finite braid-invariant sequences of random variables allow us in Section 8 to construct, for any Lévy process which is self-invariant by conjugation, a planar Yang-Mills field associated with it. This construction differs from all the previous constructions since it uses



neither the notion of uniform or Ashtekar-Lewandowski measure nor the notion of stochastic differential equations. This allows us to consider any self-invariant by conjugation Lévy processes, where before, one had to consider Lévy processes with density with respect to the Haar measure and which were invariant by conjugation by the structure group  $G$ . In Section 10 and 11, using the results of Section 9, we prove that any Planar Markovian Holonomy Field is a Planar Yang-Mills Field. We show that one can characterize them according to the law of simple loops.

As we proved that any Planar Markovian Holonomy Field is a Planar Yang-Mills Field, is it possible to show that any Markovian Holonomy Field is a Yang-Mills Field? In Section 12, we answer partly to this question. First we recall the notion of Markovian Holonomy Fields. We construct the free boundary condition expectation which is a bridge between Markovian Holonomy Fields and Planar Markovian Holonomy Fields. Using the results shown previously, we are able to prove our last main result, namely Theorem 28: the restriction of a Markovian Holonomy Field on the set of contractible curves is equal to the restriction of a Yang-Mills field on the same set of paths.

In order to get a more accurate idea of the results shown in this article and the different notions defined in it, one can refer to the diagram page 116.

**1.4. Notations.** — Throughout this paper,  $M$  is either a smooth compact surface, possibly with boundary, or the plane  $\mathbb{R}^2$ . We denote by  $G$  a compact Lie group, with the usual convention that a compact Lie group of dimension 0 is a finite group. The Lie algebra of  $G$  is denoted by  $\mathfrak{g}$ , and the connected component of the neutral element by  $G_0$ . The neutral element will be denoted either by 1 or  $e$ . We endow  $G$  with a bi-invariant distance  $d_G$ . We denote by  $\mathcal{M}(G)$  the space of finite Borel positive measures on  $G$ .

For each  $n \geq 1$ , the group  $G$  acts by diagonal conjugation on  $G^n$ :

$$g \cdot (g_1, \dots, g_n) = (g^{-1}g_1g, \dots, g^{-1}g_ng),$$

for any  $g \in G$  and any  $n$ -tuple  $(g_1, \dots, g_n)$  of elements of  $G$ . We denote by  $[(g_1, \dots, g_n)]$  the equivalence class of  $(g_1, \dots, g_n)$  in  $G^n$  under the diagonal conjugation action.

Each time we will have to use a constant function equal to 1, we will denote it by 1. Besides, if  $\mu$  is a finite measure on a measurable space  $(\Omega, \mathcal{A})$  and if  $\mathcal{B} \subset \mathcal{A}$  is a sub- $\sigma$ -field, by  $\mu|_{\mathcal{B}}$ , we denote the image of  $\mu$  by the identity map:  $(\Omega, \mathcal{A}) \rightarrow (\Omega, \mathcal{B})$ .

## 2. Backgrounds: paths, random multiplicative functions on paths

**2.1. Paths.** — Let  $M$  be either a smooth compact surface (possibly with boundary) or the plane  $\mathbb{R}^2$ .

A *measure of area* on  $M$  is a smooth non-vanishing density on  $M$ , that is, a Borel measure which has a smooth positive density with respect to the Lebesgue measure in any coordinate chart. It will often be denoted by  $vol$ . We call  $(M, vol)$  a *measured surface*. Any measure of area is the Riemannian density of a Riemannian metric on  $M$ . We will denote by  $\gamma_0$  the standard Riemannian metric on  $\mathbb{R}^2$ .

We endow  $M$  with a Riemannian metric  $\gamma$ .

**Definition 1.** — *A parametrized path on  $M$  is a continuous curve  $c : [0, 1] \rightarrow M$  which is either constant or Lipschitz continuous with speed bounded below by a positive constant.*

Two parametrized paths can give the same drawing on  $M$  but with different speed and we will only consider equivalence classes of paths.

**Definition 2.** — *Two parametrized paths on  $M$  are equivalent if they differ by an increasing bi-Lipschitz homeomorphism of  $[0, 1]$ . An equivalence class of parametrized paths is called a path and the set of paths on  $M$  is denoted by  $P(M)$ .*

Actually, the notion of path does not depend on  $\gamma$ . Two parametrized paths  $pp_1$  and  $pp_2$  which represent the same path  $p$  share the same endpoints. It is thus possible to define the endpoints of  $p$  as the endpoints of any representative of  $p$ . If  $p$  is a path, by  $\underline{p}$  (resp.  $\bar{p}$ ) we denote the starting point (resp. the arrival point) of  $p$ . From now on, we will not make any difference between a path  $p$  and any parametrized path  $pp \in p$ .

**Definition 3.** — *A path is simple either if it is injective on  $[0, 1]$  or if it is injective on  $[0, 1[$  and  $\underline{p} = \bar{p}$ .*

Later, we will need the following subset of paths.

**Definition 4.** — *We define  $\text{Aff}_\gamma(M)$  to be the set of paths on  $M$  which are piecewise geodesic paths.*

The set of paths  $\text{Aff}_{\gamma_0}(\mathbb{R}^2)$  will be denoted by  $\text{Aff}(\mathbb{R}^2)$ . An other set of paths will be very important for our study: the set of loops.

**Definition 5.** — *A loop  $l$  is a path such that  $\underline{l} = \bar{l}$ . A smooth loop is a loop whose image is an oriented smooth 1-dimensional submanifold of  $M$ . The set of loops is denoted by  $L(M)$ . Let  $m$  be a point of  $M$ . A loop  $l$  is based at  $m$  if  $\underline{l} = m$ . The set of loops based at  $m$  is denoted by  $L_m(M)$ .*

Let  $p$  be a path, let  $pp_1$  be a representative of  $p$ . The inverse of  $p$ , denoted by  $p^{-1}$ , is the equivalence class of the parametrized path  $t \mapsto pp_1(1 - t)$ . Concatenation is also defined for paths. Let  $p_1$  and  $p_2$  be two paths such that  $\bar{p}_1 = \underline{p}_2$ , let  $pp_1$  (resp.

$pp_2$ ) be a representative of  $p_1$  (resp.  $p_2$ ). The concatenation of  $p_1$  and  $p_2$  denoted by  $p_1p_2$  is the equivalence class of the parametrized path:

$$pp_1 \cdot pp_2 : t \mapsto \begin{cases} pp_1(2t) & \text{if } t \leq 1/2, \\ pp_2(2t - 1) & \text{if } t > 1/2. \end{cases}$$

**Definition 6.** — *A set of paths  $P$  is connected if any couple of endpoints of elements of  $P$  can be joined by a concatenation of elements of  $P$ .*

**Definition 7.** — *Let  $P$  be a set of paths. Two loops  $l$  and  $l'$  are elementarily equivalent in  $P$  if there exist three paths,  $a, b, c \in P$  such that  $\{l, l'\} = \{ab, acc^{-1}b\}$ . We say that  $l$  and  $l'$  are equivalent in  $P$  if there exists a finite sequence  $l = l_0, \dots, l_n = l'$  such that  $l_i$  is elementarily equivalent to  $l_{i+1}$  for any  $i \in \{0, \dots, n-1\}$ . We will write it  $l \simeq_P l'$ .*

**Definition 8.** — *A lasso is a loop  $l$  such that one can find a simple loop  $m$ , the meander, and a path  $s$ , the spoke, such that  $l = sms^{-1}$ .*

A loop has a well-defined origin and orientation. A cycle is a loop in which one forgets about the endpoint. In a non-oriented cycle, the endpoint and the orientation are forgotten.

**Definition 9.** — *We say that two loops  $l_1$  and  $l_2$  are related if and only if they can be decomposed as:  $l_1 = cd$ ,  $l_2 = dc$ , with  $c$  and  $d$  two paths.*

*The set of equivalence classes for the relation defined on  $L(M)$  is the set of cycles. The operation of inversion is compatible with this equivalence. A non-oriented cycle is a pair  $\{l, l^{-1}\}$  where  $l$  is a cycle.*

*A cycle is simple if any loop which represents it is simple.*

We need a notion of convergence of paths in order to define the continuity of random holonomy fields. The definition makes use of the Riemannian metric  $\gamma$ , yet the notion of convergence with fixed endpoints will not depend on the choice of the Riemannian metric. We denote by  $d_\gamma$  the distance on  $M$  which is associated with  $\gamma$ .

**Definition 10.** — *Let  $p_1$  and  $p_2$  be two paths of  $M$ . Let  $l(p_1)$  (resp.  $l(p_2)$ ) be the length of the path  $p_1$  (resp.  $p_2$ ). We define the distance between  $p_1$  and  $p_2$  as:*

$$d_l(p_1, p_2) = \inf_{pp_1 \in p_1, pp_2 \in p_2} \sup_{t \in [0,1]} \left[ d_\gamma(pp_1(t), pp_2(t)) \right] + |l(p_1) - l(p_2)|.$$

*The topology induced by  $d_l$  does not depend on the choice of  $\gamma$ .*

*Let  $(p_n)_{n \geq 0}$  be a sequence of paths on  $M$ . Let  $p$  be a path on  $M$ . The sequence  $(p_n)_{n \geq 0}$  converges to  $p$  with fixed endpoints if and only if:*

- $d_l(p_n, p) \rightarrow 0$  as  $n \rightarrow +\infty$ ,
- $\forall n \geq 0, \overline{p_n} = \overline{p}$  and  $\underline{p_n} = \underline{p}$ .

With this notion comes the notion of density, and the following interesting lemma.

**Lemma 1.** — *The set of paths  $\text{Aff}_\gamma(M)$  is dense in  $P(M)$  for the convergence with fixed endpoints.*

One has to be careful when working with the convergence with fixed endpoints. For example, the set of paths whose images are concatenation of horizontal and vertical segments is **not** dense in  $P(\mathbb{R}^2)$ . Indeed, one condition in order to have the convergence with fixed endpoints is that the length of the paths converges to the length of the limit path. But, for any path  $p$  which can be written as a concatenation of horizontal and vertical segments, the following inequality holds:

$$l(p) \geq \| \bar{p} - p \|_1,$$

where  $\| \cdot \|_1$  is the usual  $L^1$  norm on  $\mathbb{R}^2$ . This inequality does not hold for a general path  $p$ .

## 2.2. Measures on the set of multiplicative functions. —

*2.2.1. Definitions.* — Let  $P$  be a subset of  $P(M)$ , and let  $L$  be a set of loops in  $P$ .

**Definition 11.** — *A function  $h$  from  $P$  to  $G$  is multiplicative if and only if:*

- $h(c^{-1}) = h(c)^{-1}$  for any path  $c$  in  $P$  such that  $c^{-1} \in P$ ,
- $h(c_1c_2) = h(c_2)h(c_1)$  for any paths  $c_1$  and  $c_2$  in  $P$  which can be concatenated and such that  $c_1c_2 \in P$ .

We denote by  $\text{Mult}(P, G)$  the set of multiplicative functions from  $P$  to  $G$ .

A function from  $L$  to  $G$  is pre-multiplicative over  $P$  if and only if:

- it is multiplicative,
- for any  $l$  and  $l'$  in  $L$  which are equivalent in  $P$ , we have:  $h(l) = h(l')$ .

We denote by  $\text{Mult}_P(L, G)$  the set of pre-multiplicative functions over  $P$ .

We will often make the following slight abuse of notation.

**Notation 1.** — *Let  $c$  be a path in  $P$ . If a multiplicative function  $h \in \text{Mult}(P, G)$  is not specified in a formula,  $h(c)$  will stand for the function on  $\text{Mult}(P, G)$  which is the evaluation on  $c$ :*

$$\begin{aligned} h(c) : \text{Mult}(P, G) &\rightarrow G \\ h &\mapsto h(c). \end{aligned}$$

The notion of equivalence of loop, as stated in Definition 7, is important due to the following remark.

**Remark 1.** — *Let  $h$  be in  $\text{Mult}(P, G)$  and let  $l, l'$  be loops in  $P$ . A simple induction and the multiplicative property of  $h$  imply that if  $l \simeq_P l'$  then  $h(l) = h(l')$ .*

Let  $P$  be a set of paths, and let  $Q$  be a freely generating subset of  $P$  in the sense that:

- any path in  $P$  is a finite concatenation of elements of  $Q$ ,
- no element of  $Q$  can be written as a non-trivial finite concatenation of paths in  $Q \cup Q^{-1}$ ,
- $Q \cap Q^{-1} = \emptyset$ .

Then we have the identification:

$$(1) \quad \text{Mult}(P, G) \simeq G^Q.$$

This is the **edge paradigm** for multiplicative functions. The novelty of the approach we have in this paper is to put the emphasis on the **loop paradigm** for gauge-invariant random holonomy fields. The first paradigm is to be considered for general random holonomy fields on surface, yet the second seems to be more appropriate for gauge-invariant random holonomy fields on the plane.

**Remark 2.** — *All the following definitions and propositions are about multiplicative functions on a set of paths  $P$ . All of them extend to  $G^T$ , with  $T \subset \mathbb{R}$ . Indeed, if  $P = \cup_{r \in T} \{c_r, c_r^{-1}\}$ , with  $c_r$  being the path on the plane based at 0 and going clockwise once around the circle of center  $(0, r)$  and radius  $r$ , we have:*

$$\text{Mult}(P, G) \simeq G^T.$$

We will now endow the space of multiplicative functions with a  $\sigma$ -field in order to be able to speak about measures on  $\text{Mult}(P, G)$ .

**Definition 12.** — *The Borel  $\sigma$ -field  $\mathcal{B}$  on  $\text{Mult}(P, G)$  is the smallest  $\sigma$ -field such that for any paths  $c_1, \dots, c_n$  and any continuous function  $f : G^n \rightarrow \mathbb{R}$ , the mapping  $h \mapsto f(h(c_1), \dots, h(c_n))$  is measurable.*

**Definition 13.** — *A random holonomy field  $\mu$  on the set  $P$  is a measure on  $(\text{Mult}(P, G), \mathcal{B})$ .*

If  $P = P(M)$ , we call it also a random holonomy field on  $M$ . Let  $\mu$  be a random holonomy field on  $P$ . We define the weight of  $\mu$  as  $\mu(1)$ .

One can define a regularity notion for random holonomy fields.

**Definition 14.** — *A random holonomy field on  $P$  is stochastically continuous if for any sequence  $(p_n)_{n \geq 0}$  of elements of  $P$  which converges with fixed endpoints to  $p \in P$ ,*

$$(2) \quad \int_{\text{Mult}(P, G)} d_G(h(p_n), h(p)) \mu(dh) \xrightarrow[n \rightarrow \infty]{} 0.$$

*Let  $\text{vol}$  be a measure of area on  $M$ . The measure  $\mu$  is locally stochastically  $\frac{1}{2}$ -Hölder continuous if for any compact set  $S \subset M$  there exists  $K > 0$  such that for*

any simple loop  $l \in P$  such that  $l \subset S$ , of length smaller than  $\frac{1}{K}$ , bounding a disk  $D$ :

$$(3) \quad \int_{\text{Mult}(P,G)} d_G(1, h(l)) \mu(dh) \leq K \sqrt{\text{vol}(D)},$$

where  $1$  is the neutral element of  $G$ . This notion does not depend on  $\text{vol}$ .

A family of random holonomy fields,  $(\mu)_{\mu \in \mathcal{F}}$ , with each  $\mu$  defined on some set  $P_\mu$ , is uniformly locally stochastically  $\frac{1}{2}$ -Hölder continuous if the constant  $K$  in equation (3) is independent of the random holonomy field in  $\mathcal{F}$ .

### 2.2.2. Construction of random holonomy fields I. —

**Notation 2.** — Let  $J$  and  $K$  be two subsets of  $P(M)$  such that  $J \subset K$ . The restriction function from  $\text{Mult}(K, G)$  to  $\text{Mult}(J, G)$  will be denoted by  $\rho_{J,K}$ . If  $M \subset M'$  are two surfaces, we denote by  $\rho_{M,M'}$  the restriction function  $\rho_{P(M), P(M')}$ . The notation is set such that for any  $J \subset K \subset L \subset P(M)$ ,  $\rho_{J,K} \circ \rho_{K,L} = \rho_{J,L}$ .

The fact that  $G$  is a compact group allows us to construct measures on the set of multiplicative functions by taking projective limits of random holonomy fields on finite subsets of paths. This behavior is very different from what can be observed for gaussian measures on Banach spaces, as for the Gaussian free field.

Indeed, in [21], Proposition 2.2.3, Lévy showed, for the case where  $\mathcal{F}$  is a collection of finite subsets of  $P$ , the next proposition using an application of Carathéodory's extension theorem. We give a proof based on the Riesz-Markov's theorem, proof which shows clearly why we only consider compact groups.

**Proposition 2.** — Let  $\mathcal{F}$  be a collection of subsets of paths on  $M$ . We denote by  $P$  their union. Suppose that, when ordered by the inclusion,  $\mathcal{F}$  is directed: for any  $J_1$  and  $J_2$  in  $\mathcal{F}$ , there exists  $J_3 \in \mathcal{F}$  such that  $J_1 \cup J_2 \subset J_3$ . For any  $J \in \mathcal{F}$ , let  $m_J$  be a probability measure on  $(\text{Mult}(J, G), \mathcal{B})$ . Assume that the probability spaces  $(\text{Mult}(J, G), \mathcal{B}, m_J)$  endowed with the restriction mappings  $\rho_{J,K}$  for  $J \subset K$  form a projective system. This means that for any  $J_1$  and  $J_2$  in  $\mathcal{F}$  such that  $J_1 \subset J_2$ , one has

$$m_{J_1} = m_{J_2} \circ \rho_{J_1, J_2}^{-1}.$$

Then there exists a unique probability measure  $m$  on  $(\text{Mult}(P, G), \mathcal{B})$  such that for any  $J \in \mathcal{F}$ ,

$$m_J = m \circ \rho_{J,P}^{-1}.$$

*Proof.* — We endow  $G^P$  with the product topology. As an application of Tychonoff's theorem it is a compact space. A consequence of this is that  $\text{Mult}(P, G)$ , endowed with the restricted topology, is also a compact space as it is closed in  $G^P$ . Besides, the  $\sigma$ -field  $\mathcal{B}$  is the Borel  $\sigma$ -field on  $\text{Mult}(P, G)$ . Let us consider  $A$  the set

of cylinder continuous functions, that is the set of functions  $\mathbf{f} : \mathcal{Mult}(P, G) \rightarrow \mathbb{R}^+$  of the form:

$$\mathbf{f} : h \mapsto f(h(p_1), \dots, h(p_n)),$$

for some  $n \in \mathbb{N}$ , some  $p_1, \dots, p_n \in P$ , and some continuous function  $f : G^n \rightarrow \mathbb{R}$ .

The set  $A$  is a subalgebra of the algebra  $C(\mathcal{Mult}(P, G), \mathbb{R})$  of real-valued continuous functions on  $\mathcal{Mult}(P, G)$ . This subalgebra separates the points of  $\mathcal{Mult}(P, G)$  and contains a non-zero constant function. Due to the Stone-Weierstrass's theorem,  $A$  is dense in  $C(\mathcal{Mult}(P, G), \mathbb{R})$ . Any function  $\mathbf{f}$  in  $A$  depends only on a finite number of paths, so that there exists some  $J \in \mathcal{F}$  such that  $\mathbf{f}$  can be seen as a continuous function on  $\mathcal{Mult}(J, P)$ . We define:

$$m(\mathbf{f}) = m_J(\mathbf{f}),$$

which does not depend on the chosen  $J \in \mathcal{F}$  thanks to the projectivity and multiplicative properties.

We have defined a positive linear functional  $m$  on  $A$ , the norm of which is bounded by the total weight of any of the measures  $(m_J)_{J \in \mathcal{F}}$ . Thus  $m$  can be extended on  $C(\mathcal{Mult}(P, G), \mathbb{R})$  and an application of the Riesz-Markov's theorem allows us to consider  $m$  as a measure on  $(\mathcal{Mult}(P, G), \mathcal{B})$ . This is the projective limit of  $(m_J)_{J \in \mathcal{F}}$ .  $\square$

The notion of locally stochastically  $\frac{1}{2}$ -Hölder continuity allows us to have an extension theorem from some subsets of paths to their closure, as shown in the proof of Corollary 3.3.2 of [21].

**Theorem 1.** — *Let  $\mu_{\text{Aff}_\gamma(M)}$  be a random holonomy field on  $\text{Aff}_\gamma(M)$ . If it is locally stochastically  $\frac{1}{2}$ -Hölder continuous then there exists a unique stochastically continuous random holonomy field  $\mu$  on  $M$  such that:*

$$\mu_{\text{Aff}_\gamma(M)} = \mu \circ \rho_{\text{Aff}_\gamma(M), P(M)}^{-1}.$$

**2.2.3. Gauge-invariance.** — For any subset  $P$  of  $P(M)$ , a natural group acts on  $\mathcal{Mult}(P, G)$ : it is what we will call the **gauge group** and which we now describe. Let us fix a subset  $P$  of  $P(M)$  which will stay fixed until Section 2.2.4.

**Definition 15.** — *Let  $V = \{x \in M, \exists p \in P, x = \underline{p} \text{ or } x = \overline{p}\}$  be the set of endpoints of  $P$ . We define the partial gauge group associated with  $P$  by setting  $J_P = G^V$ . If  $P = P(M)$ , this group is called the gauge group of  $M$ . The group  $J_P$  acts by gauge transformations on the space  $\mathcal{Mult}(P, G)$ : if  $j \in J_P$ , the action of  $j$  on  $h \in \mathcal{Mult}(P, G)$  is given by:*

$$\forall c \in P, (j \bullet h)(c) = j_{\overline{c}}^{-1} h(c) j_{\underline{c}}.$$

Let  $P = \{c_1, \dots, c_n\}$  be a finite set of paths on  $M$ . Looking only at the value on  $c_i$  for  $i \in \{1, \dots, n\}$ , we have the inclusion:  $\mathcal{Mult}(P, G) \subset G^n$ . The gauge action of

$J_P$  on  $\mathcal{M}ult(P, G)$  extends naturally to an action on  $G^n$  by  $(j \bullet \mathbf{g})_i = j_{c_i}^{-1} \mathbf{g}_i j_{c_i}$  for any  $i \in \{1, \dots, n\}$ .

**Remark 3.** — If  $l_1, \dots, l_n$  are loops based at a point  $m$ , the partial gauge group is nothing but  $G$  and the corresponding action on  $G^n$  is the diagonal conjugation:

$$j \bullet (g_1, \dots, g_n) = (j^{-1} g_1 j, \dots, j^{-1} g_n j).$$

We now define a sub- $\sigma$ -field of  $\mathcal{B}$ : the *invariant  $\sigma$ -field*.

**Definition 16.** — On  $\mathcal{M}ult(P, G)$ , the invariant  $\sigma$ -field, denoted by  $\mathcal{I}$ , is the smallest  $\sigma$ -field such that for any paths  $c_1, \dots, c_n$  in  $P$  and any continuous function  $f : G^n \rightarrow \mathbb{R}$  invariant under the action of  $J_{\{c_1, \dots, c_n\}}$  on  $G^n$  defined in Definition 15, the mapping  $h \mapsto f(h(c_1), \dots, h(c_n))$  is measurable.

If  $M$  is the disjoint union of two smooth compact surfaces,  $M = M_1 \sqcup M_2$  then  $\mathcal{M}ult(P(M), G) \simeq \mathcal{M}ult(P(M_1), G) \times \mathcal{M}ult(P(M_2), G)$ . Besides, let  $\mathcal{I}$  (respectively  $\mathcal{I}_1, \mathcal{I}_2$ ) be the invariant  $\sigma$ -field on  $\mathcal{M}ult(P(M), G)$  (respectively  $\mathcal{M}ult(P(M_1), G), \mathcal{M}ult(P(M_2), G)$ ). We have  $\mathcal{I} \simeq \mathcal{I}_1 \otimes \mathcal{I}_2$ .

Diffeomorphisms between surfaces give rise to some examples of functions which are measurable with respect to the Borel and the invariant  $\sigma$ -fields. Given  $M$  and  $M'$  two smooth compact surfaces, suppose that we are given a diffeomorphism  $\psi$  from  $M$  to  $M'$ , we can construct, for any  $h$  in  $\mathcal{M}ult(P(M'), G)$ , a natural multiplicative function  $\psi^*$  on  $M$  by using  $\psi$ :

$$(\psi^* h)(p) = h(\psi(p)), \forall p \in P(M).$$

This defines a function  $\psi^* : \mathcal{M}ult(P(M'), G) \rightarrow \mathcal{M}ult(P(M), G)$ . It is an easy exercise to verify that  $\psi^*$  is measurable for the Borel and the invariant  $\sigma$ -fields. From now on, we denote also by  $\psi$  the application  $\psi^*$ .

**Definition 17.** — Let  $\mu$  be a random holonomy field on  $P$ . Let  $P_1$  and  $P_2$  be two families of paths in  $P$ . We will say that  $(h(p))_{p \in P_1}$  and  $(h(p))_{p \in P_2}$  are  $\mathcal{I}$ -independent if and only if, for any finite family  $(p_{1,i})_{i=1}^n$  in  $P_1$ , any finite family  $(p_{2,i})_{i=1}^m$  in  $P_2$ , and any continuous function  $f_1 : G^n \rightarrow \mathbb{R}$  (resp.  $f_2 : G^m \rightarrow \mathbb{R}$ ) invariant under the action of  $J_{\{p_{1,1}, \dots, p_{1,n}\}}$  (resp.  $J_{\{p_{2,1}, \dots, p_{2,m}\}}$ ), the following equality holds:

$$(4) \quad \mu \left[ f_1 \left( (h(p_{1,i}))_{i=1}^n \right) f_2 \left( (h(p_{2,j}))_{j=1}^m \right) \right] = \mu \left[ f_1 \left( (h(p_{1,i}))_{i=1}^n \right) \right] \mu \left[ f_2 \left( (h(p_{2,j}))_{j=1}^m \right) \right].$$

This is equivalent to say that under  $\mu$ , the two  $\sigma$ -fields  $\sigma(h(p) : p \in P_1) \cap \mathcal{I}$  and  $\sigma(h(p) : p \in P_2) \cap \mathcal{I}$  are independent.



**Remark 4.** — One has to be careful when dealing with invariant  $\sigma$ -fields. For instance, consider the diagonal action of  $G$  by conjugation on  $G^2$ . The invariant  $\sigma$ -field on  $G^2$  which we denote by  $\mathcal{I}_{(2)}$  is different from the product  $\mathcal{I} \otimes \mathcal{I}$  where  $\mathcal{I}$  is the invariant  $\sigma$ -field of  $G$ . To see why, we only have to consider the symmetric group  $\mathfrak{S}_3$ . If it was true that  $\mathcal{I}_{(2)} = \mathcal{I} \otimes \mathcal{I}$ , we would have the equality:

$$\#\{[(\sigma, \sigma')], (\sigma, \sigma') \in \mathfrak{S}_3^2\} = \left(\#\{[\sigma], \sigma \in \mathfrak{S}_3\}\right)^2.$$

It is fairly easy to see that the l.h.s. is equal to eleven and the r.h.s. to nine.

Let  $(X, Y)$  be a random vector with value in  $(G^2, \mathcal{I}_{(2)})$ , such that  $X$  is independent of  $Y$ . In the light of the discussion we just had, we see that the knowledge of the laws of  $X$  and  $Y$  as  $(G, \mathcal{I})$  random variables **does not allow** us to reconstruct the law of the couple  $(X, Y)$ .

On the invariant  $\sigma$ -field on  $\text{Mult}(P, G)$ , any measure is of course invariant by the gauge transformations. Explicitly, for any measure  $\mu$  on  $(\text{Mult}(P, G), \mathcal{I})$ , for any measurable continuous function  $f$  from  $(\text{Mult}(P, G), \mathcal{I})$  to  $\mathbb{R}$ , and for any  $j \in J_P$ :

$$(5) \quad \int_{\text{Mult}(P, G)} f(j \bullet h) d\mu(h) = \int_{\text{Mult}(P, G)} f(h) d\mu(h).$$

The following definition is less trivial as the following class of gauge-invariant measures is not equal to the collection of all measures.

**Definition 18.** — Let  $\mu$  be a random holonomy field on  $P$ . We say that  $\mu$  is invariant under gauge transformations if and only if the equality (5) holds for any continuous function  $f$  from  $(\text{Mult}(P, G), \mathcal{B})$  to  $\mathbb{R}$ , and for any  $j \in J_P$ .

**Remark 5.** — Let  $\mu$  be a gauge-invariant random holonomy field on  $P$ . Let  $p$  a path in  $P$  which is not a loop:  $\underline{p} \neq \bar{p}$ . Then under  $\frac{\mu}{\mu(\underline{1})}$ ,  $h(p)$  has the law of a Haar random variable. Indeed, applying the gauge transformation which is equal to 1 everywhere except at  $\underline{p}$  or  $\bar{p}$ , where its value is set to be an arbitrary element of  $G$ , we see that the law of  $h(p)$  is invariant by left- and right-multiplication.

Actually there exists a one-to-one correspondence between measures on  $(\text{Mult}(P, G), \mathcal{I})$  and gauge-invariant measures on  $(\text{Mult}(P, G), \mathcal{B})$ . This proposition is an analog of the results of [9]. For any integer  $n$ , for any continuous function  $f$  on  $G^n$  and any set of paths  $\{c_1, \dots, c_n\}$  in  $P$ , we define the function  $\hat{f}_{J_{c_1, \dots, c_n}}$  such that, for any  $g_1, \dots, g_n$  in  $G$ :

$$(6) \quad \hat{f}_{J_{c_1, \dots, c_n}}(g_1, \dots, g_n) = \int_{J_{c_1, \dots, c_n}} f(j \bullet (g_1, \dots, g_n)) dj,$$

where  $dj$  is the Haar measure on  $J_{c_1, \dots, c_n}$ .

**Proposition 3.** — *For any measure  $\mu$  on  $(\text{Mult}(P, G), \mathcal{I})$ , there exists a unique gauge-invariant random holonomy field on  $P$  which will be denoted either by  $\hat{\mu}$  or  $\widehat{\mu}$ , such that:*

$$\hat{\mu}_{\mathcal{I}} = \mu.$$

*Proof.* — The uniqueness of  $\hat{\mu}$  follows from the upcoming Proposition 5. Let us prove its existence. We will define  $\hat{\mu}$  by the fact that for any measurable function  $f : G^n \rightarrow \mathbb{R}^+$  and any  $n$ -tuple  $c_1, \dots, c_n$  of elements of  $P$ :

$$\hat{\mu}\left(f(h(c_1), \dots, h(c_n))\right) = \mu\left(\hat{f}_{J_{c_1, \dots, c_n}}(h(c_1), \dots, h(c_n))\right).$$

The existence of  $\hat{\mu}$  is not totally trivial. For any finite set of paths in  $P$ ,  $P_1 = \{c_1, \dots, c_n\}$ , we have the natural inclusion  $\iota : \text{Mult}(P_1, G) \subset G^n$  by looking at the evaluations on  $c_1, \dots, c_n$ . The equalities  $\hat{\mu}_{P_1}(f) = \mu\left(\hat{f}_{J_{c_1, \dots, c_n}}(h(c_1), \dots, h(c_n))\right)$  for any continuous function on  $G^n$  define a linear positive functional on  $C(G^n)$ . By compactness of  $G^n$ , applying the theorem of Riesz-Markov, it gives a measure  $\hat{\mu}_{P_1}$  on  $G^n$ , the support of which is easily seen to be a subset of  $\iota(\text{Mult}(P_1, G))$ . We can thus look at the induced measure on  $\text{Mult}(P_1, G)$  named  $\hat{\mu}_{|\text{Mult}(P_1, G)}$ . The family of measures  $(\hat{\mu}_{|\text{Mult}(P_1, G)})_{P_1 \subset P, P_1 \text{ finite}}$  forms a projective family of measures for the inclusion of sets. Thus, by Proposition 2, it defines a measure on  $(\text{Mult}(P, G), \mathcal{B})$ .  $\square$

Using the notation of (6) we just introduced, we can better understand the notion of  $\mathcal{I}$ -independent in a special case.

**Remark 6.** — *Let  $\mu$  be a gauge-invariant random holonomy field on  $P$  and let  $P_1$  and  $P_2$  be two sets of paths such that their sets of endpoints  $V_{P_1}$  and  $V_{P_2}$  are disjoint. Then  $(h(p))_{p \in P_1}$  and  $(h(p))_{p \in P_2}$  are  $\mathcal{I}$ -independent if and only if they are independent. Indeed, let us suppose that they are  $\mathcal{I}$ -independent. If  $f$  and  $g$  are real-valued continuous functions on  $G^n$  and  $G^m$  respectively, we denote by  $f \otimes g$  the function from  $G^n \times G^m$  to  $\mathbb{R}$  defined by:*

$$f \otimes g(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}) = f(x_1, \dots, x_n)g(x_{n+1}, \dots, x_{n+m}).$$

*With this notation, and the notation of (6), as the two families  $P_1$  and  $P_2$  have disjoint sets of endpoints,*

$$\widehat{(f \otimes g)}_{J_{P_1 \cup P_2}} = \hat{f}_{J_{P_1}} \otimes \hat{g}_{J_{P_2}},$$

where the partial gauge group was defined in Definition 15. Thus, using the gauge-invariance of  $\mu$ ,

$$\begin{aligned}
\mu \left[ f((h(p))_{p \in P_1}) g((h(p))_{p \in P_2}) \right] &= \mu \left[ (f \otimes g)((h(p))_{p \in P_1}, (h(p))_{p \in P_2}) \right] \\
&= \mu \left[ \widehat{(f \otimes g)}_{J_{P_1 \cup P_2}}((h(p))_{p \in P_1}, (h(p))_{p \in P_2}) \right] \\
&= \mu \left[ \widehat{f}_{J_{P_1}} \otimes \widehat{g}_{J_{P_2}}((h(p))_{p \in P_1}, (h(p))_{p \in P_2}) \right] \\
&= \mu \left[ \widehat{f}_{J_{P_1}}((h(p))_{p \in P_1}) \right] \mu \left[ \widehat{g}_{J_{P_2}}((h(p))_{p \in P_2}) \right] \\
&= \mu \left[ f((h(p))_{p \in P_1}) \right] \mu \left[ g((h(p))_{p \in P_2}) \right].
\end{aligned}$$

Let us introduce the main ingredient in order to construct gauge-invariant holonomy fields: the **loop paradigm** for multiplicative functions.

**Lemma 2.** — *Let us suppose that  $P$  is connected, stable by concatenation and inversion. Let  $m$  be an endpoint of a path in  $P$ . Let  $L_m$  be the set of loops in  $P$  based at  $m$ . The **loop paradigm** for the multiplicative functions is the fact that:*

$$(7) \quad \text{Mult}(P, G)/J_P \simeq \text{Mult}_P(L_m, G)/J_{L_m}.$$

*Proof.* — There exists a natural restriction function:

$$r : \text{Mult}(P, G)/J_P \mapsto \text{Mult}_P(L_m, G)/J_{L_m}.$$

Let us show that there exists an application

$$\iota : \text{Mult}_P(L_m, G)/J_{L_m} \rightarrow \text{Mult}(P, G)/J_P,$$

such that  $r \circ \iota = id$  and  $\iota \circ r = id$ .

For any endpoint  $v$  of  $P$ , let  $q_v$  be a path in  $P$  joining  $m$  to  $v$ . We set  $q_m$  to be the trivial path. Then, for any path  $p$  in  $P$  we define  $l(p) = q_{\underline{p}} p q_{\overline{p}}^{-1}$ . One can look at the Figure 5 to have a better understanding of  $l(p)$ . For any  $h$  in  $\text{Mult}_P(L_m, G)$ , we define for any path  $p$ ,

$$\iota(h)(p) = h(l(p)).$$

This is a multiplicative function. Let us show, for example, that it is compatible with the concatenation operation. For any  $h \in \text{Mult}_P(L_m, G)$ , and any paths  $p$  and  $p'$  in  $P$  such that  $\overline{p} = \underline{p'}$ , the following sequence of equalities holds:

$$\begin{aligned}
\iota(h)(pp') &= h(l(pp')) = h(q_{\underline{p}} p p' q_{\overline{p'}}^{-1}) = h(q_{\underline{p}} p q_{\overline{p}}^{-1} q_{\underline{p'}} p' q_{\overline{p'}}^{-1}) \\
&= h(q_{\underline{p'}} p' q_{\overline{p'}}^{-1}) h(q_{\underline{p}} p q_{\overline{p}}^{-1}) = h(l(p')) h(l(p)) = \iota(h)(p') \iota(h)(p).
\end{aligned}$$

Thus  $\iota$  is an application from  $\text{Mult}_P(L_m, G)$  to  $\text{Mult}(P, G)$ . This application  $\iota$  defines a function, that we will also call  $\iota$  from  $\text{Mult}_P(L_m, G)/J_{L_m}$  to  $\text{Mult}(P, G)/J_P$ . Indeed, if  $j \in J_{L_m}$ ,  $h \in \text{Mult}_P(L_m, G)$  and  $p \in P$

$$\iota(j \bullet h)(p) = j \bullet h(l(p)) = j(m)^{-1} h(l(p)) j(m) = \tilde{j} \bullet \iota(h)(p),$$

where  $\tilde{j}$  is the function constant equal to  $j$ . Let us show that  $\iota \circ r = id$ : for any  $h \in \text{Mult}(P, G)/J_P$ :

$$\begin{aligned} (\iota(r(h))(p))_{p \in P} &= r(h)(l(p)) = \left( h(q_{\underline{p}} p q_{\overline{p}}^{-1}) \right)_{p \in P} \\ &= \left( h(q_{\overline{p}})^{-1} h(p) h(q_{\underline{p}}) \right)_{p \in P} \\ &= (h(p))_{p \in P} \end{aligned}$$

The equality  $r \circ \iota = id$  is even easier. □

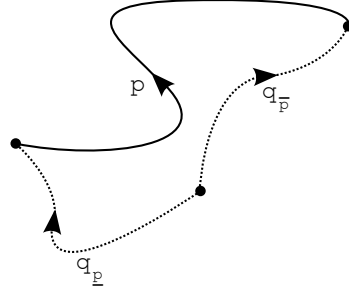


FIGURE 5. Construction of the loop  $l(p)$ .

From the proof of Lemma 2, one also gets the following lemma.

**Lemma 3.** — *Suppose that  $P$  is connected, stable by concatenation and inversion. Let  $m$  be an endpoint of  $P$  and let  $L_m$  be the subset of all loops in  $P$  based at  $m$ . There exists an application:*

$$\iota : \text{Mult}_P(L_m, G) \rightarrow \text{Mult}(P, G)$$

which is measurable for the Borel  $\sigma$ -field and such that, for any loop  $l \in L_m$ , the following diagram is commutative:

$$\begin{array}{ccc} \text{Mult}_P(L_m, G) & \xrightarrow{\iota} & \text{Mult}(P, G) \\ & \searrow h(l) & \swarrow h(l) \\ & & G \end{array}$$

One consequence of Lemma 2 is Lemma 2.1.5 in [21].

**Lemma 4.** — *Suppose that  $P$  is connected, stable by concatenation and inversion. Let  $m$  be an endpoint of  $P$ . Then for any paths,  $c_1, \dots, c_n$  in  $P$  and any measurable function  $f : G^n \rightarrow \mathbb{R}$ , invariant under the action of  $J_{c_1, \dots, c_n}$  on  $G^n$ , there exist  $n$  loops  $l_1, \dots, l_n$  in  $P$  based at  $m$  and a measurable function  $\tilde{f} : G^n \rightarrow \mathbb{R}$  invariant under the diagonal action of  $G$  such that:*

$$f(h(c_1), \dots, h(c_n)) = \tilde{f}(h(l_1), \dots, h(l_n)).$$

Lemma 4 allows us to reduce the family of variables that  $\mathcal{I}$  has to make measurable: we only have to look at finite collections of loops based at the same point, which leads us to the Definition 2.1.6 of [21].

**Proposition 4.** — *Let us suppose that  $P$  is connected, stable by concatenation and inversion. Let  $m$  be an endpoint of  $P$ . The invariant  $\sigma$ -field  $\mathcal{I}$  on  $\text{Mult}(P, G)$  is the smallest  $\sigma$ -field such that for any positive integer  $n$ , any loops  $l_1, \dots, l_n$  based at  $m$ , and any continuous function  $f : G^n \rightarrow \mathbb{R}$  invariant under the diagonal action of  $G$ , the mapping  $h \mapsto f(h(l_1), \dots, h(l_n))$  is measurable.*

Another consequence of Lemma 4 is:

**Proposition 5.** — *Let  $P$  be a connected set of paths, stable by concatenation and inversion. Let  $m$  be an endpoint in  $P$  and  $L_m$  the set of loops in  $P$  based at  $m$ . If  $\mu$  and  $\nu$  are two stochastically continuous gauge-invariant random holonomy fields on  $P$ , the two assertions are equivalent:*

1.  $\mu$  and  $\nu$  are equal,
2. there exist  $m \in M$  and  $A_m$  a dense subset of  $L_m$  for the convergence with fixed endpoints, such that for any integer  $n$ , any  $n$ -tuple of loops  $l_1, \dots, l_n$  in  $A_m$ , and any continuous function  $f : G^n \rightarrow \mathbb{R}$  invariant under the diagonal action of  $G$ ,

$$\int_{\text{Mult}(P, G)} f(h(l_1), \dots, h(l_n)) d\mu(h) = \int_{\text{Mult}(P, G)} f(h(l_1), \dots, h(l_n)) d\nu(h).$$

If the random holonomy fields are not stochastically continuous, the proposition still holds with  $A_m = L_m$ .

**Remark 7.** — *The first consequence of this proposition is the **change of base point invariance property** of gauge-invariant random holonomy fields. For the sake of simplicity, let us consider  $\mu$  a gauge-invariant random holonomy field on  $M$ . Let us consider a bijection  $\psi : M \rightarrow M$  and let us consider for any point  $x \in M$ ,  $p_x$  a path from  $\psi(x)$  to  $x$ . Then the random holonomy field which has the law of:*

$$(h(p_x))_{x \in M} \bullet [h(p)]_{p \in P}$$

*under  $\mu$ , is still gauge invariant. The last proposition shows that  $\mu$  and the new random holonomy field are equal: for any paths  $p_1, \dots, p_n$ , we have the equality in law:*

$$\left( \left[ (h(p_x))_{x \in M} \bullet [h(p)]_{p \in P} \right] (p_i) \right)_{i=1}^n = (h(p_i))_{i=1}^n$$

*under  $\mu$ . Thus, for example, if  $l_1, \dots, l_n$  are  $n$  loops based at  $m$ , and if  $s$  is a path from  $m'$  to  $m$ , under a gauge-invariant measure  $\mu$ ,  $(h(sl_1s^{-1}), \dots, h(sl_ns^{-1}))$  has the same law as  $(h(l_1), \dots, h(l_n))$ .*

2.2.4. *Construction of random holonomy field II: gauge-invariant case.* — In this section, for the sake of simplicity, we will suppose that  $M$  is connected. However, all results could be extended easily to the non-connected case. Thanks to Lemma 2 and Proposition 3, constructing an invariant random holonomy field  $\mu$  becomes easier. Let  $P$  be a connected set of paths, stable by concatenation and inversion.

**Proposition 6.** — *Let  $m$  be an endpoint of  $P$ . Suppose that for any finite subset  $L$  of loops in  $P$  based at  $m$ , we are given a gauge-invariant random holonomy field  $\mu_L$  on  $L$  such that, when endowed with the natural restriction functions,  $((\text{Mult}_P(L, G), \mathcal{B}), \mu_L)$  is a projective family. Then there exists a unique gauge-invariant random holonomy field  $\mu$  on  $P$  such that for any finite subset of loops in  $P$  based at  $m$ , one has*

$$\mu_L = \mu \circ \rho_{L,P}^{-1}.$$

*Proof.* — The uniqueness of such a measure comes from a direct application of Proposition 5.

Let  $L_m$  be the set of loops in  $P$  based at  $m$ . Let us prove the existence of  $\mu$ . Using Proposition 2, we can consider the projective limit  $\mu_{L_m}$  of  $(\mu_{L_f})_{L_f \subset L_m, \text{finite}}$ , defined on  $(\text{Mult}_P(L_m, G), \mathcal{B})$ . The measure  $\mu_{L_m}$  is gauge-invariant. The set  $P$  satisfies the assumption of Lemma 3, thus there exists a measurable application  $\iota$  from  $\text{Mult}_P(L_m, G)$  to  $\text{Mult}(P, G)$ . We can then look at the following measure:

$$\mu = ((\mu_{L_m} \circ \iota^{-1})|_{\mathcal{I}})^{\widehat{}}$$

where we remind the reader that  $(\ )^{\widehat{}}$  is the notation for the extension of measure from the invariant  $\sigma$ -field to the Borel  $\sigma$ -field given by Proposition 3. By definition, it is defined on the Borel  $\sigma$ -field on  $\text{Mult}(P, G)$  and it is gauge-invariant.

If  $L$  is a finite subset of loops in  $P$  based at  $m$ , thanks to the definitions of  $\iota$  and  $\mu_{L_m}$ ,  $(\mu_{L_m} \circ \iota^{-1}) \circ \rho_{L,P}^{-1} = \mu_L$ . Thanks to the gauge-invariance of  $\mu_L$ ,  $((\mu_L)|_{\mathcal{I}})^{\widehat{}} = \mu_L$ , which leads us to the conclusion:  $\mu_L = \mu \circ \rho_{L,P}^{-1}$ .  $\square$

In fact, what will be used is the combination of this last proposition and Theorem 1.

**Proposition 7.** — *Let  $\gamma$  be a Riemannian metric on  $M$ . Let  $m \in M$ . Suppose that for any finite subset of loops  $L$  in  $\text{Aff}_{\gamma}(M)$  based at  $m$ , we are given  $\mu_L$ , a gauge-invariant random holonomy field on  $L$ , such that  $((\text{Mult}_P(L, G), \mathcal{B}), \mu_L)$  is a projective family of uniformly locally stochastically  $\frac{1}{2}$ -Hölder continuous. Then there exists a unique stochastically continuous gauge-invariant random holonomy field  $\mu$  on  $M$  such that:*

$$\mu_L = \mu \circ \rho_{L,P(M)}^{-1}.$$

*2.2.5. Restriction and extension of the group.* — Let  $P$  be a connected set of paths, stable by concatenation and inversion. Let  $m$  be an endpoint of  $P$  and let  $L_m$  be the set of loops in  $P$  based at  $m$ . Let  $H$  be a closed subgroup of  $G$ .

There exists a natural injection:

$$i_P : (\mathcal{M}ult(P, H), \mathcal{B}) \rightarrow (\mathcal{M}ult(P, G), \mathcal{B}).$$

Thus, we can always push forward any  $H$ -valued random holonomy field by  $i_P$  in order to create a  $G$ -valued random holonomy field. Of course, if a  $G$ -valued holonomy field on  $P$ , say  $\mu$ , is such that there exists a closed group  $H \subset G$  such that for any path  $p \in P$ , one has  $h(p) \in H$   $\mu$  a.s., then we can restrict the group to  $H$ : for any finite  $P_f \subset P$  it defines a measure on  $\mathcal{M}ult(P_f, H)$  and we can take the projective limit thanks to Proposition 2.

In the gauge-invariant setting, what can be done ? First of all, if  $\mu$  is a gauge-invariant holonomy field on  $H$ ,  $\mu \circ i_P^{-1}$  is not in general a gauge-invariant holonomy field on  $G$ . The simplest counterexample is to consider  $P$  to be reduced to a single loop: a  $G$ -valued random variable can be  $H$ -invariant but not  $G$ -invariant by conjugation. Thus, in order to extend the structure group from  $H$  to  $G$  of a  $H$ -gauge-invariant holonomy field  $\mu$ , one has to consider:

$$(8) \quad ((\mu \circ i_P^{-1})|_{\mathcal{I}})^{\widehat{}}$$

the gauge-invariant extension (see Proposition 3) to  $\mathcal{B}$  of the restriction on the invariant  $\sigma$ -field  $\mathcal{I}$  of  $\mu \circ i_P^{-1}$ .

Thus, the natural injection is replaced by the following map:

$$\mu \mapsto ((\mu \circ i_P^{-1})|_{\mathcal{I}})^{\widehat{}}.$$

**Notation 3.** — In the following, for any  $H$ -valued random holonomy field, and any set of paths  $P$ , we will use the notation  $\mu \circ \hat{i}_P^{-1}$  instead of  $((\mu \circ i_P^{-1})|_{\mathcal{I}})^{\widehat{}}$ .

Now, let us consider the problem of restricting a gauge-invariant random holonomy field  $\mu$ . Thanks to Lemma 4, the only important objects are loops based at  $m$ . Hence the question: what can be done with a  $G$ -valued random holonomy field such that for any loop or for any simple loop  $l \in L_m$ ,  $\mu$  a.s.  $h(l) \in H$  ? An important remark is that it does not imply that for any path  $p \in P$ ,  $\mu$  a.s.  $h(p) \in H$ . Indeed, as seen in Remark 5, for any  $p$  such that  $\bar{p} \neq \underline{p}$ , under  $\mu/\mu(1)$ ,  $h(p)$  has the law of a Haar random variable on  $G$ . Nevertheless, the following result is true.

**Proposition 8.** — Let  $\mu$  be a  $G$ -valued gauge-invariant holonomy field such that for any loop  $l \in L_m$ ,  $h(l) \in H$ ,  $\mu$  a.s. Then there exists an  $H$ -valued gauge-invariant holonomy field  $\mu_H$  such that:

$$\mu = \mu_H \circ \hat{i}_P^{-1}.$$

Suppose that  $P$  is equal to  $P(M)$ , that  $\mu$  is stochastically continuous and that  $H$  is a close subgroup. Then, the result is still true if for any simple loop  $l \in \text{Aff}_\gamma(\mathbf{M}) \cup L_m(M)$ ,  $h(l) \in H$ ,  $\mu$  a.s.

**Remark 8.** — An important remark is that  $\mu_H$  is **not** unique.

We give below the loop-erasure lemma used in the last proof which is taken from Proposition 1.4.9 in [21].

**Lemma 5.** — Let  $(M, \gamma)$  be a Riemannian compact surface. Let  $c$  be a loop in  $\text{Aff}_\gamma(M)$ . There exists in  $\text{Aff}_\gamma(M)$  a finite sequence of lassos  $l_1, \dots, l_p$  and a simple loop  $d$  with the same endpoints as  $c$  such that:

$$c \simeq l_1 \dots l_p d.$$

*Proof of Proposition 8.* — We only prove the second case, under the assumption that  $P$  is equal to  $P(M)$  and  $\mu$  is stochastically continuous. Let  $\gamma$  be a Riemannian metric on  $M$ . Suppose that for any simple loop  $l \in \text{Aff}_\gamma(\mathbf{M}) \cap L_m(M)$ ,  $h(l) \in H$ ,  $\mu$  a.s. We may assume that  $H$  is the smallest group for which this property is satisfied. Then, thanks to the gauge-invariance of  $\mu$ ,  $H$  must be normal in  $G$ . Using the multiplicativity property, for every lasso  $l$  in  $\text{Aff}_\gamma(M)$  based at  $m$ ,  $h(l) \in H$ ,  $\mu$  a.s. As a consequence of Lemma 5, for any loop  $l \in \text{Aff}_\gamma(M)$  based at  $m$ ,  $h(l) \in H$ ,  $\mu$  a.s. Thus, by Lemma 1, the stochastic continuity of  $\mu$ , and the fact that  $H$  is closed, for any  $l \in L(M)$ ,  $h(l) \in H$ ,  $\mu$  a.s.

Let  $m$  be a point in  $M$ . Using  $\mu$ , one can define, for any finite subset  $L_f$  of  $L_m(M)$ , a gauge-invariant measure  $\mu_{L_f}$  on  $\text{Mult}_{P(M)}(L(M), H)$ . As a consequence of Proposition 6, there exists a unique  $H$ -valued gauge-invariant random holonomy field  $\mu_H$  on  $M$  such that  $\mu_{L(M)} = \mu_H \circ \rho_{L(M), P(M)}^{-1}$ . It can be easily verified that  $\mu = \mu_H \circ \hat{i}_{P(M)}^{-1}$ .  $\square$

### 3. Graphs

**3.1. Definitions and simple facts.** — Later we will need the definition of graph for the construction of special families of random fields: Planar Markovian Holonomy Fields. The graphs we look at are not only combinatorial ones: we insist that the faces are homeomorphic to an open disk of  $\mathbb{R}^2$ .

**Definition 19.** — A pre-graph on  $M$  is a triple  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  such that:

- $\mathbb{E}$ , the set of edges, is a non-empty finite set of simple paths on  $M$ , stable by inversion, such that two edges which are not each other's inverse meet, if at all, only at some of their endpoints,
- $\mathbb{V}$ , the set of vertices, is the finite subset of  $M$  equal to  $\bigcup_{e \in \mathbb{E}} \{\bar{e}, \underline{e}\}$ ,
- $\mathbb{F}$ , the set of faces, is the set of the connected components of  $M \setminus \bigcup_{e \in \mathbb{E}} e([0, 1])$ .



Any pre-graph  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  whose bounded faces  $F \in \mathbb{F}$  are homeomorphic to an open disk of  $\mathbb{R}^2$ , is called a graph on  $M$ .

**Remark 9.** — If  $\mathbb{G}$  is a graph on  $M$  then  $\partial M$  can be represented by a concatenation of edges in  $\mathbb{E}$ .

Due to the last definition, any pre-graph  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  is characterized by its set of edges  $\mathbb{E}$ . Thus, in order to construct a graph, we will only define its set of edges. We will often use the following graph.

**Example 1.** — Let  $l$  be a simple loop on  $\mathbb{R}^2$ . We denote by  $\mathbb{G}(l)$  the finite planar graph composed of  $l$  and  $l^{-1}$  as unique edges.

When  $M$  is homeomorphic to a sphere, we will consider that  $(\{m\}, \emptyset, M \setminus \{m\})$  is a graph for any  $m \in M$ . For a non-connected surface, a graph is a collection of graphs on each connected component of this surface. A finite graph on  $\mathbb{R}^2$  will be also called a *finite planar graph*; its set of faces is composed of one unbounded face denoted by  $F_\infty$  and a set  $\mathbb{F}^b$  of bounded faces.

**Definition 20.** — Let  $\mathbb{G}$  be a graph on  $M$ ,  $P(\mathbb{G})$  is the set of paths obtained by concatenating edges of  $\mathbb{G}$ . The set of loops in  $P(\mathbb{G})$  is denoted by  $L(\mathbb{G})$  and if  $v \in \mathbb{V}$ ,  $L_v(\mathbb{G})$  is the set of loops in  $L(\mathbb{G})$  based at  $v$ .

A graph is connected if and only if any two points of  $\mathbb{E}$  are the endpoints of a path in  $P(\mathbb{G})$ .

For any smooth compact surface with boundary  $M$  embedded in  $\mathbb{R}^2$ , a graph on  $M$  can be considered as a finite planar graph. This kind of graphs, of interest later, will be called embedded graphs on  $\mathbb{R}^2$ .

**Definition 21.** — An embedded graph on  $\mathbb{R}^2$  is a graph on a smooth compact surface with boundary  $M$  embedded in  $\mathbb{R}^2$ .

The two definitions of graphs on  $\mathbb{R}^2$  seen here are in fact almost equivalent. An embedded graph is obviously a finite graph and one can prove, using Propositions 1.3.24 and 1.3.26 of [21], the following result.

**Proposition 9.** — Every finite planar graph on  $\mathbb{R}^2$  is a subgraph of an embedded graph.

The intersection of a graph  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  with a subset  $A$  of  $\mathbb{R}^2$  is the pre-graph  $(\mathbb{V}', \mathbb{E}', \mathbb{F}')$ , denoted by  $\mathbb{G} \cap A$ , such that  $\mathbb{E}' = \{e \in \mathbb{E}, e \cap A \not\subseteq \{e, \bar{e}\}\}$ .

Let us give the general definition of a planar graph. For any positive real  $r$ , let  $\mathbb{D}(0, r)$  be the closed ball of center  $(0, 0)$  and radius  $r$  in  $\mathbb{R}^2$ .

**Definition 22.** — A planar graph is a triple  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  of vertex, edges and faces for which exists an increasing unbounded sequence of positive reals  $(r_n)_{n \in \mathbb{N}}$  such that for each integer  $n$ ,  $\mathbb{G} \cap \mathbb{D}(0, r_n)$  is a finite planar graph.

**Exemple 2.** — We consider  $\mathbb{N}^2$  as a planar graph, the edges being the vertical and horizontal segments between nearest neighbors.

Sometimes, one wants to consider graphs whose edges are in a given subset  $A$  of  $P(M)$ . We denote by  $\mathcal{G}(A)$  the set of graphs  $\mathbb{G} = \{\mathbb{V}, \mathbb{E}, \mathbb{F}\}$  such that  $\mathbb{E} \subset A$ .

In the notions of graph exposed above, the edges are non-oriented, which means that there is no preference between  $e$  and  $e^{-1}$  for any edge  $e$ . We will need later to use an orientation on the graph.

**Definition 23.** — An orientation on a graph  $\mathbb{G}$  is the data of a subset  $\mathbb{E}^+$  of  $\mathbb{E}$  such that  $\mathbb{E}^+ \cap (\mathbb{E}^+)^{-1} = \emptyset$  and  $\mathbb{E}^+ \cup (\mathbb{E}^+)^{-1} = \mathbb{E}$ . Given an orientation  $\mathbb{E}^+$  on  $\mathbb{G}$ , for each subset  $J$  of  $\mathbb{E}$ , we denote by  $J^+$  the set  $J \cap \mathbb{E}^+$ .

**3.2. Graphs and homeomorphisms.** — We will need in the following to understand the action of homeomorphisms on the set of graphs. For that, an important notion introduced in [21] by Lévy is the cyclic order of the outgoing edges at a vertex.

**Definition 24.** — Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a finite planar graph. Let  $v$  be a vertex and let  $\mathbb{E}_v$  be the set of edges  $e \in \mathbb{E}$  such that  $\underline{e} = v$ . For any  $e \in \mathbb{E}_v$ , let  $e_p$  be a parametrized path which represents  $e$ . We then define:

$$r_0 = \min \left\{ \left\| v - e_p \left( \frac{1}{2} \right) \right\|, e \in \mathbb{E}_v \right\}.$$

Let  $r \in ]0, r_0[$ . For each  $e \in \mathbb{E}_v$ , we define  $s_e(r) \in [0, \frac{1}{2}]$  as the first time  $e_p$  hits the boundary of  $\mathbb{D}(0, r)$ :

$$s_e(r) = \inf \left\{ t \in \left[ 0, \frac{1}{2} \right], \left\| v - e_p(t) \right\| = r \right\}.$$

The cyclic permutation of  $\mathbb{E}_v$  corresponding to the cyclic order of the points  $\{e_p(s_e(r)), e \in \mathbb{E}_v\}$  on the circle  $\partial\mathbb{D}(0, r)$  does not depend on the chosen  $r \in ]0, r_0[$ : it is the cyclic order of the edges at the vertex  $v$  denoted by  $\sigma_v$ .

Let  $\mathbb{G}$  and  $\mathbb{G}'$  be two finite planar graphs. Let  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a homeomorphism which sends  $\mathbb{G}$  on  $\mathbb{G}'$ . It induces a bijection  $\mathcal{S}_{\mathbb{G}}^\psi$  from the set  $\mathbb{V}$  of vertices of  $\mathbb{G}$  to the set  $\mathbb{V}'$  of vertices of  $\mathbb{G}'$ . It also induces a bijection  $\mathcal{E}_{\mathbb{G}}^\psi$  from the set  $\mathbb{E}$  of edges of  $\mathbb{G}$  to the set  $\mathbb{E}'$  of edges of  $\mathbb{G}'$ . These bijections are defined by:

$$\begin{aligned} \mathcal{S}_{\mathbb{G}}^\psi(v) &= \psi(v), \text{ for any } v \in V, \\ \mathcal{E}_{\mathbb{G}}^\psi(e) &= \psi(e), \text{ for any } e \in E. \end{aligned}$$

**Definition 25.** — Let  $\psi$  and  $\psi'$  be two homeomorphisms of  $\mathbb{R}^2$ . Let  $\mathbb{G}$  be a finite planar graph. We will say that two homeomorphisms  $\psi$  and  $\psi'$  are equivalent on  $\mathbb{G}$  if and only if  $\mathcal{S}_{\mathbb{G}}^\psi = \mathcal{S}_{\mathbb{G}}^{\psi'}$  and  $\mathcal{E}_{\mathbb{G}}^\psi = \mathcal{E}_{\mathbb{G}}^{\psi'}$ .

A consequence of Jordan-Schönflies's theorem is the Heffter-Edmonds-Ringel rotation principle, stated in Theorem 3.2.4 of [23]. Using the notions early defined, we can state it as follows.

**Theorem 2.** — *Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  and  $\mathbb{G}' = (\mathbb{V}', \mathbb{E}', \mathbb{F}')$  be two connected finite planar graphs such that the following assertions hold:*

1. *there exists a bijection  $\mathcal{S} : \mathbb{V} \rightarrow \mathbb{V}'$ ,*
2. *there exists a bijection  $\mathcal{E} : \mathbb{E} \rightarrow \mathbb{E}'$  such that for any  $e \in \mathbb{E}$ ,  $\mathcal{E}(e^{-1}) = \mathcal{E}(e)^{-1}$ ,*
3. *for any edge  $e \in \mathbb{E}$ ,  $\mathcal{S}(\underline{e}) = \underline{\mathcal{E}(e)}$ ,*
4. *for any vertex  $v \in \mathbb{V}$ ,  $\sigma_{\mathcal{S}(v)} = \mathcal{E} \circ \sigma_v \circ \mathcal{E}^{-1}$ .*

*Then there exists a homeomorphism  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\psi(\mathbb{G}) = \mathbb{G}'$  and  $\psi$  induces the two bijections  $\mathcal{S}$  and  $\mathcal{E}$ .*

If one considers only piecewise affine edges, the theorem can be applied to pregraph with affine edges.

Latter we will need the notion of diffeomorphisms at infinity. The motivation will appear in Lemma 25 where we show that the free boundary condition expectation on the plane associated with a Markovian Holonomy Field is a Planar Markovian Holonomy Field.

**Definition 26.** — *A homeomorphism  $\psi$  of  $\mathbb{R}^2$  is a diffeomorphism at infinity if there exists a real  $R$  such that  $\psi|_{\mathbb{D}(0,R)^c}$  is a diffeomorphism.*

In fact each time we have a homeomorphism from a close domain delimited by a Jordan curve to an other domain delimited by an other Jordan curve, we can extend it as a diffeomorphism at infinity. Using the Carathéodory's theorem for Jordan curves, we can suppose that both domains are the unit disk. In this case, the result follows from the following lemma.

**Lemma 6.** — *Let  $\mathbb{D}$  be the closed disk of center 0 and radius 1. Let  $\psi : \partial\mathbb{D} \rightarrow \partial\mathbb{D}$  be a homeomorphism. There exists a diffeomorphism  $\Psi : \mathbb{D}^c \rightarrow \mathbb{D}^c$  such that for any  $x \in \partial\mathbb{D}$ ,*

$$\lim_{y \rightarrow x} \Psi(y) = \psi(x).$$

*Proof.* — Let  $(\eta_r)_{r>1}$  be a smooth even approximation to the identity when  $r$  goes to one:  $(\eta_r)_{r \geq 1}$  is a family of smooth even functions from  $\mathbb{R}$  to  $\mathbb{R}$  such that for any  $r > 1$ :

$$\eta_r \geq 0, \int_{\mathbb{R}} \eta_r = 1, \int_{]1-r, r-1[^c} \eta_r = 0.$$

There is a natural bijection  $\Phi$  between homeomorphisms of  $\partial\mathbb{D}$  and the set  $Hom_{\partial\mathbb{D}}^{\mathbb{R}}$  of strictly increasing or decreasing continuous functions  $f$  from  $\mathbb{R}$  to  $\mathbb{R}$  such that  $f - Id$  is 1-periodic. For any real  $r \geq 1$ , the convolution with  $\eta_r$  sends

$Hom_{\partial\mathbb{D}}^{\mathbb{R}}$  on itself. Let  $\psi : \partial\mathbb{D} \rightarrow \partial\mathbb{D}$  be a homeomorphism of the circle. We define the smooth function  $\Psi$  equal to:

$$\begin{aligned} \Psi : \quad \mathbb{D}^c &\rightarrow \mathbb{D}^c \\ re^{2i\pi\theta} &\mapsto re^{2i\pi(\Phi(\psi)*\eta_r)(\theta)}. \end{aligned}$$

As  $\psi$  is continuous on the disk,  $\Phi(\psi)$  is uniformly continuous. Thus  $\Phi(\psi) * \eta_r$  converges uniformly to  $\Phi(\psi)$  as  $r$  tends to 1. This implies that for any  $x \in \partial\mathbb{D}$ ,  $\lim_{y \rightarrow x} \Psi(y) = \psi(x)$ .

The convolution with  $\eta_r$  sending  $Hom_{\partial\mathbb{D}}^{\mathbb{R}}$  on itself implies that  $\Psi$  is bijective. It remains to show that the Jacobian of  $\Psi$  is strictly positive. Yet, for any  $x \in \partial\mathbb{D}$ , the module of  $x$  is only involved in the calculation of the module of  $\Psi(x)$ : the Jacobian matrix is then triangular. Besides, using the fact that  $\eta_r$  is even for any  $r$ , and that  $\Phi(\psi)$  is strictly increasing, it is easy to see that the derivative of  $\Phi(\psi) * \eta_r$  is strictly positive. These facts imply that the Jacobian matrix is invertible:  $\Psi$  is a diffeomorphism.  $\square$

**3.3. Graphs and partial order.** — The piecewise affine edge graphs are interesting when one considers a special partial order on graphs studied in [21].

*Definition 27.* — Let  $\mathbb{G}$  and  $\mathbb{G}'$  be two planar graphs. We say that  $\mathbb{G}'$  is finer than  $\mathbb{G}$  if  $P(\mathbb{G}) \subset P(\mathbb{G}')$ . We denote it by  $\mathbb{G} \preceq \mathbb{G}'$ .

In fact, in Lemma 1.4.6. of [21], Lévy showed that this partial order is not directed. Yet, one can, by restricting it to a dense subspace of graphs, make it directed: for this, the edges of the graphs which we consider must be in a good subspace as defined below.

*Definition 28.* — Let  $P$  be a subset of  $P(M)$ . A good subspace  $A$  of  $P$  is a dense subset of  $P$  for the convergence with fixed endpoints such that for any finite subset  $\{c_1, \dots, c_n\}$  of  $A$ , there exists a graph  $\mathbb{G}$  such that  $\{c_1, \dots, c_n\} \subset P(\mathbb{G})$ .

If  $A$  is a good subspace,  $\mathcal{G}(A)$  endowed with  $\preceq$  is directed.

*Lemma 7.* — For any Riemannian metric  $\gamma$  on  $M$ ,  $\text{Aff}_{\gamma}(M)$  is a good subspace for  $P(M)$ .

There are other natural examples of good subspaces of  $P(M)$ . For example, Baez in [9] used the good subspace of piecewise real-analytic paths in  $P(\mathbb{R}^2)$  in order to define the Ashtekar and Lewandowski uniform measure. Another example of good subspace is used in the works [24] and [25] of Sengupta.

By definition, any path in  $M$  can be approximated by a sequence of paths in  $A$  if  $A$  is a good subspace. But  $\text{Aff}_{\gamma}$  satisfies the stronger property which roughly asserts that  $\mathcal{G}(\text{Aff}_{\gamma})$  is “dense” for a certain notion in the set of planar graphs. The next theorem is a consequence of Proposition 1.4.10. in [21].

**Theorem 3.** — Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a finite planar graph. Let  $\gamma$  be a Riemannian metric on  $\mathbb{R}^2$  and  $\text{vol}$  the measure of area associated with it. There exists a sequence of finite planar graphs  $(\mathbb{G}_n = (\mathbb{V}_n, \mathbb{E}_n, \mathbb{F}_n))_{n \in \mathbb{N}}$  in  $\mathcal{G}(\text{Aff}_\gamma(\mathbb{R}^2))$  such that:

1. for any integer  $n$ , there exists  $\psi_n$  a homeomorphism of  $\mathbb{R}^2$  such that  $\psi_n(\mathbb{G}) = \mathbb{G}_n$ .
2.  $\mathbb{V}_n = \mathbb{V}$ ,
3. for any edge  $e \in \mathbb{E}$ ,  $\psi_n(e)$  converges to  $e$  for the convergence with fixed end-points,
4. for any face  $F \in \mathbb{F}$ ,  $\text{vol}(\psi_n(F)) \xrightarrow[n \rightarrow \infty]{} \text{vol}(F)$ .

Another interesting property of  $\mathcal{G}(\text{Aff}_\gamma(\mathbb{R}^2))$  is the fact that any planar finite graph with piecewise affine edges can be sent by a piecewise smooth application to a sub-graph of the  $\mathbb{N}^2$  planar graph. We will prove this in Section 3.5, but before, we need to gather a few facts about graphs and triangulations.

**3.4. Graphs and piecewise diffeomorphisms.** — In this subsection, we will only consider finite planar graphs  $\mathbb{G}$  in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$ .

**Definition 29.** — Let  $\mathbb{G}$  be a connected finite planar graph in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$ . It is simple if the boundary of any face of  $\mathbb{G}$  is a simple loop. It is a triangulation if any bounded face is a non degenerate triangle.

**Definition 30.** — Let  $\mathbb{G}$  be a connected finite planar graph in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$ . A mesh of  $\mathbb{G}$  is a simple graph  $\mathbb{G}'$  in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$  such that  $\mathbb{G} \preceq \mathbb{G}'$ . A triangulation of  $\mathbb{G}$  is a triangulation  $\mathbb{T}$  such that  $\mathbb{G} \preceq \mathbb{T}$  and the unbounded face of  $\mathbb{T}$  is the unbounded face of  $\mathbb{G}$ .

Two triangulations are homeomorphic if they are homeomorphic as finite planar graphs.

**Definition 31.** — Let  $\mathbb{G}$  and  $\mathbb{G}'$  be two finite planar graphs. A homeomorphism  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a  $\mathbb{G} - \mathbb{G}'$  piecewise diffeomorphism if the three following assertions hold:

1.  $\phi(\mathbb{G}) = \mathbb{G}'$ ,
2. there exists a mesh  $\mathbb{G}_0$  of  $\mathbb{G}$  (resp  $\mathbb{G}'_0$  of  $\mathbb{G}'$ ) such that  $\phi(\mathbb{G}_0) = \mathbb{G}'_0$  and for any bounded face  $F$  of  $\mathbb{G}_0$ ,  $\phi|_F : F \rightarrow \phi(F)$  is a diffeomorphism whose Jacobian determinant is bounded below and above by some strictly positive real numbers and whose Jacobian determinant can also be extended on the boundary of  $F$ ,
3. let  $F_\infty$  be the unbounded face of  $\mathbb{G}_0$ . The application  $\phi|_{F_\infty} : F_\infty \rightarrow \phi(F_\infty)$  is a diffeomorphism.

We will say that  $\mathbb{G}_0$  is a good mesh for  $\phi$ .

The piecewise diffeomorphisms we will construct will always be of the following form: they will be the extension (using Lemma 6 and the discussion before) of

a piecewise affine homeomorphism from the interior of a piecewise affine Jordan curve to itself. In order to construct these piecewise affine homeomorphisms, we need to be able to triangulate two graphs in the same way. Recall the definition of equivalence defined in Definition 25.

**Proposition 10.** — *Let  $\mathbb{G}_1$  and  $\mathbb{G}_2$  be two homeomorphic simple connected finite planar graphs with piecewise affine edges. Let us choose a homeomorphism  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\phi(\mathbb{G}_1) = \mathbb{G}_2$ . There exist two triangulations,  $\mathbb{T}_1$  of  $\mathbb{G}_1$ ,  $\mathbb{T}_2$  of  $\mathbb{G}_2$  and a piecewise-diffeomorphism  $\psi$  such that:*

1.  $\psi$  and  $\phi$  are equivalent on  $\mathbb{G}_1$ ,
2.  $\psi(\mathbb{T}_1) = \mathbb{T}_2$ .

Consequently, the set of  $\mathbb{G} - \mathbb{G}'$  piecewise diffeomorphisms is not empty.

In order to prove this proposition, we will need the following result proved in the paper of Aronov-Seidel-Souvaine ([6]).

**Theorem 4.** — *Let  $Q_1$  and  $Q_2$  be two simple  $n$ -gons. Let us choose a homeomorphism  $\psi$  between  $Q_1$  and  $Q_2$ . Let  $T_1$  (resp.  $T_2$ ) be a triangulation of  $Q_1$  (resp.  $Q_2$ ). There exists  $\hat{T}_1$  (resp.  $\hat{T}_2$ ) a triangulation of  $Q_1$  (resp.  $Q_2$ ), finer than  $T_1$  (resp.  $T_2$ ) and a homeomorphism  $\psi'$  such that  $\psi$  and  $\psi'$  are equivalent on  $Q_1$  and  $\psi'(\hat{T}_1) = \hat{T}_2$ .*

In fact, the homeomorphism  $\psi$  between  $\hat{T}_1$  and  $\hat{T}_2$  in Theorem 4 can be chosen so that it is linear on each bounded face of  $\hat{T}_1$ .

**Lemma 8.** — *Let  $T_1$  and  $T_2$  be two triangulations in the plane. If they are homeomorphic there exists a function  $\psi$  linear on each bounded face of  $T_1$ , defined on the union of the bounded faces of  $T_1$  such that  $\psi$  is a homeomorphism between  $T_1$  and  $T_2$ .*

*Proof.* — Let  $T_1$  and  $T_2$  be two homeomorphic triangulations. Let  $\phi$  be a homeomorphism of  $\mathbb{R}^2$  which sends  $T_1$  on  $T_2$ . The action of the affine maps on the set of triangles on the plane is transitive. For any face  $F$  of  $T_1$ , let  $\psi|_F$  be an affine map defined on  $F$  such that  $\psi|_F(\partial F) = \phi(\partial F)$ .

For any triangle  $T$  and any  $x \in \partial T$ , and any affine map  $F$ ,  $F(x)$  depends only on the image by  $F$  of the segment  $[a, b]$  of  $T$  such that  $x \in [a, b]$ . This allows us to glue the affine maps  $(\psi|_F)_F$  and to get the desired  $\psi$ .  $\square$

We can now prove Proposition 10.

*Proof of Proposition 10.* — Let  $\mathbb{G}_1$  and  $\mathbb{G}_2$  be two simple homeomorphic connected finite planar graphs with piecewise affine edges. Let  $\phi$  be a homeomorphism such that  $\phi(\mathbb{G}_1) = \mathbb{G}_2$ . For any face  $F$  of  $\mathbb{G}_1$ ,  $F$  and  $\phi(F)$  are simple polygons. As any polygon can be triangulated, one consequence of Theorem 4 and Lemma 8 is that there exists  $\mathbb{T}_{1,F}$  (resp.  $\mathbb{T}_{2,F}$ ) a triangulation of  $F$  (resp.  $\phi(F)$ ) and  $\psi|_F$

a function defined on  $F$ , linear on each bounded face of  $\mathbb{T}_{1,F}$ , such that  $\psi|_F$  is a homeomorphism between  $\mathbb{T}_{1,F}$  and  $\mathbb{T}_{2,F}$  and such that  $\psi|_F$  and  $\phi$  are equivalent on  $\partial F$ . We define  $\mathbb{T}_1$  (resp.  $\mathbb{T}_2$ ) as the triangulation obtained by taking the union of all the triangulations  $(\mathbb{T}_{1,F})_F$  ( $(\mathbb{T}_{2,F})_F$ ). As in the proof of Lemma 8, we can glue the  $\psi|_F$  together: this gives a function  $\psi|_{F_\infty^c}$  defined on the complementary of the unbounded face  $F_\infty$  of  $\mathbb{G}$ . As  $\mathbb{G}_1$  is simple, the boundary of  $F_\infty$  is a Jordan curve. Thus, according to the discussion we had before Lemma 6, we can extend  $\psi|_{F_\infty^c}$  on  $F_\infty$  as a homeomorphism of the plane  $\psi$  such that  $\psi|_{F_\infty}$  is a diffeomorphism. By construction,  $\psi$  satisfies all the conditions to be a  $\mathbb{G}-\mathbb{G}'$  piecewise diffeomorphism. Besides  $\phi$  and  $\psi$  are equivalent on  $\mathbb{G}_1$  and  $\psi(\mathbb{T}_1) = \mathbb{T}_2$ .  $\square$

**3.5. Universality of  $\mathbb{N}^2$ .** — We have seen after Lemma 1 that the set of piecewise horizontal or vertical paths is not dense in  $P(\mathbb{R}^2)$  for the convergence with fixed endpoints. Thus, when one works with a stochastically continuous random holonomy field, it would seem that it is not possible to restrict oneself to such paths. Actually, we show in the following that, in some sense, we can always inject any graph in the  $\mathbb{N}^2$  graph. This property is crucial and it will allow us to restrict the study of Planar Markovian Holonomy Fields to the  $\mathbb{N}^2$  graph.

**Definition 32.** — Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a finite planar graph. We say that  $\mathbb{G}$  is generic if for any vertex  $v \in \mathbb{V}$ ,  $\#\mathbb{E}_v \leq 4$ , where we remind the reader that  $\mathbb{E}_v$  is the set of edges  $e \in \mathbb{E}$  such that  $e = v$ .

It is worth noticing that any finite planar graph can be approximated by a generic graph. This is illustrated in Figure 6.

**Lemma 9.** — Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a finite planar graph in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$ . Let  $v$  be a vertex of  $\mathbb{G}$ . There exists a sequence of generic graphs  $\mathbb{G}_n = (\mathbb{V}_n, \mathbb{E}_n, \mathbb{F}_n)$  in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$  such that:

1.  $v \in \mathbb{G}_n$ ,
2. there exists an injective function  $\mathcal{L}_n : L_v(\mathbb{G}) \rightarrow L_v(\mathbb{G}_n)$  such that for any loop  $l \in L_v(\mathbb{G})$ ,  $\mathcal{L}_n(l)$  converges with fixed endpoints to  $l$ .

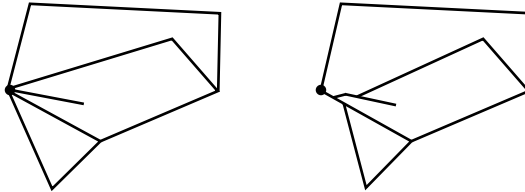


FIGURE 6. An approximation by a generic graph.

The notion of generic graph was defined so that one could send any of such graph in  $\mathbb{N}^2$ .

**Proposition 11.** — *A connected finite planar graph  $\mathbb{G}$  is generic if and only if there exists  $\psi$  a homeomorphism of  $\mathbb{R}^2$  such that  $\psi(\mathbb{G})$  is a subgraph of the  $\mathbb{N}^2$  planar graph.*

*Proof.* — One implication is straightforward. Let us consider a connected generic finite planar graph  $\mathbb{G}$ . We will prove that there exists  $\psi$  a homeomorphism such that  $\psi(\mathbb{G})$  is a subgraph of the  $\mathbb{N}^2$  planar graph. For each  $v \in \mathbb{V}$ , we choose a point  $\tilde{v}$  of  $\mathbb{N}^2$  such that the points  $(\tilde{v})_{v \in \mathbb{G}}$  are all distinct. For each of these points, we choose a subset  $\mathbb{E}_{\tilde{v}}$  of edges in  $\frac{1}{3}\mathbb{N}^2$  going out of  $\tilde{v}$  such that  $\#\mathbb{E}_{\tilde{v}} = \#\mathbb{E}_v$ . We then consider two pre-graphs:

1.  $\mathbb{G}_p$  such that  $\mathbb{E}_p = \{ e_p([0, \frac{1}{3}]), e_p \text{ represents } e \in \mathbb{E} \}$ ,
2.  $\mathbb{G}'_p: \mathbb{E}'_p = \cup_{v \in \mathbb{V}} \mathbb{E}_{\tilde{v}}$ .

Then, we define  $\mathcal{S} : v \mapsto \tilde{v}$ . Because  $\#\mathbb{E}_{\tilde{v}} = \#\mathbb{E}_v$ , and thanks to the shape of the graphs, we can choose a bijection  $\mathcal{E} : \mathbb{E}_p \rightarrow \mathbb{E}'_p$  such that the conditions of Theorem 2 hold. There exists a homeomorphism  $\psi$  such that  $\psi(\mathbb{G}_p) = \mathbb{G}'_p$  and  $\psi$  induces the two bijections  $\mathcal{S}$  and  $\mathcal{E}$ .

We define  $\mathbb{G}' = \psi(\mathbb{G})$  and approximate  $\mathbb{G}'$  in a  $\frac{1}{3^k}\mathbb{N}^2$ : for  $k$  big enough it defines a graph  $\tilde{\mathbb{G}}$  without new vertices: it is then obvious that the assumptions 1. to 4. of Theorem 2 hold for  $\mathbb{G}$  and  $\tilde{\mathbb{G}}$ . Using a dilation we can suppose that  $k = 1$ , thus by Theorem 2 there exists a homeomorphism  $\psi$  which sends  $\mathbb{G}$  to  $\mathbb{G}'$  a subgraph of the  $\mathbb{N}^2$  planar graph.  $\square$

**Corollary 1.** — *Let  $\mathbb{G}$  be a finite generic connected planar graph in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$ . There exists  $\mathbb{G}'$  a subgraph of the  $\mathbb{N}^2$  graph such that the set of  $\mathbb{G} - \mathbb{G}'$  piecewise diffeomorphisms is not empty.*

*Proof.* — Let  $\mathbb{G}$  be a finite generic connected planar graph in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$ . Due to Proposition 11 there exists a subgraph of the  $\mathbb{N}^2$  planar graph,  $\mathbb{G}'$ , such that  $\mathbb{G}$  and  $\mathbb{G}'$  are homeomorphic. By Proposition 10 the set of  $\mathbb{G} - \mathbb{G}'$  piecewise diffeomorphisms is not empty.  $\square$

## 4. Planar Markovian Holonomy Fields

We have now all the ingredients to define planar continuous and discrete Markovian Holonomy Fields: these are families of random holonomy fields on subsets of  $P(\mathbb{R}^2)$  satisfying an area-preserving homeomorphism invariance and a weak independence property. Recall that  $G$  is a compact Lie group.

**4.1. Definitions.** — We first define the strong and weak notions of Planar Markovian Holonomy Fields. Let  $l$  be any simple loop in  $\mathbb{R}^2$ . The notation  $\text{Int}(l)$  will stand for the bounded connected component of  $\mathbb{R}^2 \setminus l$ .



**Definition 33.** — A  $G$ -valued strong Planar Markovian Holonomy Field is the data, for each measure of area  $\text{vol}$  on  $\mathbb{R}^2$ , finite or infinite, of a gauge-invariant holonomy field  $\mathbb{E}_{\text{vol}}$  on  $\mathbb{R}^2$  of weight  $\mathbb{E}_{\text{vol}}(1)$  equal to 1, such that the three following axioms hold:

**P<sub>1</sub>** : Let  $\text{vol}$  and  $\text{vol}'$  be two measures of area on  $\mathbb{R}^2$ . Let  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a homeomorphism which preserves the orientation and which sends  $\text{vol}$  on  $\text{vol}'$  (i.e.  $\text{vol}' = \text{vol} \circ \psi^{-1}$ ). Let  $p_1, \dots, p_n$  be paths in  $\mathbb{R}^2$  such that for any  $i \in \{1, \dots, n\}$ ,  $p'_i = \psi(p_i)$  is a rectifiable path. Then for any continuous function  $f : G^n \rightarrow \mathbb{R}$ ,

$$\mathbb{E}_{\text{vol}} \left[ f(h(p_1), \dots, h(p_n)) \right] = \mathbb{E}_{\text{vol}'} \left[ f(h(p'_1), \dots, h(p'_n)) \right].$$

**P<sub>2</sub>** : For any measure of area  $\text{vol}$  on  $\mathbb{R}^2$ , for any simple loops  $l_1$  and  $l_2$  such that  $\text{Int}(l_1)$  and  $\text{Int}(l_2)$  are disjoint, under  $\mathbb{E}_{\text{vol}}$ , the two families:

$$\left\{ h(p), p \in P(\overline{\text{Int}(l_1)}) \right\} \text{ and } \left\{ h(p), p \in P(\overline{\text{Int}(l_2)}) \right\}$$

are  $\mathcal{I}$ -independent.

**P<sub>3</sub>** : For any measures of area on  $\mathbb{R}^2$ ,  $\text{vol}$  and  $\text{vol}'$ , if  $l$  is a simple loop such that  $\text{vol}$  and  $\text{vol}'$  are equal when restricted to the interior of  $l$ , the following equality holds:

$$\left( \mathbb{E}_{\text{vol}} \right)_{|\mathcal{M}ult(P(\overline{\text{Int}(l)}), G)} = \left( \mathbb{E}_{\text{vol}'} \right)_{|\mathcal{M}ult(P(\overline{\text{Int}(l)}), G)}.$$

As a consequence of **P<sub>1</sub>**, if we are given a diffeomorphism  $\psi$  which preserves the orientation and which sends  $\text{vol}$  on  $\text{vol}'$ , the mapping from  $\mathcal{M}ult(P(\mathbb{R}^2), G)$  to himself induced by  $\psi$  sends  $\mathbb{E}_{\text{vol}'}$  on  $\mathbb{E}_{\text{vol}}$  (i.e.  $\mathbb{E}_{\text{vol}} = \mathbb{E}_{\text{vol}'} \circ \psi^{-1}$ ).

Actually it often turns out to be easier to check that a family of random holonomy fields is a weak Planar Markovian Holonomy Field.

**Definition 34.** — A  $G$ -valued weak Planar Markovian Holonomy Field is the data, for each measure of area  $\text{vol}$  on  $\mathbb{R}^2$ , finite or infinite, of a gauge-invariant holonomy field  $\mathbb{E}_{\text{vol}}$  on  $\text{Aff}(\mathbb{R}^2)$  of weight  $\mathbb{E}_{\text{vol}}(1)$  equal to 1, such that the three following axioms hold:

**wP<sub>1</sub>** : Let  $\text{vol}$  and  $\text{vol}'$  be two measures of area on  $\mathbb{R}^2$ . Let  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a diffeomorphism at infinity which preserves the orientation and which sends  $\text{vol}$  on  $\text{vol}'$  (i.e.  $\text{vol}' = \text{vol} \circ \psi^{-1}$ ). Let  $p_1, \dots, p_n$  be paths in  $\text{Aff}(\mathbb{R}^2)$  such that for any  $i \in \{1, \dots, n\}$ ,  $p'_i = \psi(p_i)$  is in  $\text{Aff}(\mathbb{R}^2)$ . Then for any continuous function  $f : G^n \rightarrow \mathbb{R}$ ,

$$\mathbb{E}_{\text{vol}} \left[ f(h(p_1), \dots, h(p_n)) \right] = \mathbb{E}_{\text{vol}'} \left[ f(h(p'_1), \dots, h(p'_n)) \right].$$

**wP<sub>2</sub>** : *For any measure of area on  $\mathbb{R}^2$ ,  $vol$ , for any simple loops  $l_1$  and  $l_2$  such that  $\text{Int}(l_1)$  and  $\text{Int}(l_2)$  are disjoint, under  $\mathbb{E}_{vol}$ , the two families:*

$$\left\{ h(p), p \in \text{Aff}(\mathbb{R}^2) \cap P(\overline{\text{Int}(l_1)}) \right\} \text{ and } \left\{ h(p), p \in \text{Aff}(\mathbb{R}^2) \cap P(\overline{\text{Int}(l_2)}) \right\}$$

*are independent.*

**wP<sub>3</sub>** : *For any measures of area on  $\mathbb{R}^2$ ,  $vol$  and  $vol'$ , if  $l$  is a simple loop such that  $vol$  and  $vol'$  are equal when restricted to the interior of  $l$ , the following equality holds:*

$$\left( \mathbb{E}_{vol} \right)_{|\mathcal{M}ult(\text{Aff}(\overline{\text{Int}(l)}), G)} = \left( \mathbb{E}_{vol'} \right)_{|\mathcal{M}ult(\text{Aff}(\overline{\text{Int}(l)}), G)}.$$

It can seem strange that we replaced the  $\mathcal{I}$ -independence by the usual independence in **wP<sub>2</sub>**, but this was precisely the point of Remark 6: in the setting of **wP<sub>2</sub>**, the endpoints set of any set of loops in  $\overline{\text{Int}(l_1)}$  and of any set of loops in  $\overline{\text{Int}(l_2)}$  are disjoint, thus the  $\mathcal{I}$ -independence notion is the same as the usual independence. As a consequence, any strong Planar Markovian Holonomy Field defines, by restriction, a weak Planar Markovian Holonomy Field. We will see later that the two notions, when we restrict them to stochastically continuous objects, are equivalent.

By  $G$ -valued Planar Markovian Holonomy Fields, we will denote the family of strong or weak  $G$ -valued Planar Markovian Holonomy Fields.

**Definition 35.** — *Let  $(\mathbb{E}_{vol})_{vol}$  be a  $G$ -valued Planar Markovian Holonomy Field. We say that  $(\mathbb{E}_{vol})_{vol}$  is stochastically continuous if, for any measure of area  $vol$  on  $\mathbb{R}^2$ ,  $\mathbb{E}_{vol}$  is stochastically continuous.*

A discrete counterpart exists for Planar Markovian Holonomy Fields.

**Definition 36.** — *A  $G$ -valued strong discrete Planar Markovian Holonomy Field is the data, for each measure of area  $vol$ , for each finite planar graph  $\mathbb{G}$ , of a gauge-invariant holonomy field  $\mathbb{E}_{vol}^{\mathbb{G}}$  on  $P(\mathbb{G})$  of weight  $\mathbb{E}_{vol}^{\mathbb{G}}(1)$  equal to 1, such that the four following axioms hold:*

**DP<sub>1</sub>** : *Let  $vol$  and  $vol'$  be two measures of area on  $\mathbb{R}^2$ , let  $\mathbb{G}$  and  $\mathbb{G}'$  be two finite planar graphs. Let  $\psi$  be a homeomorphism which preserves the orientation, satisfies  $\psi(\mathbb{G}) = \mathbb{G}'$ , and such that for any  $F \in \mathbb{F}^b$ ,  $vol(F) = vol'(\psi(F))$ . The mapping from  $\text{Mult}(P(\mathbb{G}'), G)$  to  $\text{Mult}(P(\mathbb{G}), G)$  induced by  $\psi$  satisfies:*

$$\mathbb{E}_{vol'}^{\mathbb{G}'} \circ \psi^{-1} = \mathbb{E}_{vol}^{\mathbb{G}}.$$

**DP<sub>2</sub>** : *For any measure of area  $vol$  on  $\mathbb{R}^2$ , for any finite graph  $\mathbb{G}$ , for any simple loops  $l_1$  and  $l_2$  in  $P(\mathbb{G})$ , such that  $\text{Int}(l_1) \cap \text{Int}(l_2) = \emptyset$ , under  $\mathbb{E}_{vol}^{\mathbb{G}}$ , the two families:*

$$\left\{ h(p), p \in P(\mathbb{G}) \cap P(\overline{\text{Int}(l_1)}) \right\} \text{ and } \left\{ h(p), p \in P(\mathbb{G}) \cap P(\overline{\text{Int}(l_2)}) \right\}$$

*are  $\mathcal{I}$ -independent.*

**DP<sub>3</sub>** : For any measures of area on  $\mathbb{R}^2$ ,  $vol$  and  $vol'$ , if  $l$  is a simple loop such that  $vol$  and  $vol'$  are equal when restricted to the interior of  $l$ , if  $\mathbb{G}$  is included in  $\overline{\text{Int}(l)}$ , then the following equality holds:

$$\mathbb{E}_{vol}^{\mathbb{G}} = \mathbb{E}_{vol'}^{\mathbb{G}}.$$

**DP<sub>4</sub>** : For any finite planar graphs  $\mathbb{G}_1$  and  $\mathbb{G}_2$ , such that  $\mathbb{G}_1 \preceq \mathbb{G}_2$ :

$$\mathbb{E}_{vol}^{\mathbb{G}_2} \circ \rho_{P(\mathbb{G}_1), P(\mathbb{G}_2)}^{-1} = \mathbb{E}_{vol}^{\mathbb{G}_1},$$

where we remind the reader that

$$\rho_{P(\mathbb{G}_1), P(\mathbb{G}_2)} : \text{Mult}(P(\mathbb{G}_2), G) \rightarrow \text{Mult}(P(\mathbb{G}_1), G)$$

is the restriction map.

**Definition 37.** — A  $G$ -valued weak discrete Planar Markovian Holonomy Field is the data, for each measure of area  $vol$ , for each finite graph  $\mathbb{G}$  in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$ , of a gauge-invariant holonomy field  $\mathbb{E}_{vol}^{\mathbb{G}}$  of weight  $\mathbb{E}_{vol}^{\mathbb{G}}(1)$  equal to 1, such that the four following axioms hold:

**wDP<sub>1</sub>** : Let  $vol$  and  $vol'$  be two measures of area on  $\mathbb{R}^2$ , let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  and  $\mathbb{G}'$  be two simple connected finite planar graphs in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$ . Let  $\psi$  be a  $\mathbb{G} - \mathbb{G}'$  piecewise diffeomorphism which preserves the orientation. Suppose that for any bounded face  $F$  of  $\mathbb{G}$ ,  $vol(F) = vol'(\psi(F))$ . Then the mapping from  $\text{Mult}(P(\mathbb{G}'), G)$  to  $\text{Mult}(P(\mathbb{G}), G)$  induced by  $\psi$  satisfies:

$$\mathbb{E}_{vol'}^{\mathbb{G}'} \circ \psi^{-1} = \mathbb{E}_{vol}^{\mathbb{G}}.$$

**wDP<sub>2</sub>** : For any measure of area  $vol$  on  $\mathbb{R}^2$ , for any finite graph  $\mathbb{G}$  in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$ , for any simple loops  $l_1$  and  $l_2$  in  $P(\mathbb{G})$ , such that  $\overline{\text{Int}(l_1)} \cap \overline{\text{Int}(l_2)} = \emptyset$ , under  $\mathbb{E}_{vol}^{\mathbb{G}}$ , the two families:

$$\left\{ h(p), p \in P(\mathbb{G}) \cap P(\overline{\text{Int}(l_1)}) \right\} \text{ and } \left\{ h(p), p \in P(\mathbb{G}) \cap P(\overline{\text{Int}(l_2)}) \right\}$$

are independent.

**wDP<sub>3</sub>** : For any measures of area on  $\mathbb{R}^2$ ,  $vol$  and  $vol'$ , if  $l$  is a simple loop such that  $vol$  and  $vol'$  are equal when restricted to the interior of  $l$ , if  $\mathbb{G}$  is included in the closure of the interior of  $l$ , then the following equality holds:

$$\mathbb{E}_{vol}^{\mathbb{G}} = \mathbb{E}_{vol'}^{\mathbb{G}}.$$

**wDP<sub>4</sub>** : For any finite planar graphs  $\mathbb{G}_1$  and  $\mathbb{G}_2$  in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$ , such that  $\mathbb{G}_1 \preceq \mathbb{G}_2$ :

$$\mathbb{E}_{vol}^{\mathbb{G}_2} \circ \rho_{P(\mathbb{G}_1), P(\mathbb{G}_2)}^{-1} = \mathbb{E}_{vol}^{\mathbb{G}_1},$$

where again  $\rho_{P(\mathbb{G}_1), P(\mathbb{G}_2)} : \text{Mult}(P(\mathbb{G}_2), G) \rightarrow \text{Mult}(P(\mathbb{G}_1), G)$  is the restriction map.

In the definition of Planar Markovian Holonomy Fields, we introduced the axioms  $\mathbf{P}_3$ ,  $\mathbf{wP}_3$  in order to allow for the measure of area  $vol$  to have an infinite weight, that is  $vol(\mathbb{R}^2) \leq \infty$ . In case we had only considered measures of area  $vol$  such that  $vol(\mathbb{R}^2) = \infty$ , using the tools developed in Section 5, axioms  $\mathbf{P}_3$ ,  $\mathbf{wP}_3$  could be deduced from the others. The axioms  $\mathbf{DP}_3$  and  $\mathbf{wDP}_3$  actually can be directly deduced respectively from  $\mathbf{DP}_1$  and  $\mathbf{wDP}_1$  by considering the identity function of the plane. Yet, in order to have a similar formulation for continuous and discrete objects we preferred to keep them in the definition.

As for the continuous objects, any  $G$ -valued strong discrete Planar Markovian Holonomy Field defines, by restriction, a weak discrete Planar Markovian Holonomy Field. By  $G$ -valued discrete Planar Markovian Holonomy Fields, we will denote the family of  $G$ -valued strong or weak discrete Planar Markovian Holonomy Fields.

In any assertion about  $G$ -valued discrete Planar Markovian Holonomy Field, the reader will have to understand that, in the case we are working with a weak discrete Planar Markovian Holonomy Field, all the graphs must be in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$ .

**Remark 10.** — Let  $(\mathbb{E}_{vol}^G)_{\mathbb{G}, vol}$  be a  $G$ -valued discrete Planar Markovian Holonomy Field. As an application of Proposition 2, axiom  $\mathbf{DP}_4$  or  $\mathbf{wDP}_4$  allows us to define for any measure of area and any possibly infinite planar graph  $\mathbb{G}$ , a unique gauge-invariant holonomy field  $\mathbb{E}_{vol}^G$  on  $P(\mathbb{G})$  of weight  $\mathbb{E}_{vol}^G(1)$  equal to 1, such that, for any finite graph  $\mathbb{G}_f \preceq \mathbb{G}$ ,  $\mathbb{E}_{vol}^G \circ \rho_{P(\mathbb{G}_f), P(\mathbb{G})}^{-1} = \mathbb{E}_{vol}^{G_f}$ .

Besides, let  $\gamma$  be a Riemannian metric on  $\mathbb{R}^2$ . As a consequence of Lemma 7, we have seen that the set of finite planar graphs  $\mathcal{G}(\text{Aff}_\gamma)$  was directed. Thus:

$$\left\{ \left( \text{Mult}(P(\mathbb{G}), G), \mathcal{B}, \mathbb{E}_{vol}^G \right)_{\mathbb{G} \in \mathcal{G}(\text{Aff}_\gamma)}, \left( \rho_{P(\mathbb{G}), P(\mathbb{G}')} \right)_{\mathbb{G}, \mathbb{G}' \in \mathcal{G}(\text{Aff}_\gamma(\mathbb{R}^2)), \mathbb{G} \preceq \mathbb{G}'} \right\}$$

is a projective family. Hence, for any measure of area  $vol$  and any Riemannian metric  $\gamma$ , we can define a unique gauge-invariant holonomy field on  $\text{Aff}_\gamma(\mathbb{R}^2)$  of weight equal to 1, which we denote by  $\mathbb{E}_{vol, \gamma}$ , such that for any finite planar graph  $\mathbb{G}_f \in \mathcal{G}(\text{Aff}_\gamma(\mathbb{R}^2))$ ,  $\mathbb{E}_{vol, \gamma} \circ \rho_{P(\mathbb{G}_f), \text{Aff}_\gamma(\mathbb{R}^2)}^{-1} = \mathbb{E}_{vol}^{G_f}$ .

For all the objects we have just defined (weak/strong, discrete or not, Planar Markovian Holonomy Fields), we can define an oriented object. For this, we modify  $\mathbf{P}_1$ ,  $\mathbf{wP}_1$ ,  $\mathbf{DP}_1$  and  $\mathbf{wDP}_1$  so that  $\psi$  is now an orientation preserving homeomorphism. It defines oriented weak/strong, discrete or not, Planar Markovian Holonomy Fields.

The notion of area-dependent continuity and locally stochastically  $\frac{1}{2}$ -Hölder continuity for  $G$ -valued discrete Planar Markovian Holonomy Field is the exact replica of Definition 3.2.8 in [21] for discrete Markovian Holonomy Fields.

**Definition 38.** — Let  $(\mathbb{E}_{vol}^G)_{\mathbb{G}, vol}$  be a family of  $G$ -valued holonomy fields such that for any measure of area  $vol$  and any finite planar graph  $\mathbb{G}$ ,  $\mathbb{E}_{vol}^G$  is an holonomy field on  $\mathbb{G}$ . It is locally stochastically  $\frac{1}{2}$ -Hölder continuous if for any measure of area  $vol$  on  $\mathbb{R}^2$ ,  $(\mathbb{E}_{vol}^G)_{\mathbb{G}}$  is a uniformly locally  $\frac{1}{2}$ -Hölder continuous family of random holonomy fields.

It is continuously area-dependent if, for any sequence of finite planar graphs  $\mathbb{G}_n$  which are the images of a common graph  $\mathbb{G}$  by a sequence of homeomorphisms  $\psi_n$  ( $\psi_n(\mathbb{G}) = \mathbb{G}_n$ ) and such that  $vol(\psi_n(F))$  tends to  $vol(F)$  as  $n$  tends to infinity for any bounded face  $F$  of  $\mathbb{G}$ , the following convergence holds:

$$\mathbb{E}_{vol}^{\mathbb{G}_n} \circ \psi_n^{-1} \xrightarrow[n \rightarrow \infty]{} \mathbb{E}_{vol}^G,$$

where we denote by  $\psi_n$  the induced map from  $\text{Mult}(P(\mathbb{G}_n), G)$  to  $\text{Mult}(P(\mathbb{G}), G)$ .

It is regular if it is locally stochastically  $\frac{1}{2}$ -Hölder continuous and continuously area-dependent.

We define also the stochastic continuity in law.

**Definition 39.** — Let  $(\mathbb{E}_{vol}^G)_{\mathbb{G}, vol}$  be a family of  $G$ -valued holonomy fields such that for any measure of area  $vol$  and any finite planar graph  $\mathbb{G}$ ,  $\mathbb{E}_{vol}^G$  is an holonomy field on  $\mathbb{G}$ . We say that  $(\mathbb{E}_{vol}^G)_{\mathbb{G}, vol}$  is stochastically continuous in law if the following condition holds for any measure of area  $vol$ :

for any integer  $m$ , any finite graph  $\mathbb{G}$ , any sequence of finite graphs  $(\mathbb{G}_n)_{n \geq 0}$ , and any sequence of  $m$ -tuples of loops in  $\mathbb{G}_n$ ,  $(l_k^n)_{k=1}^m$ , if there exists a  $m$ -tuple of loops in  $\mathbb{G}$ ,  $(l_k)_{k=1}^m$  such that for any  $i \in \{1, \dots, k\}$ ,  $l_i^n$  converges with fixed endpoints to  $l_i$  when  $n$  goes to infinity, then the law of  $(h(l_k^n))_{k=1}^m$  under  $\mathbb{E}_{vol}^{\mathbb{G}_n}$  converges to the law of  $(h(l_k))_{k=1}^m$  under  $\mathbb{E}_{vol}^G$  when  $n$  tends to infinity.

We have to be careful that  $\mathbf{DP}_1$  is not just a discrete version of  $\mathbf{P}_1$  since in  $\mathbf{DP}_1$  we do not require that  $vol'$  is the image of  $vol$  by  $\psi$ . Thus, it is not obvious yet that any Planar Markovian Holonomy Field, when restricted to graphs, defines a discrete Planar Markovian Holonomy Field. For now, we define the notion of constructibility but later, in Theorem 6 and Theorem 21, we will show that, under some regularity conditions, any Planar Markovian Holonomy Field is constructible.

**Definition 40.** — Let  $(\mathbb{E}_{vol})_{vol}$  be a weak (resp. strong)  $G$ -valued Planar Markovian Holonomy Field. It is constructible if the family of measures:

$$\left( (\mathbb{E}_{vol})_{|\text{Mult}(P(\mathbb{G}), G)} \right)_{\mathbb{G}, vol}$$

is a weak (resp. strong)  $G$ -valued discrete Planar Markovian Holonomy Field.

**Remark 11.** — If  $(\mathbb{E}_{vol})_{vol}$  is a constructible stochastically continuous  $G$ -valued Planar Markovian Holonomy Field, its restriction to graphs defines a stochastically continuous in law discrete  $G$ -valued Planar Markovian Holonomy Field.

We have seen, in Remark 10, that, given a discrete Planar Markovian Holonomy Field  $(\mathbb{E}_{vol}^{\mathbb{G}})_{vol, \mathbb{G}}$ , we could define a family of probability measures  $(\mathbb{E}_{vol, \gamma})_{vol, \gamma}$ . If the discrete Planar Markovian Holonomy Field is locally stochastically  $\frac{1}{2}$ -Hölder continuous, so is  $\mathbb{E}_{vol, \gamma}$  for any Riemannian metric  $\gamma$  and any measure of area  $vol$ . By Theorem 1, one can extend  $\mathbb{E}_{vol, \gamma}$  as a stochastically continuous random holonomy field on  $\mathbb{R}^2$ , called  $\mathbb{E}_{vol}^{(\gamma)}$  and then one can prove (Proposition 3.4.1 in [21]) that  $\mathbb{E}_{vol}^{(\gamma)}$  only depends on  $vol$  and not on  $\gamma$  if the discrete Markovian Holonomy Field is continuously area-dependent. We have thus defined  $(\mathbb{E}_{vol})_{vol}$  a family of random holonomy fields on  $\mathbb{R}^2$ .

It is easy to check that  $(\mathbb{E}_{vol})_{vol}$  is a stochastically continuous strong (resp. weak) Planar Markovian Holonomy Field if  $(\mathbb{E}_{vol}^{\mathbb{G}})_{vol, \mathbb{G}}$  is a strong (resp. discrete) Planar Markovian Holonomy Field. This leads us to the following theorem which is a modification of Theorem 3.2.9 in [21].

**Theorem 5.** — *Let  $(\mathbb{E}_{vol}^{\mathbb{G}})_{vol, \mathbb{G}}$  be a strong (resp. weak)  $G$ -valued discrete Planar Markovian Holonomy Field. If it is continuously area-dependent and locally stochastically  $\frac{1}{2}$ -Hölder continuous then there exists a unique stochastically continuous strong (resp. weak)  $G$ -valued Planar Markovian Holonomy Field  $(\mathbb{E}_{vol})_{vol}$  such that, for any finite graph  $\mathbb{G}$  and any measure of area  $vol$ ,  $(\mathbb{E}_{vol}^{\mathbb{G}})_{vol, \mathbb{G}}$  is the restriction to  $\text{Mult}(P(\mathbb{G}), G)$  of  $\mathbb{E}_{vol}$ :*

$$\mathbb{E}_{vol} \circ \rho_{P(\mathbb{G}), P(M)}^{-1} = \mathbb{E}_{vol}^{\mathbb{G}}$$

This theorem, with the help of Remark 11 shows that:

**Corollary 2.** — *Any  $G$ -valued discrete Planar Markovian Holonomy Field which is continuously area-dependent and locally stochastically  $\frac{1}{2}$ -Hölder continuous is stochastically continuous in law.*

In the rest of the paper, we will almost only work with stochastically continuous in law discrete Markovian Holonomy Fields.

*4.1.1. Example: the index field.* — In this section, we define one of the simplest Planar Markovian Holonomy Field.

For any parametrized loop  $l$ , for any  $x$  in  $\mathbb{R}^2 \setminus l([0, 1])$ , the index of  $l$  with respect to  $x$  is defined as the integer:

$$\mathbf{n}_l(x) = \frac{1}{2i\pi} \oint_l \frac{dz}{z - x}.$$

Actually, one needs to approximate uniformly  $l$  by piecewise smooth loops and take the limit. Given the Banchoff-Pohl inequality proved in [28], the index of any rectifiable loop is square-integrable, thus integrable since it takes values in  $\mathbb{N}$ . Since any loop is bounded,  $\mathbf{n}_l$  vanishes outside a ball, so that  $\mathbf{n}_l$  is also integrable

against any measure of area  $vol$ . Besides if  $l_1$  and  $l_2$  are based at the same point, using the additivity of the curve integral, we get:

$$(9) \quad \mathbf{n}_{l_1 l_2} = \mathbf{n}_{l_1} + \mathbf{n}_{l_2}.$$

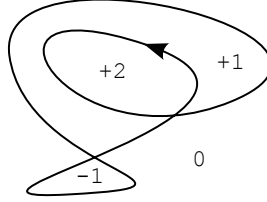


FIGURE 7. The index of a curve.

Actually, if one does not want to use the definition with the curve integral, one can define the index field by first constructing the holonomy field, on  $\mathbf{Aff}(\mathbb{R}^2)$ , which sends  $l$  on  $\mathbf{n}_l$ . In this setting, the index can be defined as a combinatorial object. Using the  $L^2$  norm on the set of  $\mathbb{N}$ -valued functions on the plane, this discrete Planar Markovian Holonomy Field is locally stochastically  $\frac{1}{2}$ -Hölder continuous: by Theorem 1, we can extend it in order to get a stochastically continuous planar holonomy field on the plane.

Let  $D$  be an element of the Lie algebra  $\mathfrak{g}$  of  $G$ . We can now define the index field driven by  $D$ .

**Definition 41.** — *The index field driven by  $D$  is the only Planar Markovian Holonomy Field  $(\mathbb{E}_{vol})_{vol}$  such that for any measure of area  $vol$ , any loops  $l_1, \dots, l_n$  based at the same point, and any continuous function  $f$  from  $G^n$  to  $\mathbb{R}$  invariant by diagonal conjugation, we have:*

$$\mathbb{E}_{vol} \left[ f(h(l_1), \dots, h(l_n)) \right] = f \left( e^{D \int_{\mathbb{R}^2} \mathbf{n}_{l_1}(x) vol(dx)}, \dots, e^{D \int_{\mathbb{R}^2} \mathbf{n}_{l_n}(x) vol(dx)} \right).$$

Such a Planar Markovian Holonomy Field exists. Indeed, let us define for any  $D$  in  $\mathfrak{g}$  and any measure of area  $vol$ :

$$\begin{aligned} \text{ind}_{vol}^D : P(\mathbb{R}^2) &\longrightarrow G \\ p &\mapsto \begin{cases} e^{D \int_{\mathbb{R}^2} \mathbf{n}_p(x) vol(dx)}, & \text{if } p \in L(\mathbb{R}^2), \\ 1, & \text{otherwise.} \end{cases} \end{aligned}$$

The equation (9) implies that  $\text{ind}_D \in \text{Mult}(P(\mathbb{R}^2), G)$ . The index field driven by  $D$  is then given by:

$$\forall vol, \quad \mathbb{E}_{vol} = \left( \left( \delta_{\text{ind}_{vol}^D} \right)_{|\mathcal{I}} \right)^{\widehat{\quad}}$$

where we remind the reader that  $\hat{\mu}$  was defined in Proposition 3. The reader can check that all the axioms are satisfied by this Planar Markovian Holonomy Field. An interesting fact is that it is stochastically continuous and constructible.

This Planar Markovian Holonomy Field plays the role of a drift in the theory of holonomy field. Indeed, we can drift any holonomy field on the plane by any index field driven by a central element of the Lie algebra.

**Lemma 10.** — *Let  $\mu$  be a holonomy field on the plane. Let  $vol$  be a measure of area. Let  $D$  be an element of the center of  $\mathfrak{g}$ . There exists a Planar Markovian Holonomy Field  $\mu^{D,vol}$  such that for any loops  $l_1, \dots, l_n$  based at the same point, and any continuous function  $f$  from  $G^n$  to  $\mathbb{R}$  invariant by diagonal conjugation, we have:*

$$(10) \quad \mu^{D,vol} \left[ f(h(l_1), \dots, h(l_n)) \right] = \mu \left[ f \left( e^{D \int_{\mathbb{R}^2} \mathfrak{n}_{l_1}(x) vol(dx)} h(l_1), \dots, e^{D \int_{\mathbb{R}^2} \mathfrak{n}_{l_n}(x) vol(dx)} h(l_n) \right) \right].$$

*Any regularity which holds for  $\mu$  holds for  $\mu^{D,vol}$ . Besides, if  $(\mathbb{E}_{vol})_{vol}$  is a Planar Markovian Holonomy Field, so is  $(\mathbb{E}_{vol}^{D,vol})_{vol}$ .*

**4.2. Restriction and extension of the group.** — We have seen in Section 2.2.5 how to restrict and extend the group on which a gauge-invariant holonomy field is defined. We will apply this to discrete Planar Markovian Holonomy Fields. Actually, Propositions 12 and 13 can be extended to Planar Markovian Holonomy Fields. Let  $H$  be a closed subgroup of  $G$ .

*4.2.1. Extension.* — Let  $(\mathbb{E}_{vol}^G)_{vol,G}$  be a  $H$ -valued discrete Planar Markovian Holonomy Field. Recall the notation  $\mu \circ \hat{i}_P^{-1}$  defined in Notation 3. Following Section 2.2.5, in the gauge-invariant setting, for any finite planar graph  $\mathbb{G}$  and any measure of area  $vol$ , we can see  $\mathbb{E}_{vol}^G$  as a  $G$ -valued gauge-invariant holonomy field on  $P(\mathbb{G})$  by studying  $\mathbb{E}_{vol}^G \circ \hat{i}_{P(\mathbb{G})}^{-1}$ . It is not difficult to verify next proposition.

**Proposition 12.** — *Let  $(\mathbb{E}_{vol}^G)_{vol,G}$  be a  $H$ -valued discrete Planar Markovian Holonomy Field. The family  $(\mathbb{E}_{vol}^G \circ \hat{i}_{P(\mathbb{G})}^{-1})_{vol,G}$  is a  $G$ -valued discrete Planar Markovian Holonomy Field. The regularities are the same for the  $H$ -valued and the  $G$ -valued random holonomy fields.*

*4.2.2. Restriction.* —

**Proposition 13.** — *Let  $(\mathbb{E}_{vol}^G)_{vol,G}$  be a  $G$ -valued stochastically continuous discrete Planar Markovian Holonomy Field. Let us suppose that for any finite planar graph  $\mathbb{G}$ , any vertex  $v$  of  $\mathbb{G}$ , any measure of area  $vol$ , and any simple loop  $l \in L_v(\mathbb{G})$ ,*

$$\mathbb{E}_{vol}^G \text{ a.s., } h(l) \in H,$$



then there exists a  $H$ -valued stochastically continuous discrete Planar Markovian Holonomy Field  $(\tilde{\mathbb{E}}_{vol}^{\mathbb{G}})_{vol, \mathbb{G}}$  such that:

$$\mathbb{E}_{vol}^{\mathbb{G}} = \tilde{\mathbb{E}}_{vol}^{\mathbb{G}} \circ \hat{i}_{P(\mathbb{G})}^{-1},$$

for any finite graph  $\mathbb{G}$  and any measure of area  $vol$ .

The proof of the restriction property is more difficult than one might think because of Remark 8: indeed when one restricts the group on which a random holonomy field is defined, there is no uniqueness. In fact, one can show that **the natural choice we made in Proposition 8 does not allow one to define a  $H$ -valued Markovian Holonomy Field**. Indeed, for any connected finite planar graph  $\mathbb{G}$  and any measure of area  $vol$ , the natural restriction would be defined by:

$$\tilde{\mathbb{E}}_{vol}^{\mathbb{G}} = \left( \widehat{\left( (\mathbb{E}_{vol}^{\mathbb{G}})_{|L_v(\mathbb{G})} \circ \iota \right)_{|\mathcal{I}_H}} \right),$$

with  $v$  any vertex of  $\mathbb{G}$ ,  $\iota : \text{Mult}(L_v(\mathbb{G}), H) \rightarrow \text{Mult}(P, H)$  any map given by Lemma 3,  $\mathcal{I}_H$  is the  $H$ -invariant  $\sigma$ -field and  $\widehat{\phantom{x}}$  being the gauge-invariant extension (where the gauge group is now built on  $H$ ) given by Proposition 3.

Let  $l$  and  $l'$  be two simple loops in  $P(\mathbb{G})$ , with  $\underline{l} = v$ , such that  $\overline{\text{Int}(l)} \cap \overline{\text{Int}(l')} = \emptyset$  as shown in the Figure 8.

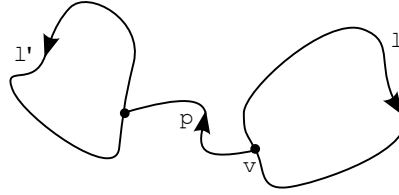


FIGURE 8. Two simple loops  $l, l'$  with a path  $p$  joining  $\underline{l}$  to  $\underline{l}'$ .

If the family of measures  $(\tilde{\mathbb{E}}_{vol}^{\mathbb{G}})_{vol, \mathbb{G}}$  just defined above was a discrete Planar Markovian Holonomy Field, then  $h(l)$  and  $h(l')$  would be either independent or  $\mathcal{I}_H$ -independent when working with the weak or strong version. But if  $p$  is the path from  $v$  to  $\underline{l}'$  used to define  $\iota$  and if  $f_1, f_2$  are two continuous functions on  $H$  invariant by conjugation  $H$ , we have:

$$\tilde{\mathbb{E}}_{vol}^{\mathbb{G}} \left( f_1(h(l)) f_2(h(l')) \right) = \mathbb{E}_{vol}^{\mathbb{G}} \left[ f_1(h(l)) f_2(h(pl'p^{-1})) \right].$$

In the r.h.s. appear the two loops  $l$  and  $pl'p^{-1}$  which are not of null intersection (as they share at least  $v$ ), and only appear functions invariant by conjugation by  $H$ . This does not allow us to split the expectation into two parts neither in the setting of weak nor strong discrete Planar Markovian Holonomy Fields.

The discussion we just had is only a heuristic, yet, page 87, we will explain how to construct an example of  $G$ -valued stochastically continuous discrete Planar

Markovian Holonomy Field  $(\mathbb{E}_{vol}^{\mathbb{G}})_{vol, \mathbb{G}}$  such that for any simple loop  $l$ ,  $h(l) \in H$  a.s., with  $H$  a subgroup of  $G$ , and for which we will be able to prove that the family  $(\tilde{\mathbb{E}}_{vol}^{\mathbb{G}})_{vol, \mathbb{G}}$  constructed as above is not a discrete Planar Markovian Holonomy Field. The proof of Proposition 13, relies heavily on a theorem which will prove later, namely Theorem 18. Thus, the proof will be given page 88.

### 5. Weak constructibility and locality

In this section, we are going to prove that any weak continuous Planar Markovian Holonomy Fields is constructible. For this, we will need a proposition which follows directly from an important theorem of Moser and Dacorogna in [11]. We denote by  $\text{Leb}$  the Lesbegue measure on  $\mathbb{R}^2$ .

**Proposition 14.** — *Let  $Q$  be a simple  $n$ -gon in  $\mathbb{R}^2$ . Let  $f$  and  $g$  be two strictly positive functions on  $Q$  which are in  $C^1(Q) \cap C^0(\overline{Q})$ . Suppose that:*

$$\int_Q f d\text{Leb} = \int_Q g d\text{Leb}.$$

*Then there exists  $\phi \in \text{Dif}^1(Q) \cap \text{Dif}^0(\overline{Q})$ , a homeomorphism of  $\overline{Q}$  which restricts to a diffeomorphism of  $Q$ , such that:*

$$g \cdot \text{Leb}|_Q = (f \cdot \text{Leb}|_Q) \circ \phi^{-1}.$$

*and  $\phi(x) = x$  for any  $x \in \partial Q$ .*

*Proof.* — In [11], page 15 the authors define for any integer  $k$ , a property  $(H_k)$  for open subsets of  $\mathbb{R}^n$  for any integer  $k \geq 1$ . They show in Theorem 7 of the same paper, that for any integer  $k \geq 1$ , any open domain  $\Omega$  which satisfies  $(H_k)$ , any positive functions  $f$  and  $g$  in  $C^k(\Omega)$  with  $f + \frac{1}{f}$  and  $g + \frac{1}{g}$  bounded and satisfying:

$$\int_{\Omega} f d\text{Leb} = \int_{\Omega} g d\text{Leb},$$

there exists  $\phi \in \text{Dif}^1(\Omega) \cap \text{Dif}^0(\overline{\Omega})$  with  $\phi(x) = x$  on  $\partial\Omega$  such that:

$$g \cdot d\text{Leb}|_{\Omega} = (f \cdot d\text{Leb}|_{\Omega}) \circ \phi^{-1}.$$

Besides, Proposition A.2 of the same paper asserts that any domain with Lipschitz boundary satisfies  $(H_k)$  for every  $k \geq 1$ . The case of a  $n$ -gon follows from this discussion.  $\square$

**Theorem 6.** — *Any  $G$ -valued weak Planar Markovian Holonomy Field is constructible.*

*Proof.* — Let  $(\mathbb{E}_{vol})_{vol}$  be a  $G$ -valued weak Planar Markovian Holonomy Field. Let  $(\mathbb{E}_{vol}^{\mathbb{G}})_{vol, \mathbb{G}}$  be the family of random holonomy fields that we get by restricting  $(\mathbb{E}_{vol})_{vol}$  on each  $P(\mathbb{G})$ . As explained before Definition 40, we only have to check

that the axiom **wDP**<sub>1</sub> is satisfied by  $(\mathbb{E}_{vol}^{\mathbb{G}})_{vol, \mathbb{G}}$ . Let  $vol$  and  $vol'$  be two measures of area on  $\mathbb{R}^2$ . Consider  $\mathbb{G}$  and  $\mathbb{G}'$  two simple connected finite planar graphs in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$ . Let  $\psi$  be a  $\mathbb{G} - \mathbb{G}'$  piecewise diffeomorphism. Let us suppose that for any bounded face  $F$  of  $\mathbb{G}$ ,  $vol(F) = vol'(\psi(F))$ .

Let  $F'_\infty$  be the unbounded face of  $\mathbb{G}'$ . Our goal is to construct a diffeomorphism at infinity  $\Phi$  on  $\mathbb{R}^2$  such that:

$$(11) \quad vol'_{|(F'_\infty)^c} = (vol \circ (\Phi \circ \psi)^{-1})_{|(F'_\infty)^c},$$

$$(12) \quad \Phi|_{\mathbb{G}'} = Id|_{\mathbb{G}'}$$

Let us suppose that we managed to do so. As  $\mathbb{G}'$  is a simple graph, the boundary of  $F'_\infty$  is a simple loop. Applying the axiom **wP**<sub>3</sub> and using the condition (11):

$$\mathbb{E}_{vol'}^{\mathbb{G}'} = \mathbb{E}_{vol \circ (\Phi \circ \psi)^{-1}}^{\mathbb{G}'}$$

Yet by condition (12),  $\mathbb{G}' = \psi(\mathbb{G}) = \Phi \circ \psi(\mathbb{G})$ . Thus, as an application of axiom **wP**<sub>1</sub>, we get:

$$\mathbb{E}_{vol'}^{\mathbb{G}'} = \mathbb{E}_{vol \circ (\Phi \circ \psi)^{-1}}^{\mathbb{G}'} = \mathbb{E}_{vol \circ (\Psi \circ \phi)^{-1}}^{\Phi \circ \psi(\mathbb{G})} = \mathbb{E}_{vol}^{\mathbb{G}},$$

which is the desired equality.

Let us construct now a diffeomorphism at infinity  $\Phi$  on  $\mathbb{R}^2$  satisfying the two conditions (11) and (12). This will be done by applying twice the Proposition 14. Let  $\mathbb{G}_0$  be a good mesh for  $\psi$ . First we regularize the measure  $vol \circ \psi^{-1}$ , which does not have a smooth density, by applying the proposition in each face of the mesh  $\mathbb{G}_0$ . Then we transport the resulting measure of area on  $vol'$  by applying again Proposition 14 for each bounded face of  $\mathbb{G}$ . We remind the reader that  $\mathbb{G}$  is simple: its faces are  $n$ -gons.

Let us fix a measure of area  $vol''$  such that, for any bounded face  $F$  of  $\mathbb{G}_0$ ,  $vol''(\psi(F)) = vol(F)$ . Let us consider any bounded face  $F$  of  $\mathbb{G}_0$ . By definition of a  $\mathbb{G} - \mathbb{G}'$  piecewise diffeomorphism and the definition of a good mesh for  $\psi$ ,  $\psi|_{F_0}$  is a diffeomorphism from  $F_0$  to  $\psi(F_0)$  which are two simple  $n$ -gon. Thus,  $\widetilde{vol}|_{\psi(F_0)} = vol|_{F_0} \circ (\psi|_{F_0})^{-1}$  defines a measure with smooth density on  $\psi(F_0)$ . Using the condition on the Jacobian determinant of  $\psi$ , this smooth density can be extended as a continuous function on  $\overline{\psi(F_0)}$ . Using Proposition 14, we can consider  $\phi|_{\psi(F_0)} \in Diff^1(\psi(F_0)) \cap Diff^0(\overline{\psi(F_0)})$  with  $\phi|_{\psi(F_0)}(x) = x$  for any  $x \in \partial\psi(F_0)$  such that:

$$vol''_{|\psi(F_0)} = \widetilde{vol}|_{\psi(F_0)} \circ (\phi|_{\psi(F_0)})^{-1}.$$

Let us finally set  $\phi|_{\psi(F_\infty)} = Id|_{\psi(F_\infty)}$ . Thanks to the boundary condition on  $\phi|_{\psi(F)}$  for any face  $F$  of  $\mathbb{G}_0$ , we can glue together all the homeomorphisms  $\phi|_{\psi(F)}$  constructed for each face  $F$  of  $\mathbb{G}_0$ . It defines a diffeomorphism at infinity  $\phi_1$  on  $\mathbb{R}^2$  such that  $vol''_{|F'_\infty} = (vol \circ (\phi_1 \circ \psi)^{-1})_{|F'_\infty}$  and  $(\phi_1)|_{\mathbb{G}'} = Id|_{\mathbb{G}'}$ .

For any face  $F$  of  $\mathbb{G}'$ , we have:

$$\text{vol}''(F) = \text{vol}(\psi^{-1}(F)) = \text{vol}'(F).$$

Besides,  $\mathbb{G}'$  is a simple graph. We can apply Proposition 14 for any face  $F$  of  $\mathbb{G}'$  in order to transport  $\text{vol}''|_F$  on  $\text{vol}'|_F$ . Applying the same arguments (gluing the homeomorphisms as we just did) allows us to construct a homeomorphism  $\phi_2$  such that:

$$\begin{aligned} (\text{vol}')|_{(F'_\infty)^c} &= (\text{vol}'' \circ \phi_2^{-1})|_{(F'_\infty)^c} \\ (\phi_2)|_{\mathbb{G}'} &= \text{Id}|_{\mathbb{G}'}. \end{aligned}$$

The diffeomorphism at infinity  $\Phi = \phi_2 \circ \phi_1$  satisfies the two conditions (11) and (12). This concludes the proof.  $\square$

## 6. Group of reduced loops

Our goal is to construct Planar Markovian Holonomy Fields. For this purpose, we will need the group of reduced loops. Indeed, we have seen that, in order to construct a gauge invariant holonomy field on  $P$ , it is enough to construct a measure on  $\text{Mult}_P(L, G)$  for any set  $L$  of loops of  $P$ : this was the loop paradigm explained in Lemma 2. If  $L$  is the set of loops  $L_m(\mathbb{G})$ , where  $\mathbb{G}$  is a finite planar graph and  $m$  is a vertex of  $\mathbb{G}$ , and if  $P$  is equal to  $P(\mathbb{G})$ , then:

$$\text{Mult}_P(L, G) = \text{Hom}(\pi_1(\mathbb{G}, m), G^\vee),$$

where  $G^\vee$  is the group based on the same set as  $G$ , endowed with the multiplication  $\cdot_\vee$  such that  $x \cdot_\vee y = yx$  for any  $x, y \in G$ . This shows the importance of the group  $\pi_1(\mathbb{G}, m)$ .

**6.1. Definition and facts.** — The group of based reduced loops  $RL_v(\mathbb{G})$  is the fundamental group of  $\mathbb{G}$  based at  $v$ :  $RL_v(\mathbb{G}) = \pi_1(\mathbb{G}, v)$ . For convenience we look at it with a combinatorial point of view, as does Lévy in section 1.3.4 of [21]. In this section, in order to simplify the presentation, we will suppose  $\mathbb{G}$  connected.

Let  $l$  be a loop in  $P(\mathbb{G})$ . Recall the definition of equivalence of paths explained in Definition 7. The equivalence class in  $P(\mathbb{G})$ , denoted by  $[l]_\simeq$ , contains a unique element of shortest combinatorial length, which is said to be reduced. Besides, if  $l_1$  and  $l_2$  are two loops in  $P(\mathbb{G})$  based at  $m$ ,  $[l_1 l_2]_\simeq$  depends only on  $[l_1]_\simeq$  and  $[l_2]_\simeq$ .

Thus, it is equivalent to speak about equivalence classes or about reduced paths and the set of reduced paths is endowed with an internal operation.

**Definition 42.** — *The set of reduced loops in  $P(\mathbb{G})$  based at  $v$  will be denoted by  $RL_v(\mathbb{G})$ . Let  $l_1$  and  $l_2$  be two loops in  $RL_v(\mathbb{G})$ , we define  $l_1 \times l_2 = [l_1 l_2]_\simeq$ .*

Endowed with this operation,  $RL_v(\mathbb{G})$  is a group. The existence of the inverse of a loop  $l$  based at  $v$  is due to the fact that  $[ll^{-1}]_\simeq = [1_v]_\simeq$ , where  $1_v$  is the trivial

path constant to  $v$ . We will denote the reduced product of  $l_1$  with  $l_2$  by  $l_1 l_2$  rather than  $l_1 \times l_2$ .

A simple, yet crucial lemma about lassos is the following.

**Lemma 11.** — *Two lassos based at the same point and whose meanders represent the same cycle are conjugated in  $RL_v(\mathbb{G})$ .*

*Proof.* — Let  $l$  and  $l'$  be two lassos based at  $v$ . They can be written as  $l = sms^{-1}$  and  $l' = s'm's'^{-1}$ , where  $s$  and  $s'$  are respectively the spoke of  $l$  and  $l'$ . As the loops  $m$  and  $m'$  are related, there exist  $c$  and  $d$  two paths such that  $m = cd$  and  $m' = dc$ . Let us denote by  $p$  the loop  $s'c^{-1}s^{-1}$ , then  $l' = plp^{-1}$ .  $\square$

**Definition 43.** — *A loop in a planar graph  $\mathbb{G}$  is called a facial lasso if it is a lasso and its meander represents a facial cycle  $\partial F$  of  $\mathbb{G}$ .*

The exact definition of facial cycle, a cycle which represents the boundary of a face, is given in [21], in Definition 1.3.13. In the following, we address the problem of creating families of lassos which generate the whole group  $RL_m(\mathbb{G})$ . Lemma 12 provides a solution of this problem which is not adapted to our context, but will nevertheless be the departure point of our discussion. In order to state it, we need the definition of a spanning tree.

**Definition 44.** — *Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a finite planar graph. A spanning tree  $T$  is a subset of  $\mathbb{E}$  such that:*

- if an edge  $e$  is in  $T$ ,  $e^{-1}$  is also in  $T$ ,
- the set of non degenerate loops in  $T$  is empty,
- $\mathbb{V} = \{\underline{e}, e \in \mathbb{E}\}$ .

**Lemma 12.** — *Let  $\mathbb{G}$  be a finite planar graph and  $v$  a vertex of  $\mathbb{G}$ . Let  $T \subset \mathbb{E}$  be a spanning tree of  $\mathbb{G}$  rooted at  $v$ . If  $u$  and  $w$  are vertices of  $\mathbb{G}$ , we set  $[u, w]_T$  to be the unique injective path in  $T$  joining  $u$  to  $w$ . Let  $\mathbb{E}^+$  be an orientation of  $\mathbb{G}$ . The group  $RL_v(\mathbb{G})$  is freely generated by the loops  $\{l_{e,T} : e \in (\mathbb{E} \setminus T)^+\}$  where for any edge  $e$ ,  $l_{e,T}$  is equal to  $[v, \underline{e}]_T e [\bar{e}, v]_T$ .*

The loops  $l_{e,T}$  defined above are actually lassos.

*Proof.* — We only have to prove that  $(\{l_{e,T} : e \in (\mathbb{E} \setminus T)^+\}, RL_v(\mathbb{G}))$  satisfies the universal property of free group: given any function  $f$  from  $\{l_{e,T} : e \in (\mathbb{E} \setminus T)^+\}$  to a group  $G$  there exists a homomorphism  $\phi : RL_v(\mathbb{G}) \rightarrow G$  such that

$$\phi(l_{e,T}) = f(l_{e,T}),$$

for any  $e \in (\mathbb{E} \setminus T)^+$ .

Let  $G$  be any group, and let 1 be its neutral element. Let  $\mathbb{E}^+$  be an orientation of  $\mathbb{G}$ . We recall (1), in Subsection 2.2, which shows that one can construct a multiplicative function from  $P(\mathbb{G})$  to  $G$  by specifying the value on  $\mathbb{E}^+$ . In the definition of multiplicative functions, we asked that the function reverses the order

of multiplication. Only for this proof, we will suppose that it preserves the order. This means that if  $g \in \mathcal{M}ult(P(\mathbb{G}), G)$ , then for any path  $p_1$  and  $p_2$  in  $P$  which can be concatenated,  $g(p_1 p_2) = g(p_1)g(p_2)$ . Let  $f$  be a function from  $\{l_{e,T} : e \in (\mathbb{E} \setminus T)^+\}$  to  $G$ . We define the element  $\phi$  in  $G^{\mathbb{E}^+}$  by:

$$\phi(e) = \begin{cases} f(l_{e,T}), & \text{if } e \in (\mathbb{E} \setminus T)^+, \\ 1, & \text{otherwise.} \end{cases}$$

This defines an element of  $\mathcal{M}ult(P(\mathbb{G}), G)$ , thus, it can be seen as a homeomorphism from  $RL_v(\mathbb{G})$  to  $G$ . Beside, for any path  $p$  in  $T$ ,  $\phi(p) = 1$ . Let  $e$  be any element of  $(\mathbb{E} \setminus T)^+$ . Then, we have:

$$\phi(l_{e,T}) = \phi([v, \underline{e}]_T e [\bar{e}, v]_T) = \phi([v, \underline{e}]_T) \phi(e) \phi([\bar{e}, v]_T) = f(l_{e,T}).$$

The universal property of free group holds:  $RL_v(\mathbb{G})$  is the free group generated by  $\{l_{e,T} : e \in (\mathbb{E} \setminus T)^+\}$ .  $\square$

**6.2. Example:**  $RL_0(\mathbb{N}^2)$ . — We define in the following section a family of facial lassos on  $\mathbb{N}^2$ . This family will be the one used in Section 10. Thus, even if this family can be studied with the help of Proposition 15, we give an elementary proof that it generates  $RL_0(\mathbb{N}^2)$ .

**Notation 4.** — Let  $(i, j)$  and  $(k, l)$  be couples of reals such that  $i = k$  or  $j = l$ . We denote by  $(i, j) \rightarrow (k, l)$  the straight line from  $(i, j)$  to  $(k, l)$ . If  $j = l$  and  $k = i + 1$  it will also be denoted by  $e_{i,j}^r$ ; if  $i = k$  and  $l = j + 1$  it will also be denoted by  $e_{i,j}^u$ .

**Definition 45.** — Let  $i, j$  be two non negative integers. Let  $\partial c_{i,j}$  be the loop in  $L(\mathbb{N}^2)$  defined by:

$$\begin{aligned} \partial c_{i,j} &= (i, j + 1) \rightarrow (i, j) \rightarrow (i + 1, j) \rightarrow (i + 1, j + 1) \rightarrow (i, j + 1) \\ &= (e_{i,j}^u)^{-1} e_{i,j}^r e_{i+1,j}^u (e_{i,j+1}^r)^{-1}. \end{aligned}$$

Let  $p_{i,j}$  be the path in  $L_0(\mathbb{N}^2)$  defined by:

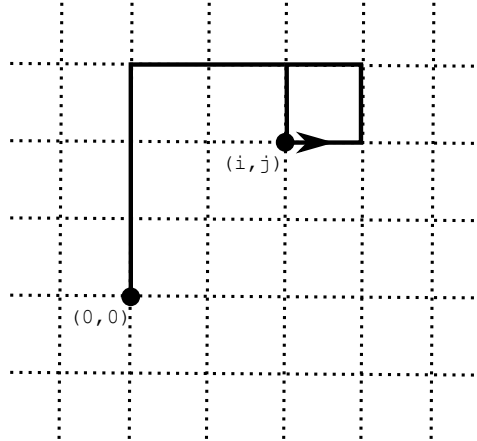
$$p_{i,j} = (0, 0) \rightarrow (0, j + 1) \rightarrow (i, j + 1) = e_{0,0}^u \dots e_{0,j}^u e_{0,j+1}^r \dots e_{i-1,j+1}^r.$$

Let  $L_{i,j}$ , the reduced loop based at 0, be:

$$L_{i,j} = [p_{i,j} \partial c_{i,j} p_{i,j}^{-1}]_{\simeq}.$$

One can refer to Figure 9 to have a clear representation of the lasso  $L_{i,j}$ .

**Lemma 13.** — The family  $(L_{i,j})_{(i,j) \in \mathbb{N}^2}$  is a freely generating subset of  $RL_0(\mathbb{N}^2)$ .

FIGURE 9. The lasso  $L_{i,j}$ .

*Proof.* — We only have to work with the finite planar graph:

$$\mathbb{G} = \mathbb{N}^2 \cap \{(x, y), x \leq k, y \leq k'\},$$

where  $k$  and  $k'$  are any positive integers. We remind the reader that the intersection of a graph with a set was defined before Definition 22. Lemma 12 ensures that  $RL_0(\mathbb{G})$  is a free group of rank  $k \times k'$ . Let  $l$  be a loop in  $RL_0(\mathbb{G})$ . We endow the graph  $\mathbb{G}$  with the following orientation: from bottom to top, from left to right. Let  $T$  be the tree defined by:

$$T = \left\{ (e_{i,j}^r)^{\pm 1}, i \in \{1, \dots, k-1\}, j \in \{0, \dots, k'\} \right\} \cup \left\{ (e_{0,j}^u)^{\pm 1}, j \in \{0, \dots, k'-1\} \right\}.$$

The root of  $T$  will be chosen to be  $(0, 0)$ .

One can look at Figure 10 to have a better idea of the graph and the tree we have just constructed. Applying Lemma 12 to this situation,  $l$  can be written as the reduced concatenation of some elements of  $\{l_{e,T}^{\pm 1}\}_{e \in (\mathbb{E} \setminus T)^+}$ , where  $\mathbb{E}$  is the set of edges of  $\mathbb{G}$ .

Moreover  $(\mathbb{E} \setminus T)^+$  is equal to  $\{e_{i,j}^u, i \in \{1, \dots, k\}, j \in \{0, \dots, k'-1\}\}$  and

$$l_{e_{i,j}^u, T} = L_{i,0} L_{i,1} \dots L_{i,j}.$$

Thus,  $(L_{i,j})_{(i,j) \in \mathbb{N}^2}$  is a generating subset of  $\mathbb{G}$  whose cardinal is  $k \times k'$ .  $\square$

### 6.3. Family of generators of the group of reduced loops on a planar graph. —

We remind the reader that we assume  $\mathbb{G}$  connected in this section. As  $RL_v(\mathbb{G})$  is a group, it is interesting to know some generating families. We have seen an example in the last subsection and we will generalize, for planar graphs, section 2.9 in [21] about tame generators. Moreover the proof does not follow the one given in [21] but rather uses a recursive decomposition of graphs.

The next definition follows Definition 2.4.6. of [21].

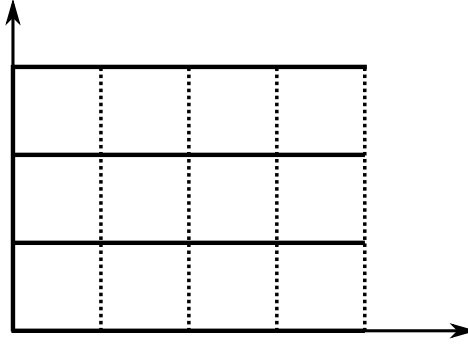


FIGURE 10. The covering tree  $T$ .

**Definition 46.** — Let  $T$  be a spanning tree rooted at  $v$ , and  $l_{e,T}$  be the loops defined in Lemma 12. Let  $F$  be a bounded face of  $\mathbb{G}$  and let  $c_F$  be a simple loop representing the facial cycle associated with  $F$ : it can be written as  $c_F = e_1 \dots e_n$ . We define the reduced path  $\mathbf{l}_{c_F,T} = l_{e_1,T} \dots l_{e_n,T}$  in  $RL_v(\mathbb{G})$ .

For each rooted spanning tree  $T$  and each choice of loops  $(c_F)_{F \in \mathbb{F}^b}$  such that  $c_F$  is a representative of the facial cycle associated with  $F$ , we have defined a new family of loops:

$$(\mathbf{l}_{c_F,T})_{F \in \mathbb{F}^b}.$$

The family  $(c_F)_{F \in \mathbb{F}}$  is called a family of facial loops of  $\mathbb{G}$ . The difference with Definition 2.4.7 in [21] is that the choice of  $c_F$  is not given by the choice of  $T$ . There is freedom to choose the base point of  $c_F$  where we want.

A remark that we will often use is that, when one changes the root of  $T$  from  $v$  to  $v'$ , this has the effect to conjugate the family  $(\mathbf{l}_{c_F,T})_{F \in \mathbb{F}^b}$  by  $[v', v]_T$ . This comes from the fact that, for any covering tree  $T$ , any vertices  $v, v'$  and  $v''$ , we have the equality in the set of reduced paths,  $[v, v'']_T = [v, v']_T [v', v'']_T$ . Proposition 15 and Lemma 14 give the two most important properties of this new families of reduced loops.

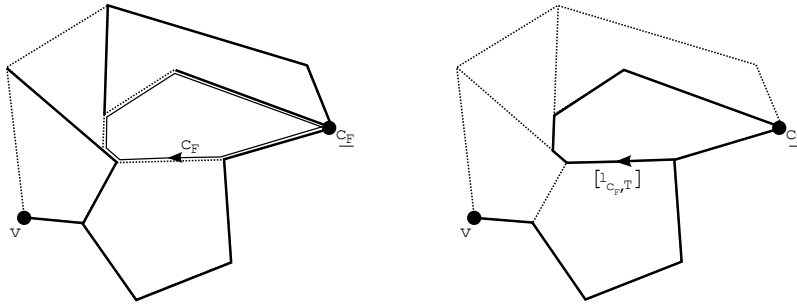


FIGURE 11. A graph, a spanning tree, a facial cycle: the associated reduced facial lasso.



**Proposition 15.** — For any vertex  $v$ , for any spanning tree  $T$  rooted at  $v$ , and any family of facial loops  $(c_F)_{F \in \mathbb{F}^b}$ , the following assertions hold:

1. for any bounded face  $F$ ,  $\mathbf{l}_{c_F, T}$  is a facial lasso based at  $v$  whose meander represents the facial cycle  $\partial F$ ,
2.  $(\mathbf{l}_{c_F, T})_{F \in \mathbb{F}^b}$  freely generates  $RL_v(\mathbb{G})$ .

*Proof.* — 1. The equality  $\mathbf{l}_{c_F, T} = [v, \underline{c}_F]_T c_F [v, \underline{c}_F]_T^{-1}$  allows us to see that  $\mathbf{l}_{c_F, T}$  is a lasso of meander  $c_F$  and spoke  $[v, \underline{c}_F]_T$ .

2. As seen in Lemma 12,  $RL_v(\mathbb{G})$  is a free group of rank  $\#\mathbb{F}^b$ . Thus, we have only to show that  $(\mathbf{l}_{c_F, T})_{F \in \mathbb{F}^b}$  generates  $RL_v(\mathbb{G})$ . Thanks to Lemma 12, it suffices to show that for every  $e \in \mathbb{E} \setminus T$ ,  $l_{e, T}$  is a product of elements of the form  $\mathbf{l}_{c_F, T}^{\pm 1}$ .

Let  $e$  be an edge not in  $T$ . As  $T$  is a tree, there exist  $c$ ,  $p$  and  $p'$  three simple paths in  $T$  which do not intersect, except at the point  $\bar{c} = \underline{p} = \underline{p}'$ , such that:

- $[v, \underline{c}]_T = cp$ ,
- $[v, \bar{c}]_T = cp'$ ,
- the meander  $m$  of  $l_{e, T}$  is  $pep'^{-1}$ .

Let  $v'$  be any point of  $\mathbb{G}$  inside the meander of  $l_{e, T}$ . Because  $T$  is a tree,  $[v, v']_T$  must begin with the path  $c$ . If not, it would create a non degenerate loop in  $T$ . Define  $\mathbb{G}'$  (resp.  $T'$ ) the restriction of  $\mathbb{G}$  (resp.  $T$ ) to the closure of the inside of the meander  $m$  of  $l_{e, T}$ . We have drawn an example in Figure 12. We set  $\bar{c}$  to be the root of  $T'$ . For every bounded face  $F$  of  $\mathbb{G}$  inside  $m$ ,  $\mathbf{l}_{c_F, T} = c \mathbf{l}_{c_F, T'} c^{-1}$  where  $\mathbf{l}_{c_F, T'}$  is the facial lasso based at  $\bar{c}$  defined in  $\mathbb{G}'$  thanks to  $T'$ .

Applying Lemma 14 to  $\mathbb{G}'$  endowed with  $T'$ ,  $m$  can be written as a product of lassos of the form  $\mathbf{l}_{c_F, T'}^{\pm 1}$ , thus  $l_{e, T}$  can be written as a product of lassos of the form  $\mathbf{l}_{c_F, T}^{\pm 1}$ . □

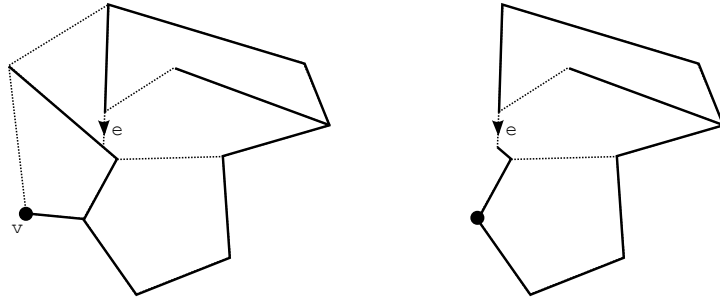


FIGURE 12. The restriction of  $\mathbb{G}$  used in Proposition 15.

**Lemma 14.** — Let  $\mathbb{G}$  be a finite planar graph. Let  $\mathbb{F}^b$  be the set of bounded faces of  $\mathbb{G}$ . Let  $v$  be a vertex on the boundary of the unbound face, and  $(F_i)_{i=1}^{\#\mathbb{F}^b}$  an

enumeration of the bounded faces. Let  $l_\infty$  be the only loop in  $P(\mathbb{G})$  based at  $v$  with anti-clockwise orientation which represents the facial cycle of the unbounded face  $F_\infty$ . For any spanning tree  $T$  rooted at  $v$ , and any family of facial loops  $(c_F)_{F \in \mathbb{F}^b}$ , there exists a permutation  $\sigma$  of  $\{1, \dots, \#\mathbb{F}^b\}$  and an application  $\epsilon : \{1, \dots, n\} \rightarrow \{-1, 1\}$  such that the equality

$$\mathbf{l}_{c_{F_{\sigma(n)}}, T}^{\epsilon(n)} \mathbf{l}_{c_{F_{\sigma(n-1)}}, T}^{\epsilon(n-1)} \dots \mathbf{l}_{c_{F_{\sigma(1)}}, T}^{\epsilon(1)} = l_\infty$$

holds in  $RL_v(\mathbb{G})$ . Besides, for any integer  $k \in \{1, \dots, n\}$ ,  $\epsilon(k)$  is equal to 1 if and only if  $c_{F_{\sigma(k)}}$  is oriented anti-clockwise.

*Proof.* — The last assertion comes from a topological index argument. Let us suppose that there exists a permutation  $\sigma$  of  $\{1, \dots, \#\mathbb{F}^b\}$  and an application  $\epsilon : \{1, \dots, n\} \rightarrow \{-1, 1\}$  such that:

$$\mathbf{l}_{c_{F_{\sigma(n)}}, T}^{\epsilon(n)} \mathbf{l}_{c_{F_{\sigma(n-1)}}, T}^{\epsilon(n-1)} \dots \mathbf{l}_{c_{F_{\sigma(1)}}, T}^{\epsilon(1)} = l_\infty$$

holds in  $RL_v(\mathbb{G})$ .

We can compute the index of  $l_\infty$ :

$$\mathbf{n}_{l_\infty} = \mathbf{n}_{c_{F_{\sigma(n)}}, T} \cdot \epsilon(n) + \dots + \mathbf{n}_{c_{F_{\sigma(1)}}, T} \cdot \epsilon(1)$$

Let  $F$  be any bounded face of  $\mathbb{G}$ . We can evaluate the last equality for any  $x \in F$ . This implies that for any  $i \in \{1, \dots, n\}$ ,  $1 = \mathbf{n}_{c_{F_{\sigma(i)}}, T} \cdot \epsilon(i)$ .

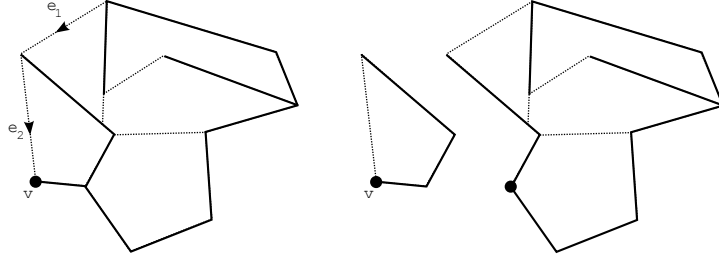
Let us show the first part of Lemma 14. The proof goes by induction on the number  $\#\mathbb{F}^b$  of bounded faces. For a graph with only one bounded face the result is true since  $l_\infty = l_{c_F}^\epsilon$  in  $RL_v(\mathbb{G})$ , with  $\epsilon$  being  $-1$  or  $1$ , depending on the orientation of  $c_F$ .

There exists a unique way to write  $l_\infty$  as  $p_1 e_1 p_2 e_2 \dots e_n p_n$  with  $p_i$  a path in  $T$  and  $e_i$  an edge in  $\mathbb{E} \setminus T$  bounding  $F_\infty$  for any  $i$  in  $\{1, \dots, n\}$ . Let us decompose the graph  $\mathbb{G}$  in  $n$  subgraphs. The  $i$ -th subgraph  $\mathbb{G}_i$  is the part of  $\mathbb{G}$  which is inside the meander  $m_i$  of  $l_{e_i, T}$ . The vertex  $v_i = \underline{m}_i$  will be the chosen point on the boundary of  $\mathbb{G}_i$ , then:

- the restriction of  $T$  to  $\mathbb{G}_i$ ,  $T_i$ , is still a covering tree of  $\mathbb{G}_i$ ,
- for any face  $F$  in  $\mathbb{G}_i$ ,  $\mathbf{l}_{c_F, T} = [v, v_i]_{T_i} \mathbf{l}_{c_F, T_i} [v, v_i]_{T_i}^{-1}$ , where  $\mathbf{l}_{c_F, T_i}$  is the facial lasso based at  $v_i$ .

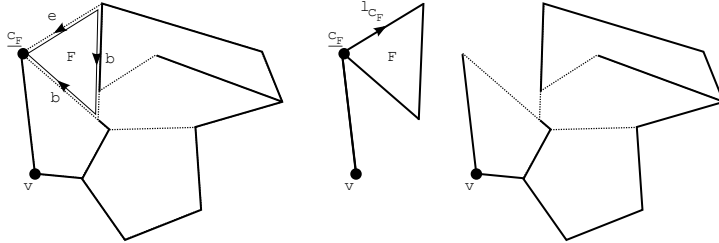
If  $n > 1$ , each of the graphs  $\mathbb{G}_i$  has strictly less than  $\#\mathbb{F}^b$  bounded faces. An example is drawn in Figure 13. By induction the result holds for  $\mathbb{G}_i$ , based at  $v_i$  and endowed with  $T_i$ . It follows that  $l_{e_i, T}$ , which is equal to  $[v, v_i]_{T_i} m_i [v, v_i]_{T_i}^{-1}$ , is an ordered product of all the facial lasso (or their inverse) associated with the faces  $F$  in  $\mathbb{G}_i$ . But the family  $(\mathbb{G}_i)_i$  induces a partition of the set of bounded faces of  $\mathbb{G}$ . As  $l_\infty = l_{e_1, T} \dots l_{e_n, T}$  it is now clear that the result holds.

It remains the case where  $n = 1$ . In this case,  $l_\infty = pep'$  in  $RL_v(\mathbb{G})$ , with  $p$  and  $p'$  two simple paths in  $T$  and  $e$  an edge in  $\mathbb{E} \setminus T$  bounding  $F_\infty$ . We have

FIGURE 13. Decomposition when  $n > 1$ .

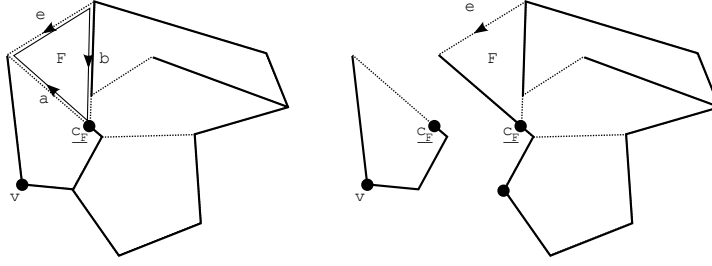
to find a new way of decomposing  $\mathbb{G}$  in order to apply the induction hypothesis. Let  $F$  be the only bounded face which is surrounded by  $e$ . We can suppose  $c_F$  turning clockwise thus it can be decomposed as  $c_F = ae^{-1}b$ . Consider the loop:  $l = [v, \underline{c}_F]_T a e^{-1} [\underline{c}, v]_T$ , where we have used the reduced product. This is a lasso and as before we only look at  $\mathbb{G}$  inside the meander  $m_l$  of  $l$ :  $\mathbb{G}_l = \mathbb{G} \cap \text{Int}(m_l)$ . We base this graph at  $v_l = \underline{m}_l$ .

First of all, if  $\mathbb{G}_l$  has the same number of faces than  $\mathbb{G}$ , as in Figure 14, then the equality  $l = l_\infty$  must hold in  $RL_v(\mathbb{G})$ , and thus one has  $l_\infty = ([v, \underline{e}]_T b [\underline{c}_F, v]_T) \mathbf{l}_{c_F}^{-1}$  in  $RL_v(\mathbb{G})$ . The path  $l' = [[v, \underline{e}]_T b [\underline{c}_F, v]_T]_{\approx}$  is a loop based at  $v$  which represents in  $RL_v(\mathbb{G})$  the facial cycle of the unbounded face of the graph obtained when one removes  $e$  to  $\mathbb{G}$ . On this graph,  $T$  is still a covering tree, and this graph has one less bounded face. The induction hypothesis allows us to conclude.

FIGURE 14. Decomposition when  $\mathbb{G}_l$  has the as many faces as  $\mathbb{G}$ .

In the case where  $\mathbb{G}_l$  has less faces than  $\mathbb{G}$ , as in Figure 15, the restriction of  $T$  in  $m_l$  is not a covering tree. We will define  $T_l$  to be the restriction of  $T$  in  $m_l$  to which one adds all the edges in the path  $a$ , rooted at  $v_l$ . With this modifications,  $T_l$  is a covering tree of  $\mathbb{G}_l$  and for any bounded face  $F$  of  $\mathbb{G}_l$ , we have  $\mathbf{l}_{c_F, T} = s_l \mathbf{l}_{c_F, T_l} s_l^{-1}$ , where  $\mathbf{l}_{c_F, T_l}$  is the facial lasso based at  $v_l$  defined in  $\mathbb{G}_l$  thanks to  $T_l$  and  $s_l$  is the spoke of  $l$ .

We define also  $l' = [[v, \underline{c}_F]_T a [\bar{e}, v]_T]_{\approx}$  and  $\mathbb{G}_{l'}$  the part of  $\mathbb{G}$  inside the meander  $m_{l'}$  of  $l'$ . By  $T_{l'}$ , we denote the restriction of  $T$  to  $\mathbb{G}_{l'}$ . In this case  $T_{l'}$  is a covering tree of  $\mathbb{G}_{l'}$ . Besides, for any face  $F$  in  $\mathbb{G}_{l'}$ ,  $\mathbf{l}_{c_F, T} = s_{l'} \mathbf{l}_{c_F, T_{l'}} s_{l'}^{-1}$ , where  $\mathbf{l}_{c_F, T_{l'}}$  is the facial lasso based at  $v_{l'} = \underline{m}_{l'}$  defined in  $\mathbb{G}_{l'}$  thanks to  $T_{l'}$  and  $s_{l'}$  is the spoke of  $l'$ .


 FIGURE 15. Decomposition when  $\mathbb{G}_l$  has less faces than  $\mathbb{G}$ .

In the case we are studying,  $\mathbb{G}_l$  and  $\mathbb{G}_{l'}$  have strictly less bounded faces than  $\mathbb{G}$ , thus we can apply the induction hypothesis. Using the link between facial lassos in  $\mathbb{G}_l$  (resp. in  $\mathbb{G}_{l'}$ ) and in  $\mathbb{G}$ , there exists an ordering on the faces of  $\mathbb{G}_l$  (resp.  $\mathbb{G}_{l'}$ ) such that  $l$  (resp.  $l'$ ) is the ordered product of the facial lassos  $(\mathbf{I}_{c_F, T}^{\pm 1})_{F \in \mathbb{F}_l^b}$  (resp.  $(\mathbf{I}_{c_F, T}^{\pm 1})_{F \in \mathbb{F}_{l'}^b}$ ), where  $\mathbb{F}_l^b$  (resp.  $\mathbb{F}_{l'}^b$ ) is the set of bounded faces of  $\mathbb{G}_l$  (resp.  $\mathbb{G}_{l'}$ ).

As in  $RL_v(\mathbb{G})$ ,  $l_\infty = [v, \underline{e}]_T e [\bar{e}, v]_T = l^{-1} l'$  and as any face  $F$  of  $\mathbb{G}$  is either in  $\mathbb{G}_l$  or in  $\mathbb{G}_{l'}$ , we can conclude that there exists a permutation  $\sigma$  of  $\{1, \dots, \#\mathbb{F}^b\}$  and an application  $\epsilon : \{1, \dots, n\} \rightarrow \{-1, 1\}$  such that:

$$\mathbf{I}_{c_{F_{\sigma(n)}}, T}^{\epsilon(n)} \mathbf{I}_{c_{F_{\sigma(n-1)}}, T}^{\epsilon(n-1)} \dots \mathbf{I}_{c_{F_{\sigma(1)}}, T}^{\epsilon(1)} = l_\infty,$$

in  $RL_v(\mathbb{G})$ . □

In Proposition 24, we will need the following proposition.

**Proposition 16.** — *Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a finite planar graph. Let  $v$  be a vertex of  $\mathbb{G}$ . Let  $l_1$  and  $l_2$  two simple loops in  $\mathbb{G}$  such that  $\overline{\text{Int}(l_1)}$  and  $\overline{\text{Int}(l_2)}$  are disjoint. There exists a spanning tree  $T$  rooted at  $v$ , such that for any family of facial loops  $(c_F)_{F \in \mathbb{F}}$  the following is valid:*

1. for every loop  $l$  inside  $l_1$ ,  $\left[ [v, \underline{l}]_T l [v, \underline{l}]_T^{-1} \right]_{\simeq}$  is a product in  $RL_v(\mathbb{G})$  of elements of  $\left\{ \mathbf{I}_{c_F, T}^{\pm 1}; F \in \mathbb{F}, F \subset \overline{\text{Int}(l_1)} \right\}$ ,
2. for every loop  $l$  inside  $l_2$ ,  $\left[ [v, \underline{l}]_T l [v, \underline{l}]_T^{-1} \right]_{\simeq}$  is a product in  $RL_v(\mathbb{G})$  of elements of  $\left\{ \mathbf{I}_{c_F, T}^{\pm 1}; F \in \mathbb{F}, F \subset \overline{\text{Int}(l_2)} \right\}$ .

*Proof.* — Consider  $\mathbb{G}$  a finite planar graph,  $v$  a vertex of  $\mathbb{G}$  and  $l_1, l_2$  two simple loops in  $\mathbb{G}$ . Suppose that  $\overline{\text{Int}(l_1)}$  and  $\overline{\text{Int}(l_2)}$  are disjoint. We can decompose  $l_1$  and  $l_2$  as a concatenation of edges of  $\mathbb{G}$ :

$$\begin{aligned} l_1 &= e_1^1 \dots e_1^n, \\ l_2 &= e_2^1 \dots e_2^m. \end{aligned}$$

The set  $\{e_1^1, \dots, e_1^{n-1}, e_2^1, \dots, e_2^{m-1}\}$  can be extended as a tree  $T$  rooted at  $v$ . Thanks to the construction, the restriction of  $T$  to  $\overline{\text{Int}(l_1)}$ ,  $T_1$ , is a spanning tree of the restriction of  $\mathbb{G}$  to  $\overline{\text{Int}(l_1)}$ ,  $\mathbb{G}_1$ . We set  $v_1$  to be equal to  $e_1^1$ . We define the root of  $T_1$  being  $v_1$ . Applying Proposition 15, for any loop  $l$  inside  $l_1$ ,

$$\left[ [v_1, \underline{l}]_{T_1} l [v_1, \underline{l}]_{T_1}^{-1} \right]_{\simeq}$$

is a product of elements of  $\left\{ \mathbf{I}_{c_F, T_1}^{\pm 1}; F \in \mathbb{F}, F \subset \overline{\text{Int}(l_1)} \right\}$ . Besides for any vertex  $w$  in  $\mathbb{G}_1$ ,  $[v, w]_T = [v, v_1]_T [v_1, w]_{T_1}$  in  $RL_v(\mathbb{G})$ . Thus for any face  $F \in \mathbb{F}$  such that  $F \subset \overline{\text{Int}(l_1)}$ ,  $\mathbf{I}_{c_F, T} = [v, v_1]_T \mathbf{I}_{c_F, T_1} [v, v_1]_T^{-1}$  in  $RL_v(\mathbb{G})$  and for any loop  $l$  inside  $l_1$ ,  $[v, \underline{l}]_T l [v, \underline{l}]_T^{-1}$  is equal in  $RL_v(\mathbb{G})$  to  $[v, v_1]_T [v_1, \underline{l}]_{T_1} l [v_1, \underline{l}]_{T_1}^{-1} [v, v_1]_T^{-1}$ . Thus  $[v, \underline{l}]_T l [v, \underline{l}]_T^{-1}$  is a product of elements of  $\left\{ \mathbf{I}_{c_F, T}^{\pm 1}; F \in \mathbb{F}, F \subset \overline{\text{Int}(l_1)} \right\}$ . The same can be done with  $l_2$ .  $\square$

**6.4. Construction of random holonomy fields III: the planar case and the group of reduced loops.** — For this subsection we take  $M$  equal to  $\mathbb{R}^2$ , yet we will use the notation  $M$  as the propositions can be to surfaces by using the general tame generators defined in section 2.4 of [21]. As we will only use the planar version, we decided to present only this one. We will expose two propositions about uniqueness and construction of random holonomy fields on the plane, using the group of reduced loops.

**Proposition 17.** — *Let  $m \in M$ . Let  $\mu$  and  $\nu$  be two stochastically continuous measures on  $(\text{Mult}(P(M), G), \mathcal{B})$  invariant by gauge transformations. The two assertions are equivalent:*

1.  $\mu$  and  $\nu$  are equal,
2. there exist  $m \in M$  and  $A_m$  a good subspace of  $L_m(M)$ , such that for any graph  $\mathbb{G}$  in  $\mathcal{G}(A_m)$  and which has  $m$  as a vertex, there exist a rooted spanning tree  $T$  and a family of facial loops of  $\mathbb{G}$ ,  $(c_F)_{F \in \mathbb{F}^b}$ , such that the law of  $(l_{c_F, T})_{F \in \mathbb{F}^b}$  is the same under  $\mu$  and under  $\nu$ .

*Proof.* — It is an easy application of the multiplicative property of random holonomy fields, Proposition 5 and Proposition 15.  $\square$

**Proposition 18.** — *Let  $\gamma$  be a Riemannian metric on  $M$  and  $m \in M$ . Suppose that for any finite planar graph  $\mathbb{G}$  in  $\mathcal{G}(\text{Aff}_\gamma(M))$  which has  $m$  as vertex, we are given a diagonal conjugation-invariant measure  $\mu_{\mathbb{G}}$  on  $\mathbb{G}^{\#\mathbb{F}^b}$ , a spanning tree  $T$  rooted at  $m$  and a choice of facial loops  $(c_F)_{F \in \mathbb{F}^b}$ . There is only one possibility to extend  $\mu_{\mathbb{G}}$  as a gauge-invariant random field on  $\mathbb{G}$ , extension which will be also called  $\mu_{\mathbb{G}}$ .*

*If  $(\mu_{\mathbb{G}})_{\mathbb{G}}$  is uniformly locally stochastically  $\frac{1}{2}$ -Hölder continuous and if for any  $\mathbb{G} \preceq \mathbb{G}'$ ,  $h(\mathbf{I}_{c_F, T})_{F \in \mathbb{F}^b}$  has the same law under  $\mu_{\mathbb{G}}$  as under  $\mu_{\mathbb{G}'}$ , there exists a unique*

stochastically continuous random holonomy field  $\mu$  on  $M$  such that for any finite graph  $\mathbb{G}$  in  $\mathcal{G}(\text{Aff}_\gamma(M))$  and with  $m$  as a vertex, the law of  $h(\mathbf{l}_{c_F,T})_{F \in \mathbb{F}^b}$  is the same under  $\mu$  and under  $\mu_{\mathbb{G}}$ .

*Proof.* — For any finite planar graph  $\mathbb{G}$ , there exists a measurable function from  $\text{Hom}(RL_m(\mathbb{G}), G)$  to  $\text{Mult}_{P(\mathbb{G})}(L_m(\mathbb{G}), G)$ . Thus, we can transport any measure from the first space to the second. Using the freeness of the generating families  $\mathbf{l}_{c_F,T}$ , the multiplicity property of random holonomy fields and Proposition 6, we can extend  $\mu_{\mathbb{G}}$  as a gauge-invariant random field on  $\mathbb{G}$ . Then an application of Proposition 7 and Lemma 7 allows us to construct the desired  $\mu$ .  $\square$

### 7. Braids and probabilities I: an algebraic point of view and finite random sequences

Let  $\mathbb{G}$  be a finite planar graph. We have constructed in the last section a set of generating family of facial lassos of  $\mathbb{G}$ . It is natural to wonder what is the transformation which sends one generating family to another. It has to be noticed that, as soon as the base point of the covering tree is chosen, for any generating family of lassos we have constructed in the last section, their product, up to some suitable permutation, is always equal to the same loop. This remark, and Artin's theorem 7 motives the study of the group of braids.

**7.1. Generators, Relations, Actions.** — We have seen a geometric definition of the braid group in the introduction of this paper. One can also define the braid group with a generator-relation presentation.

**Definition 47.** — Let  $n$  be an integer greater than 2. The braid group with  $n$  strands  $\mathcal{B}_n$  is the group with the following presentation:

$$\left\langle (\beta_i)_{i=1}^{n-1} \mid \forall i, j \in \{1, \dots, n-1\}, \begin{array}{l} |i-j|=1 \implies \beta_i \beta_j \beta_i = \beta_j \beta_i \beta_j \\ |i-j| > 1 \implies \beta_i \beta_j = \beta_j \beta_i \end{array} \right\rangle.$$

The elements  $(\beta_i)_{i=1}^{n-1}$  we defined before Figure 4 satisfy the braid group relations which now seem more natural. Let us illustrate the first relation between  $\beta_i$  and  $\beta_j$  when  $|i-j|=1$ .

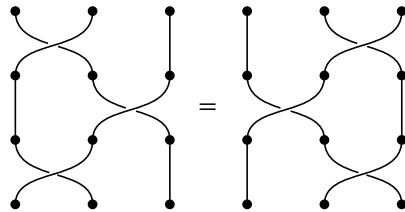


FIGURE 16. The braid relation

This presentation of the braid group is not intuitive, yet it allows us to construct some natural actions of the braid group  $\mathcal{B}_n$ : one on the free group of rank  $n$  and one on  $G^n$ .

**Definition 48.** — Let  $\mathbb{F}_n$  be the free group of rank  $n$  generated by  $e_1, \dots, e_n$ . We define the natural action of  $\mathcal{B}_n$  on  $\mathbb{F}_n$  by:

$$\begin{aligned}\beta_i e_i &= e_{i+1}, \\ \beta_i e_{i+1} &= e_{i+1} e_i e_{i+1}^{-1}, \\ \beta_i e_j &= e_j, \text{ for any } j \neq i.\end{aligned}$$

One can verify easily that the braid group relations is satisfied in this last definition: the natural action of  $\mathcal{B}_n$  on  $\mathbb{F}_n$  is well defined. There is a diagrammatic way to compute the action: one puts  $e_1, \dots, e_n$  at the bottom of a diagram representing  $\beta$ , then we propagate these  $e_1, \dots, e_n$  in the diagram from the bottom to the top, with the rule that at each crossing, the value on the string which is behind does not change, and the value on the upper string is conjugated by the value of the other so that the product from right to left remains unchanged. At the end we get a  $n$ -uple  $(f_1, \dots, f_n)$  at the top of the diagram: the braid sends  $e_i$  on  $f_i$ .

**Definition 49.** — Consider  $G$  an arbitrary group, we call the natural action of  $\mathcal{B}_n$  on  $G^n$  the action such that:

$$(13) \quad \beta_i \bullet (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) = (x_1, \dots, x_{i-1}, x_i x_{i+1} x_i^{-1}, x_i, \dots, x_n),$$

for any integer  $i \in \{1, \dots, n-1\}$  and  $n$ -tuple  $(x_i)_{i=1}^n$  in  $G^n$ .

One can verify easily that the braid group relations is satisfied in this last definition: the natural action of  $\mathcal{B}_n$  on  $G^n$  is well defined. There is a diagrammatic way to compute the action: one puts  $x_1, \dots, x_n$  at the upper part of a diagram representing  $\beta$ , then we propagate these  $x_1, \dots, x_n$  in the diagram from the top to the bottom with the rule that at each crossing, the value on the string which is behind does not change, and the value on the upper string is conjugated by the value of the other so that the product from left to right remains unchanged. At the end we get a  $n$ -uple  $(y_1, \dots, y_n)$  at the bottom of the diagram: the braid sends  $(x_1, \dots, x_n)$  on  $(y_1, \dots, y_n)$ .

The definition of the action of the braid group on  $G^n$  can seem odd as it has a different form than the action on the free group: this is due to Lemma 15. Let  $h$  be a  $G$  valued multiplicative function from the free group: for us, it will mean that  $h(x^{-1}) = h(x)$  and  $h(xy) = h(y)h(x)$  for any  $x$  and  $y$  in  $\mathbb{F}_n$ . For any  $n$ -uple  $(f_1, \dots, f_n)$  of elements of  $\mathbb{F}_n$ , we define  $h(f_1, \dots, f_n) = (h(f_1), \dots, h(f_n))$ . We have the following lemma which shows how both actions are linked, and which is a consequence of the diagrammatic formulation of both actions.

**Lemma 15.** — Let  $\mathbb{F}_n$  be the free group of rank  $n$  generated by  $e_1, \dots, e_n$ . For any braid  $\beta \in \mathcal{B}_n$ , we have:

$$h(\beta \bullet (e_1, \dots, e_n)) = \beta^{-1} \bullet h(e_1, \dots, e_n).$$

With the  $n$ -diagrams picture in mind, it is obvious that the application, which sends a braid on the permutation obtained by erasing the information at each crossing, is a homomorphism: it is the one which sends  $\beta_i$  to the transposition  $(i, i + 1)$  for any integer  $i$ .

**Lemma 16.** — The operation of erasing the information at each crossing induces a natural homomorphism from  $\mathcal{B}_n$  to  $\mathfrak{S}_n$ . We will denote the image of  $\beta$  by  $\sigma_\beta$ .

**7.2. Artin theorem.** — For any braid  $\beta$  with  $n$  strands, we can associate an action  $a_\beta$  on  $\mathbb{F}_n$  which is, in fact, an automorphism. Thus, there exists a morphism from  $\mathcal{B}_n$  in  $\text{Aut}(\mathbb{F}_n)$  which is moreover injective.

In [8] and [7], Artin gave a sufficient and necessary condition for an automorphism of  $\mathbb{F}_n$  to be the induced action of a braid in  $\mathcal{B}_n$ :

**Theorem 7.** — Let  $x_1, \dots, x_n$  be a free family of generators of  $\mathbb{F}_n$ . An automorphism  $\mathbf{a}$  of  $\mathbb{F}_n$  is the induced action of a braid in  $\mathcal{B}_n$  if and only if:

**Conjugation property :** for any  $i$  in  $\{1, \dots, n\}$ ,  $\mathbf{a}(x_i)$  is a conjugate of an element of  $(x_j)_{j=1}^n$ .

**Product invariance :**  $\mathbf{a}(x_n \dots x_1) = x_n \dots x_1$ .

**Remark 12.** — Let  $(x_1, \dots, x_n)$  be a free family of generators of  $\mathbb{F}_n$ , let  $\beta$  be a braid in  $\mathcal{B}_n$  and  $a_\beta$  the induced action on  $\mathbb{F}_n$ . For each  $i$  in  $\{1, \dots, n\}$ ,  $a_\beta(x_i)$  is conjugated to  $x_{\sigma_\beta(i)}$  and this property characterizes  $\sigma_\beta$ .

**7.3. Braids and the group of reduced loops.** —

**Proposition 19.** — Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a finite planar graph. Let  $v$  be a vertex of  $\mathbb{G}$ . Let  $T$  and  $T'$  be two spanning trees of  $\mathbb{G}$  rooted at  $v$ . Let us consider  $(c_F)_{F \in \mathbb{F}^b}$  and  $(c'_F)_{F \in \mathbb{F}^b}$  two families of facial loops turning anti-clockwise. There exists an enumeration of the bounded faces  $(F_i)_{i=1}^{\#\mathbb{F}^b}$ , a braid  $\beta$  in  $\mathcal{B}_{\#\mathbb{F}^b}$  such that:

$$\beta \bullet \left( \mathbf{l}_{c_{F_i}, T} \right)_{i=1}^{\#\mathbb{F}^b} = \left( \mathbf{l}_{c'_{F_{\sigma(i)}}, T'} \right)_{i=1}^{\#\mathbb{F}^b},$$

where  $\sigma = \sigma_\beta$  (see Lemma 16 for the definition of  $\sigma_\beta$ ).

*Proof.* — For any bounded face  $F$  of  $\mathbb{G}$ , the first part of Proposition 15 asserts that  $\mathbf{l}_{c_F, T}$  and  $\mathbf{l}_{c'_F, T'}$  are facial lassos based at  $v$  whose meanders represent the facial cycle  $\partial F$  oriented anti-clockwise. By Lemma 11, we deduce that  $\mathbf{l}_{c'_F, T'}$  is conjugated to  $\mathbf{l}_{c_F, T}$  in  $RL_v(\mathbb{G})$ . Besides, thanks to Lemma 14, we can find an enumeration of the bounded faces  $(F_i)_{i=1}^{\#\mathbb{F}^b}$  and a permutation  $\sigma$  of  $\{1, \dots, \#\mathbb{F}^b\}$  such that:



1.  $\mathbf{l}_{c_{F_n}, T} \mathbf{l}_{c_{F_{n-1}}, T} \cdots \mathbf{l}_{c_{F_1}, T} = l_\infty$ ,
2.  $\mathbf{l}'_{c'_{F_{\sigma(n)}}, T'} \mathbf{l}'_{c'_{F_{\sigma(n-1)}}, T'} \cdots \mathbf{l}'_{c'_{F_{\sigma(1)}}, T'} = l_\infty$ ,

in  $RL_v(\mathbb{G})$ , where  $l_\infty$  is the facial loop based at  $v$ , turning anti-clockwise, representing the facial cycle  $\partial F_\infty$ .

Besides, Proposition 15 tells us that both  $(\mathbf{l}_{c_F})_{F \in \mathbb{F}^b}$  and  $(\mathbf{l}'_{c'_F})_{F \in \mathbb{F}^b}$  are free families of generators of the free group  $RL_v(\mathbb{G})$ . A natural automorphism of  $RL_v(\mathbb{G})$  is defined by:

$$\forall i \in \{1, \dots, \#\mathbb{F}^b\}, \mathbf{a}(\mathbf{l}_{c_{F_i}, T}) = \mathbf{l}'_{c'_{F_{\sigma(i)}}, T'}.$$

This automorphism of free group satisfies the conditions of Artin's theorem, Theorem 7. There exists a braid  $\beta$  such that  $\mathbf{a}$  is equal to  $\mathbf{a}_\beta$ , the action induced by  $\beta$  on the free group of rank  $\#\mathbb{F}^b$ . Using Remark 12, it is straightforward to see that  $\sigma$  is equal to  $\sigma_\beta$ .  $\square$

**7.4. Braids and finite sequence of random variables.** — In the last section we have understood the transformation between families of loops of the form  $(\mathbf{l}_{c_F, T})_F$ . In the context of holonomy fields, a random variable is associated with any loop. Thus we study the action of the braid groups on finite sequence of random variables. When one has to deal with non-commutative random variables (i.e. random variables in a non-commutative group), this action is in some sense more appropriate than the symmetrical group action which is often studied in mathematical literature. This leads to a theory of braidability which is more efficient than the exchangeability concept for sequences of random variables in a non-commutative group.

**Notation 5.** — Consider  $G$  an arbitrary topological group. For any  $n \geq 2$ , the braid group  $\mathcal{B}_n$  acts on the set of sequences of  $n$   $G$ -valued random variables according to the formula:

$$(14) \quad \beta_i \bullet (X_1, \dots, X_{i-1}, X_i, X_{i+1}, \dots, X_n) = (X_1, \dots, X_{i-1}, X_i X_{i+1} X_i^{-1}, X_i, \dots, X_n)$$

for any  $i \in \{1, \dots, n-1\}$ .

Let  $n$  be an integer greater than 1. Let  $\beta$  be a braid in  $\mathcal{B}_n$ . We remind the reader that  $\sigma_\beta$  was defined in Lemma 16.

**Definition 50.** — Let  $(X_1, \dots, X_n)$  be a finite sequence of  $G$ -valued random variables. It is purely invariant by braids if for any braid  $\beta \in \mathcal{B}_n$  one has the equality in law:

$$\beta \bullet (X_1, \dots, X_n) = \sigma_\beta \bullet (X_1, \dots, X_n),$$

where  $\sigma \bullet (X_1, \dots, X_n) = (X_{\sigma^{-1}(1)}, \dots, X_{\sigma^{-1}(n)})$  for any permutation  $\sigma \in \mathfrak{S}_n$ .

It is invariant by braids if for any braid  $\beta \in \mathcal{B}_n$ , one has the equality in law:

$$\beta \bullet (X_1, \dots, X_n) = (X_1, \dots, X_n).$$

One of the main concept we will need in the following is the notion of support and various notions of invariance by conjugation. Let  $G$  be an arbitrary topological group. If  $m$  is a probability measure on  $G$ , the support of  $m$  is the smallest closed subset of  $G$  of measure 1 for  $m$ . It will be denoted by  $\text{Supp}(m)$ . The closure of the subgroup generated by the support of  $m$  is denoted by  $H_m$ .

If  $X$  is a  $G$ -valued random variable, and  $m$  its law, we define  $\text{Supp}(X) = \text{Supp}(m)$  and  $H_X = H_m$ .

**Definition 51.** — Let  $T$  be a finite index set of cardinal strictly greater than 1. Let  $(X_t)_{t \in T}$  be a sequence of  $G$ -valued random variables. We say that  $(X_t)_{t \in T}$  is auto-invariant by conjugation if for any different elements  $i$  and  $j$  in  $T$ , and for any  $g \in \text{Supp}(X_j)$ , we have the equality in law:

$$(15) \quad gX_i g^{-1} = X_i.$$

This definition can be extended to collections of measures on  $G$ .

**Proposition 20.** — A finite sequence of independent  $G$ -valued random variables is auto-invariant by conjugation if and only if it is purely invariant by braids.

*Proof.* — Let  $(X_1, \dots, X_n)$  be a finite sequence of  $G$ -valued random variables.

Let us suppose that  $(X_1, \dots, X_n)$  is auto-invariant by conjugation and let us prove that it is purely invariant by braids. As  $\beta \mapsto \sigma_\beta$  is a morphism, and using the independence of the variables, we just have to show that  $(X_2, X_2^{-1}X_1X_2)$  and  $(X_2, X_1)$  have the same law. We recall that the permutation associated with  $\beta_1$  is the transposition  $(1, 2)$ . This result follows from the invariance by conjugation of the law of  $X_1$  by any element  $g$  in the support of  $X_2$ .

Now, let us suppose that  $(X_1, \dots, X_n)$  is purely invariant by braids and let us prove that it is auto-invariant by conjugation. Let  $i$  and  $j$  be two distinct integers in  $\{1, \dots, n\}$ . Let us suppose that  $i < j$ . Let  $\beta_{(i,j)}$  be the braid defined by:

$$\beta_{(i,j)} = \beta_i^{-1} \dots \beta_{j-2}^{-1} \beta_{j-1} \dots \beta_i.$$

An example of such a braid is shown in Figure 17. We have the following equality in law:

$$(X_i, X_j) = (X_i, X_i X_j X_i^{-1}).$$

By disintegration and using the independence of the variables, one gets the desired result. □

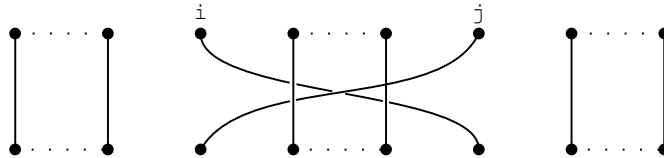


FIGURE 17. The braid  $\beta_{(i,j)}$ .

The proof of Proposition 20 is straightforward, but looking at the following equality in law:

$$(X_2^{-1}X_1X_2, X_2^{-1}X_1^{-1}X_2X_1X_2) = (X_1, X_2),$$

where  $(X_1, X_2)$  is an auto-invariant by conjugation couple of random variables, one can see that it gives identities which, at first glance, do not seem trivial. A last important remark to be made about Proposition 20 is that there exist finite sequences of non-independent  $G$ -valued random variables which are purely invariant by braids.

## 8. Planar Yang-Mills fields

**8.1. Construction of pure  $G$ -valued planar Yang-Mills fields.** — First of all, we recall the definition of Lévy processes that we introduced in Section 1.1.

**Definition 52.** — A Lévy process  $(Z_t)_{t \geq 0}$  is a random process from  $\mathbb{R}^+$  to  $G$ , left continuous with right limit with independent and stationary left increments. This means:

- $\forall t_0 < \dots < t_n, (Z_{t_i}Z_{t_{i-1}}^{-1})_{i=1}^n$  are independent,
- $\forall s < t, Z_tZ_s^{-1}$  has the same law as  $Z_{t-s}$ .

We say that  $(Z_t)_{t \geq 0}$  is invariant by conjugation by  $G$ , or conjugation-invariant, if and only if for any  $g \in G$ , the process  $(g^{-1}Z_tg)_{t \geq 0}$  has the same law as  $(Z_t)_{t \geq 0}$ .

There is a correspondence between continuous semi-groups of convolution of probability measures and Lévy processes. The goal of this section is to construct a family of Planar Markovian Holonomy Fields called the planar Yang Mills fields. For this, we are going to construct in Proposition 21, for any finite planar graph  $\mathbb{G}$ , an holonomy field on  $\mathbb{G}$  associated with any Lévy process which is invariant by conjugation by  $G$ . In Propositions 22 and 24, we show that these holonomy fields allow us to define an holonomy field on  $\mathbb{R}^2$  which is a strong Planar Markovian Holonomy Field. In Section 8.2, we weaken the condition on the Lévy process by using our past discussion about the extension of the structure group.

**Proposition 21.** — Let  $\mathbb{G}$  be a finite graph on  $\mathbb{R}^2$ . Let  $\text{vol}$  be a measure of area. Let  $Y = (Y_t)_{t \geq 0}$  be a Lévy process on  $G$  invariant by conjugation. There exists a unique holonomy field  $\mathbb{E}_{\text{vol}}^{Y, \mathbb{G}}$  on  $\mathbb{G}$  such that for any rooted spanning tree  $T$  of  $\mathbb{G}$ , any family  $(c_F)_{F \in \mathbb{F}^b}$  of facial loops of  $\mathbb{G}$ , each oriented anti-clockwise, under  $\mathbb{E}_{\text{vol}}^{Y, \mathbb{G}}$ :

1. the random variables  $(h(\mathbf{l}_{c_F, T}))_{F \in \mathbb{F}^b}$  are independent,
2. for any  $F \in \mathbb{F}^b$ ,  $h(\mathbf{l}_{c_F, T})$  has the same law as  $Y_{\text{vol}(F)}$ .

*Proof.* — Let  $\mathbb{G}$  be a finite planar graph. For any rooted spanning tree  $T$ , and any family  $(c_F)_{F \in \mathbb{F}^b}$  of facial loops oriented anti-clockwise, we define the measure

$\mathbb{E}_{vol,T,(c_F)_{F \in \mathbb{F}^b}}^{Y,\mathbb{G}}$  on  $(\text{Mult}(P(\mathbb{G}), G), \mathcal{B})$  as the unique gauge-invariant measure such that, under  $\mathbb{E}_{vol,T,(c_F)_{F \in \mathbb{F}^b}}^{Y,\mathbb{G}}$ :

1. the random variables  $(h(\mathbf{l}_{c_F,T}))_{F \in \mathbb{F}^b}$  are independent,
2. for any  $F \in \mathbb{F}^b$ ,  $h(\mathbf{l}_{c_F,T})$  has the same law as  $Y_{vol(F)}$ .

The definition makes sense by applying the first part of Proposition 18, as  $\otimes_{F \in \mathbb{F}^b} m_{vol(F)}$  is invariant by diagonal conjugation. We will show that the probability measure  $\mathbb{E}_{vol,T,(c_F)_{F \in \mathbb{F}^b}}^{Y,\mathbb{G}}$  neither depends on the choice of  $T$ , nor on the choice of  $(c_F)_{F \in \mathbb{F}^b}$ . Thanks to the uniqueness property in this last definition, we have to prove that given another rooted spanning tree  $T'$  and another family of facial loops  $(c'_F)_{F \in \mathbb{F}^b}$  oriented anti-clockwise, under  $\mathbb{E}_{vol,T,(c_F)_{F \in \mathbb{F}^b}}^{Y,\mathbb{G}}$ ,  $(h(\mathbf{l}_{c'_F,T'}))_{F \in \mathbb{F}^b}$  has the same law as  $(h(\mathbf{l}_{c_F,T}))_{F \in \mathbb{F}^b}$ .

First of all, we prove that one can suppose that  $T$  and  $T'$  are rooted at the same vertex  $v$  of  $\mathbb{G}$ . Let  $v$  be the root of  $T$ , let  $v'$  a vertex of  $\mathbb{G}$  and let  $\tilde{T}$  be the tree  $T$  rooted at  $v'$ . When we change the root of  $T$  from  $v$  to  $v'$  we conjugate every of the  $\mathbf{l}_{c_F,T}$  by the same path  $[v',v]_T$ . By Remark 7, as  $\mathbb{E}_{vol,T,(c_F)_{F \in \mathbb{F}^b}}^{Y,\mathbb{G}}$  is gauge-invariant, this does not change the law of  $h(\mathbf{l}_{c_F,T})$ . Thus, under  $\mathbb{E}_{vol,T,(c_F)_{F \in \mathbb{F}^b}}^{Y,\mathbb{G}}$ ,  $(h(\mathbf{l}_{c_F,\tilde{T}}))_{F \in \mathbb{F}^b}$  has the same law as  $(h(\mathbf{l}_{c_F,T}))_{F \in \mathbb{F}^b}$ , namely  $\otimes_{F \in \mathbb{F}^b} m_{vol(F)}: \mathbb{E}_{vol,T,(c_F)_{F \in \mathbb{F}^b}}^{Y,\mathbb{G}} = \mathbb{E}_{vol,\tilde{T},(c_F)_{F \in \mathbb{F}^b}}^{Y,\mathbb{G}}$ .

Now let us assume that  $T$  and  $T'$  are rooted at the same vertex. By Proposition 19, there exists an enumeration  $(F_i)_{i=1}^{\#\mathbb{F}^b}$  of the bounded faces of  $\mathbb{G}$ , a braid  $\beta$  in  $\mathcal{B}_{\#\mathbb{F}^b}$  such that:

$$\beta \bullet \left( \mathbf{l}_{c_{F_i},T} \right)_{i=1}^{\#\mathbb{F}^b} = \left( \mathbf{l}_{c'_{F_{\sigma_\beta(i)}},T'} \right)_{i=1}^{\#\mathbb{F}^b}.$$

Using Lemma 15,  $h \left( \beta \bullet \left( \mathbf{l}_{c_{F_i},T} \right)_{i=1}^{\#\mathbb{F}^b} \right) = \beta^{-1} \bullet \left( h(\mathbf{l}_{c_{F_i},T}) \right)_{i=1}^{\#\mathbb{F}^b}$ , and thus:

$$\beta^{-1} \bullet \left( h(\mathbf{l}_{c_{F_i},T}) \right)_{i=1}^{\#\mathbb{F}^b} = \sigma_{\beta^{-1}} \bullet \left( h(\mathbf{l}_{c'_{F_i},T'}) \right)_{i=1}^{\#\mathbb{F}^b}.$$

Applying the Proposition 20, under  $\mathbb{E}_{vol,T,(c_F)_{F \in \mathbb{F}^b}}^{Y,\mathbb{G}}$ , the following equality in law holds:

$$\beta^{-1} \bullet \left( h(\mathbf{l}_{c_{F_i},T}) \right)_{i=1}^{\#\mathbb{F}^b} = \sigma_{\beta^{-1}} \bullet \left( h(\mathbf{l}_{c_{F_i},T}) \right)_{i=1}^{\#\mathbb{F}^b}.$$

From this, we get the equality in law under  $\mathbb{E}_{vol,T,(c_F)_{F \in \mathbb{F}^b}}^{Y,\mathbb{G}}$ :

$$\left( h(\mathbf{l}_{c'_{F_i},T'}) \right)_{i=1}^{\#\mathbb{F}^b} = \left( h(\mathbf{l}_{c_{F_i},T}) \right)_{i=1}^{\#\mathbb{F}^b},$$

which is what we expected.  $\square$

This proposition allows us not to have to choose a special rooted tree for each graph in order to construct planar Yang-Mills fields. More importantly, it will allow us to show the independence property and the area-preserving diffeomorphism invariance of the family of random holonomy fields which we will construct thanks to Proposition 18.

**Proposition 22.** — *Let  $(Y_t)_{t \geq 0}$  be a  $G$ -valued Lévy process invariant by conjugation. There exists a unique family of gauge-invariant stochastically continuous random holonomy fields  $(\mathbb{E}_{vol}^Y)_{vol}$  of weight equal to 1, such that for any finite planar graph  $\mathbb{G}$ , for any rooted spanning tree  $T$  of  $\mathbb{G}$ , and any family of facial loops  $(c_F)_{F \in \mathbb{F}^b}$  oriented anti-clockwise, under  $\mathbb{E}_{vol}^Y$ :*

1. *the random variables  $(h(\mathbf{1}_{c_F, T}))_{F \in \mathbb{F}^b}$  are independent,*
2. *for any  $F \in \mathbb{F}^b$ ,  $h(\mathbf{1}_{c_F, T})$  has the same law as  $Y_{vol(F)}$ .*

*The family  $(\mathbb{E}_{vol}^Y)_{vol}$  is the discrete planar Yang-Mills field associated with  $(Y_t)_{t \geq 0}$ .*

In order to prove this result, we will use the following statement, from [21], which allows us to bound the distance of a Lévy process to the neutral element.

**Proposition 23.** — *Let  $(Y_t)_{t \geq 0}$  be a Lévy process on a compact Lie group  $G$ . Then there exists a constant  $K$  such that:*

$$\forall t \geq 0, \mathbb{E}[d_G(1, Y_t)] \leq K\sqrt{t}.$$

*Proof of Proposition 22.* — Let  $vol$  be a measure of area on the plane. Let  $m$  be a point in  $\mathbb{R}^2$ . In order to prove the result, we will apply Proposition 18 to the family of measures  $(\mathbb{E}_{vol}^{Y, \mathbb{G}})_{\mathbb{G}}$  and  $\gamma = \gamma_0$ . Then we will extend the result to finite planar graphs. For this extent, we have to prove a compatibility condition and a uniform locally stochastically  $\frac{1}{2}$ -Hölder continuity property of the family  $(\mathbb{E}_{vol}^{Y, \mathbb{G}})_{\mathbb{G} \in \mathcal{G}(\text{Aff}(\mathbb{R}^2))}$ .

**Compatibility condition :** Let us consider  $\mathbb{G}_1$  and  $\mathbb{G}_2$  two graphs in  $\mathcal{G}(\text{Aff}_{\gamma_0})$  which have  $m$  as a vertex and such that  $\mathbb{G}_1 \preceq \mathbb{G}_2$ . Using Proposition 21, it is enough to show that  $\mathbb{G}_1$  satisfies the following property:

$$(\mathcal{H}) \left\{ \begin{array}{l} \text{there exists a family of facial loops } (c_F)_{F \in \mathbb{F}_1} \text{ oriented anti-clockwise and} \\ \text{a spanning tree } T_1 \text{ of } \mathbb{G}_1 \text{ rooted at } m, \text{ such that under } \mathbb{E}_{vol}^{Y, \mathbb{G}_2}, \\ 1. \text{ the random variables } (h(\mathbf{1}_{c_F, T}))_{F \in \mathbb{F}_1^b} \text{ are independent,} \\ 2. \text{ for any } F \in \mathbb{F}_1^b, h(\mathbf{1}_{c_F, T}) \text{ has the same law as } Y_{vol(F)}. \end{array} \right.$$

We show this by a kind of induction argument on the finite set  $[\mathbb{G}_1, \mathbb{G}_2] = \{\mathbb{G}, \mathbb{G}_1 \preceq \mathbb{G} \preceq \mathbb{G}_2\}$  endowed with the partial order  $\preceq$ .

It is clearly true that  $(\mathcal{H})$  holds for  $\mathbb{G} = \mathbb{G}_2$ . Consider a finite planar graph  $\mathbb{G}$  in  $[\mathbb{G}_1, \mathbb{G}_2]$  satisfying  $(\mathcal{H})$ , we will show that there exists  $\mathbb{G}' \in [\mathbb{G}_1, \mathbb{G}]$  for which  $(\mathcal{H})$  is still valid. Thanks to Proposition 21, property  $(\mathcal{H})$  will

hold for  $\mathbb{G}$  for any family of facial loops  $(c_F)_{F \in \mathbb{F}}$  oriented anti-clockwise, and any choice of spanning tree  $T$  rooted at  $m$ . As  $\mathbb{G}_1 \preceq \mathbb{G}$ , one at least of the following assertions is true:

1. there exist an edge of  $\mathbb{G}_1$ ,  $e$ , and a vertex  $v$  of  $\mathbb{G}$  of degree two such that  $v \in e((0, 1))$ ,
2. there exists a face  $F_1$  of  $\mathbb{G}_1$ , bounded or not, such that the restriction of  $\mathbb{G}$  to  $F_1$  has a unique face  $F_0$  and  $\partial F_0$  contains a sequence of the form  $ee^{-1}$  with the interior of  $e$  included in  $F_1$ ,
3. there exists a face  $F_1$  of  $\mathbb{G}_1$  which contains more than one face of  $\mathbb{G}$ .

Let us consider the three possibilities.

1. Consider a family of facial loops  $(c_F)_{F \in \mathbb{F}}$  for  $\mathbb{G}$ , oriented anti-clockwise, and a choice of spanning tree  $T$  of  $\mathbb{G}$  rooted at  $m$ . Let  $e_1$  and  $e_2$  be the two edges of  $\mathbb{G}$  such that  $e = e_1e_2$  and  $\bar{e}_1 = v$ . We consider  $\mathbb{G}'$ , the graph defined by:

$$(\mathbb{V}', \mathbb{E}', \mathbb{F}') = \left( \mathbb{V} \setminus \{v\}, \mathbb{E} \setminus \{e_1^{\pm 1}, e_2^{\pm 1}\} \cup \{(e_1e_2)^{\pm 1}\}, \mathbb{F} \right).$$

By construction  $\mathbb{G}' \in [\mathbb{G}_1, \mathbb{G}]$ . Besides,  $(c_F)_{F \in \mathbb{F}'}$  is still a family of facial loops for  $\mathbb{G}'$  oriented anti-clockwise and  $T' = \left( T \setminus \{e_1^{\pm 1}, e_2^{\pm 1}\} \right) \cup \{(e_1e_2)^{\pm 1}\}$  is a spanning tree of  $\mathbb{G}'$  rooted at  $m$ . It is now obvious that  $\mathbb{G}'$  satisfies property  $(\mathcal{H})$  with the choices of  $(c_F)_{F \in \mathbb{F}'}$  and  $T'$ .

2. We will consider that  $F_1$  is bounded, the unbounded case is similar. In this case, let  $v$  be the vertex of  $e$  of degree 1 and define  $F'_0 = F_0 \cup e((0, 1)) \cup \{v\}$ . Consider any family of facial loops for  $\mathbb{G}$  oriented anti-clockwise,  $(c_F)_{F \in \mathbb{F}}$ , such that  $c_{F_0} \neq v$ . Let us choose any spanning tree of  $\mathbb{G}$  rooted at  $m$ ,  $T$ . We consider  $\mathbb{G}'$ , the graph defined by:

$$(\mathbb{V}', \mathbb{E}', \mathbb{F}') = \left( \mathbb{V} \setminus v, \mathbb{E} \setminus \{e, e^{-1}\}, (\mathbb{F} \setminus F_0) \cup F'_0 \right).$$

The spanning tree  $T$  of  $\mathbb{G}$  must include the unoriented edge  $\{e, e^{-1}\}$  in order to cover  $v$ , thus we can define  $T' = T \setminus \{e, e^{-1}\}$ . The facial loop  $c_{F_0}$  contains the sequence  $ee^{-1}$ . We define  $c'_{F'_0}$  from  $c_{F_0}$  by removing this sequence. For any other face  $F \in \mathbb{F}'$ , we set  $c'_F = c_F$ . For any face  $F \in \mathbb{F}'$ , using the identification between  $F_0$  and  $F'_0$ ,  $\mathbf{l}_{c_F, T} = \mathbf{l}_{c'_F, T'}$  in  $RL_m(\mathbb{G})$ , and by Remark 1,  $h(\mathbf{l}_{c_F, T}) = h(\mathbf{l}_{c'_F, T'})$ . The graph  $\mathbb{G}'$  satisfies property  $(\mathcal{H})$  with the choices of  $(c'_{F'})_{F' \in \mathbb{F}'}$  and  $T'$ .

3. We will study this case under the hypothesis that  $F_1$  is bounded, the unbounded case being very similar. The key point will be the semigroup property satisfied by the marginal distributions of Lévy processes. Let  $F_r$  and  $F_l$  be two faces of  $\mathbb{G}$  contained in  $F_1$  and adjacent, sharing an

edge  $e$  on their boundaries. We can find a facial loop oriented anti-clockwise representing the boundary of  $F_r$  (resp.  $F_l$ ) of the form  $c_{F_r} = e_1 \dots e_n e$  (resp.  $c_{F_l} = e^{-1} e'_1 \dots e'_m$ ). We define  $F_{r,l} = F_r \cup F_l \cup e((0, 1))$ . We complete the family  $(c_{F_r}, c_{F_l})$  in order to have a family of facial loops  $(c_F)_{F \in \mathbb{F}}$  oriented anti-clockwise. We consider then  $\mathbb{G}'$ , the graph defined by:

$$(\mathbb{V}', \mathbb{E}', \mathbb{F}') = \left( \mathbb{V}, \mathbb{E} \setminus \{e, e^{-1}\}, (\mathbb{F} \setminus \{F_r, F_l\}) \cup F_{r,l} \right).$$

It is still a finite planar graph, and we can consider  $T$  any spanning tree of  $\mathbb{G}'$  rooted at  $m$ : it is also a spanning tree of  $\mathbb{G}$  rooted at  $m$ . We then define  $c'_{F_{r,l}} = e_1 \dots e_n e'_1 \dots e'_m$ . For any other face  $F'$  of  $\mathbb{G}'$  different from  $F_{r,l}$ , we set  $c'_{F'} = c_F$ . Once these choices made, it needs only a simple verification to check that the following equalities hold in  $RL_m(\mathbb{G})$ :

$$\begin{aligned} \mathbf{l}_{c_{F_{r,l}}, T} &= \mathbf{l}_{c_{F_r}, T} \mathbf{l}_{c_{F_l}, T}, \\ \mathbf{l}_{c'_F, T} &= \mathbf{l}_{c_F, T}, \forall F \in \mathbb{F}', F \neq F_{r,l}. \end{aligned}$$

Thus, using the multiplicativity of  $h$ :

$$\begin{aligned} h(\mathbf{l}_{c_{F_{r,l}}, T}) &= h(\mathbf{l}_{c_{F_l}, T}) \cdot h(\mathbf{l}_{c_{F_r}, T}), \\ h(\mathbf{l}_{c'_F, T}) &= h(\mathbf{l}_{c_F, T}), \forall F \in \mathbb{F}', F \neq F_{r,l}. \end{aligned}$$

Using the fact that under  $\mathbb{E}_{vol}^{Y, \mathbb{G}_2}$ ,

(a) the random variables  $(h(\mathbf{l}_{c_F, T}))_{F \in \mathbb{F}^b}$  are independent,

(b) for any  $F \in \mathbb{F}^b$ ,  $h(\mathbf{l}_{c_F, T})$  has the same law as  $Y_{vol(F)}$ ,

and using the semigroup property of the marginal distributions of the process  $Y$ , we can conclude that  $\mathbb{G}'$  satisfies  $(\mathcal{H})$  with the choices of  $(c'_F)_{F' \in \mathbb{F}'^b}$  and  $T$ .

By descending induction, it follows that  $\mathbb{G}_1$  satisfies property  $(\mathcal{H})$ .

**Uniform  $\frac{1}{2}$ -Hölder continuity :** Let  $\mathbb{G}$  be a finite planar graph. Let  $l$  be a simple loop in  $\mathbb{G}$  bounding a disk  $D$ . A consequence of what we have just seen is that the law of  $h(l)$  under  $\mathbb{E}_{vol}^{Y, \mathbb{G}}$  is the same as under  $\mathbb{E}_{vol}^{Y, \mathbb{G}(l)}$ , where  $\mathbb{G}(l)$  is the graph containing only the edge  $l$  (see Example 1). Thus:

$$\begin{aligned} \int_{\text{Mult}(P(\mathbb{G}), G)} d_G(1, h(l)) \mathbb{E}_{vol}^{Y, \mathbb{G}}(dh) &= \int_{\text{Mult}(P(\mathbb{G}(l)), G)} d_G(1, h(l)) \mathbb{E}_{vol}^{Y, \mathbb{G}(l)}(dh) \\ &= \mathbb{E} [d_G(1, Y_{vol(D)})] \leq K \sqrt{vol(D)}, \end{aligned}$$

where the last inequality comes from Proposition 23, and where  $K$  depends only on  $G$ . The family  $(\mathbb{E}_{vol}^{Y, \mathbb{G}})_{vol, \mathbb{G}}$  is uniformly locally stochastically  $\frac{1}{2}$ -Hölder continuous.

We can thus apply Proposition 18 in order to construct a random stochastically continuous holonomy field  $\mathbb{E}_{vol}^Y$  such that for any finite planar graph  $\mathbb{G} \in \mathcal{G}(\text{Aff}_\gamma(\mathbb{R}^2))$ , for any rooted spanning tree  $T$  of  $\mathbb{G}$ , and any family of facial loops  $(c_F)_{F \in \mathbb{F}^b}$  oriented anti-clockwise, under  $\mathbb{E}_{vol}^Y$ :

1. the random variables  $(h(\mathbf{l}_{c_F, T}))_{F \in \mathbb{F}^b}$  are independent,
2. for any  $F \in \mathbb{F}^b$ ,  $h(\mathbf{l}_{c_F, T})$  has the same law as  $Y_{vol(F)}$ .

We now have to prove that this property is true for any finite planar graph not necessarily with affine edges. Let  $(m_t)_{t \in \mathbb{R}^+}$  be the continuous semi-group of convolution associated with  $(Y_t)_{t \geq 0}$ . Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a finite planar graph. Let  $T$  be a rooted covering tree and let  $(c_F)_{F \in \mathbb{F}^b}$  be a family of facial loops oriented anti-clockwise. Let us consider a sequence of finite planar graph  $(\mathbb{G}_n = (\mathbb{V}_n, \mathbb{E}_n, \mathbb{F}_n))_{n \in \mathbb{N}}$  in  $\mathcal{G}(\text{Aff}_\gamma)$  and  $(\psi_n)_{n \in \mathbb{N}}$  a sequence of homeomorphisms which satisfies the conditions of Theorem 3. For any integer  $n$ ,  $(\psi_n(c_F))_{F \in \mathbb{F}^b}$  is a family of facial loops for  $\mathbb{G}_n$  which is oriented anti-clockwise, and  $\psi_n(T)$  is a spanning tree of  $\mathbb{G}_n$ . Using the discussion we had before, the law of  $(h(\mathbf{l}_{\psi_n(c_F), \psi_n(T)}))_{F \in \mathbb{F}^b}$  under  $\mathbb{E}_{vol}^Y$  is  $\bigotimes_{F \in \mathbb{F}^b} m_{vol(\psi_n(F))}$ . As for any edge  $e \in \mathbb{E}$ ,  $(\psi_n(e))_{n \geq 0}$  converges to  $e$  for the convergence with fixed endpoints, for any face  $F \in \mathbb{F}^b$ , one has:

$$\mathbf{l}_{\psi_n(c_F), \psi_n(T)} \xrightarrow[n \rightarrow \infty]{} \mathbf{l}_{c_F, T}$$

for the fixed endpoint convergence notion. Besides, using condition 4 of Theorem 3, and the continuity of  $(m_t)_{t \in \mathbb{R}^+}$ ,

$$\bigotimes_{F \in \mathbb{F}^b} m_{vol(\psi_n(F))} \xrightarrow[n \rightarrow \infty]{} \bigotimes_{F \in \mathbb{F}^b} m_{vol(F)}.$$

Thus, under  $\mathbb{E}_{vol}^Y$ , the law of  $(\mathbf{l}_{c_F, T})_{F \in \mathbb{F}^b}$  is  $\bigotimes_{F \in \mathbb{F}^b} m_{vol(F)}$ .  $\square$

Let us remark that, in the latest argument, we actually proved that the family  $(\mathbb{E}_{vol}^{Y, \mathbb{G}})_{\mathbb{G}}$  was continuously area-dependent.

Thus for any Lévy process which is invariant by conjugation, we have constructed a family of gauge-invariant stochastically continuous random holonomy fields. In the following, we will show that this family is a Planar Markovian Holonomy Field.

**Proposition 24.** — *Let  $(Y_t)_{t \geq 0}$  be a  $G$ -valued Lévy process invariant by conjugation. The holonomy field  $(\mathbb{E}_{vol}^Y)_{vol}$  is a strong stochastically continuous planar constructible Markovian Holonomy Field.*

*Proof.* — We have already show that the family  $(\mathbb{E}_{vol}^{Y, \mathbb{G}})_{vol, \mathbb{G}}$  is continuously area-dependent and locally stochastically  $\frac{1}{2}$ -Hölder continuous. By Theorem 5, it remains to check that  $(\mathbb{E}_{vol}^{Y, \mathbb{G}})_{vol, \mathbb{G}}$  satisfies the three axioms **DP**<sub>1</sub>, **DP**<sub>2</sub> and **DP**<sub>3</sub> in



Definition 36. Besides, using Proposition 22,  $(\mathbb{E}_{vol}^{Y,\mathbb{G}})_{vol}$  is stochastically continuous in law: it is enough, using a continuity argument, to show that  $\mathbf{wDP}_2$  holds instead of  $\mathbf{DP}_2$ .

**DP<sub>1</sub>** : Consider  $vol$  and  $vol'$  two measures of area on  $\mathbb{R}^2$ . Let  $\mathbb{G}$  and  $\mathbb{G}'$  be two finite planar graphs. Let  $\psi$  be a diffeomorphism which preserves the orientation. Let us suppose that  $\psi(\mathbb{G}) = \mathbb{G}'$ , and for any  $F \in \mathbb{F}^b$ ,  $vol(F) = vol'(\psi(F))$ . Let  $(c'_F)_{F \in \mathbb{F}^b}$  be a family of facial loops oriented anti-clockwise for  $\mathbb{G}'$  and let  $T'$  be a rooted spanning tree of  $\mathbb{G}'$ . We consider  $(c_F = \psi^{-1}(c'_{\psi(F)}))_{F \in \mathbb{F}^b}$  and  $T = \psi^{-1}(T')$ . The family  $(c_F)_{F \in \mathbb{F}^b}$  is a family of facial loops for  $\mathbb{G}$  which is oriented anti-clockwise and  $T$  is a rooted spanning tree of  $\mathbb{G}$ . Then, by construction and unicity, we have the equality:

$$\mathbb{E}_{vol', T', (c'_F)_{F \in \mathbb{F}^b}}^{Y, \mathbb{G}'} \circ \psi^{-1} = \mathbb{E}_{vol, T, (c_F)_{F \in \mathbb{F}^b}}^{Y, \mathbb{G}},$$

where  $\psi$  is the induced application from  $\mathcal{Mult}(P(\mathbb{G}'), G)$  to  $\mathcal{Mult}(P(\mathbb{G}), G)$  induced by  $\psi$ . By Proposition 21,

$$\mathbb{E}_{vol'}^{Y, \mathbb{G}'} \circ \psi^{-1} = \mathbb{E}_{vol}^{Y, \mathbb{G}}.$$

**wDP<sub>2</sub>** : Let  $vol$  be a measure of area on  $\mathbb{R}^2$ . Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a finite planar graph which has  $m$  as vertex. Let  $L_1$  and  $L_2$  be two simple loops in  $P(\mathbb{G})$  whose closure of the interiors are disjoint. As an application of Proposition 16, we can consider  $T$  a spanning tree rooted at  $m$ , such that for any family of facial loops  $(c_F)_{F \in \mathbb{F}}$  oriented anti-clockwise:

1. for every loop  $l$  inside  $L_1$ ,  $\left[ [m, l]_T l [m, l]_T^{-1} \right]_{\simeq}$  is a product in  $RL_m(\mathbb{G})$  of elements of  $\left\{ \mathbf{1}_{c_F, T}^{\pm 1}; F \in \mathbb{F}^b, F \subset \overline{\text{Int}(L_1)} \right\}$ ,
2. for every loop  $l$  inside  $L_2$ ,  $\left[ [m, l]_T l [m, l]_T^{-1} \right]_{\simeq}$  is a product in  $RL_m(\mathbb{G})$  of elements of  $\left\{ \mathbf{1}_{c_F, T}^{\pm 1}; F \in \mathbb{F}^b, F \subset \overline{\text{Int}(L_2)} \right\}$ .

Let  $(p_i)_{i=1}^n$  (resp.  $(p'_j)_{j=1}^{n'}$ ) be paths inside  $L_1$  (resp.  $L_2$ ). Recall the definition given by the Equality 6. For any continuous function  $f_1$  (resp.  $f_2$ ) defined on  $G^n$  (resp. on  $G^{n'}$ ), using the gauge-invariance, we have that  $\mathbb{E}_{vol}^{Y, \mathbb{G}} \left[ f_1 \left( (h(p_i))_{i=1}^n \right) f_2 \left( (h(p'_j))_{j=1}^{n'} \right) \right]$  is equal to:

$$\mathbb{E}_{vol}^{Y, \mathbb{G}} \left[ \hat{f}_1_{J_{p_1, \dots, p_n}} (h(p_1), \dots, h(p_n)) \hat{f}_2_{J_{p'_1, \dots, p'_{n'}}} (h(p'_1), \dots, h(p'_{n'})) \right].$$

Thus, it is also equal to:

$$\mathbb{E}_{vol}^{Y, \mathbb{G}} \left[ \hat{f}_1_{J_{p_1, \dots, p_n}} \left( \left( h(\tilde{l}_i) \right)_{i=1}^n \right) \hat{f}_2_{J_{p'_1, \dots, p'_{n'}}} \left( \left( h(\tilde{l}'_i) \right)_{i=1}^{n'} \right) \right].$$

where, for any  $i \in \{1, \dots, n\}$ ,  $\tilde{l}_i = [m, \underline{p}_i]_T p_i [m, \overline{p}_i]_T$  and for any  $i \in \{1, \dots, n'\}$ ,  $\tilde{l}'_i = [m, \underline{p}'_i]_T p'_i [m, \overline{p}'_i]_T$ . Recall the form of  $T$ : this implies that there exist  $(l_i)_{i=1}^n$  some loops in  $\overline{\text{Int}(L_1)}$  and  $(l'_i)_{i=1}^{n'}$  some loops in  $\overline{\text{Int}(L_2)}$  such that for any  $i \in \{1, \dots, n\}$ , and any  $j \in \{1, \dots, n'\}$ ,

$$\begin{aligned}\tilde{l}_i &= [m, \overline{l}_i]_T l_i [m, \overline{l}_i]_T^{-1}, \\ \tilde{l}'_j &= [m, \overline{l}'_j]_T l'_j [m, \overline{l}'_j]_T^{-1},\end{aligned}$$

in  $RL_m(\mathbb{G})$ . Using the properties satisfied by  $T$ , the following assertions hold:

$$\begin{aligned}\sigma\left(\left(h(\tilde{l}_i)\right)_{i=1}^n\right) &\subset \sigma\left(\left\{h(\mathbf{l}_{c_F, T}); F \in \mathbb{F}^b, F \subset \overline{\text{Int}(L_1)}\right\}\right), \\ \sigma\left(\left(h(\tilde{l}'_i)\right)_{i=1}^{n'}\right) &\subset \sigma\left(\left\{h(\mathbf{l}_{c_F, T}); F \in \mathbb{F}^b, F \subset \overline{\text{Int}(L_2)}\right\}\right).\end{aligned}$$

We recall that  $\overline{\text{Int}(L_1)} \cap \overline{\text{Int}(L_2)} = \emptyset$ , thus the two  $\sigma$ -fields:

$$\begin{aligned}\sigma\left(\left\{h(\mathbf{l}_{c_F, T}); F \in \mathbb{F}^b, F \subset \overline{\text{Int}(L_1)}\right\}\right), \\ \sigma\left(\left\{h(\mathbf{l}_{c_F, T}); F \in \mathbb{F}^b, F \subset \overline{\text{Int}(L_2)}\right\}\right)\end{aligned}$$

are independent under  $\mathbb{E}_{vol}^{Y, \mathbb{G}}$ . Thus  $\mathbb{E}_{vol}^{Y, \mathbb{G}} \left[ f_1 \left( (h(p_i))_{i=1}^n \right) f_2 \left( (h(p'_j))_{j=1}^{n'} \right) \right]$  is equal to:

$$\mathbb{E}_{vol}^{Y, \mathbb{G}} \left[ \hat{f}_{1, J_{p_1, \dots, p_n}} \left( (h(\tilde{l}_i))_{i=1}^n \right) \right] \mathbb{E}_{vol}^{Y, \mathbb{G}} \left[ \hat{f}_{2, J_{p'_1, \dots, p'_{n'}}} \left( (h(\tilde{l}'_i))_{i=1}^{n'} \right) \right],$$

thus **wDP<sub>2</sub>** is satisfied.

**DP<sub>3</sub>** : Let  $l$  be a simple loop, let  $vol$  and  $vol'$  be two measures of area on  $\mathbb{R}^2$  which agree on the interior of  $l$ . Let  $\mathbb{G}$  be a finite planar graph included in the closure of the interior of  $l$ . The bounded faces of  $\mathbb{G}$  are in the interior of  $l$  thus for any bounded face  $F$  of  $\mathbb{G}$  it is true that  $vol(F) = vol'(F)$ : by definition, it is clear that  $\mathbb{E}_{vol}^{Y, \mathbb{G}} = \mathbb{E}_{vol'}^{Y, \mathbb{G}}$ .

We have proved all we needed in order to apply Theorem 5 and thus to assert that the holonomy field  $(\mathbb{E}_{vol}^Y)_{vol}$  is a strong planar constructible Markovian Holonomy Field.  $\square$

For this construction, we used the link between the multiplicative functions on a set  $P$  and the pre-multiplicative functions on its set of loops: the loop paradigm. The edge paradigm (1) can be used to give an explicit formula for discrete planar Yang-Mills fields associated with a conjugation invariant Lévy process with density.

**Proposition 25.** — *Let  $(Y_t)_{t \geq 0}$  be a  $G$ -valued Lévy process invariant by conjugation. Let us suppose that for any positive real  $t$ ,  $Y_t$  has a density  $Q_t$  with respect to the Haar measure. Let  $(\mathbb{E}_{vol}^{Y, \mathbb{G}})_{\mathbb{G}, vol}$  be the pure discrete planar Yang-Mills field*

associated with  $(Y_t)_{t \geq 0}$ . For any finite planar graph  $\mathbb{G}$  and for any measure of area  $vol$ :

$$(16) \quad \left( \mathbb{E}_{vol}^{Y, \mathbb{G}} \right)_{\mathbb{G}, vol} (dh) = \prod_{F \in \mathbb{F}^b} Q_{vol(F)}(h(\partial F)) \bigotimes_{e \in \mathbb{E}^+} dh(e),$$

with  $\bigotimes_{e \in \mathbb{E}^+} dh(e)$  being the push forward of  $\bigotimes_{e \in \mathbb{E}^+} dg_e$  on  $Mult(P, G)$  by the edge paradigm identification. It is independent of the choice of orientation  $\mathbb{E}^+$ .

Recall the definition of  $L_{i,j}$  in Definition 45. In order to make the proof simple, we will use Theorem 22 which asserts that a stochastically continuous Markovian Holonomy Field is characterized by the law of the random sequence  $(h(L_{n,0}))_{n \in \mathbb{N}}$ .

*Proof.* — A slight modification of Section 4.3 in [21] allows us to show that:

$$(17) \quad \left( \prod_{F \in \mathbb{F}^b} Q_{vol(F)}(h(\partial F)) \bigotimes_{e \in \mathbb{E}^+} dh(e) \right)_{vol, \mathbb{G}}$$

is a planar discrete continuous in law Markovian Holonomy Field. Let  $\mathbb{G}$  be the planar graph  $\mathbb{N}^2 \cap (\mathbb{R}^+ \times [0, 1])$ . As an application of Theorem 22, we only have to check that for any positive real  $\alpha$ ,  $(h(L_{n,0}))_{n \in \mathbb{N}}$  has the same law under  $\mathbb{E}_{\alpha dx}^{Y, \mathbb{G}}$  as under  $\prod_{F \in \mathbb{F}^b} Q_{\alpha}(h(\partial F)) \bigotimes_{e \in \mathbb{E}^+} dh(e)$ . The value of  $\alpha$  will not matter, thus we set  $\alpha$  equals to 1.

By Proposition 21, under  $\mathbb{E}_{dx}^{Y, \mathbb{G}}$ ,  $(h(L_{n,0}))_{n \in \mathbb{N}}$  are i.i.d. random variables which have the same law as  $Y_1$ .

Let  $\mathbb{E}$  be the set of edges of  $\mathbb{G}$ . Under  $\bigotimes_{e \in \mathbb{E}^+} dh(e)$ ,  $(h(e))_{e \in \mathbb{E}}$  are i.i.d. and Haar distributed. Using the multiplicativity property of random holonomy fields,  $h(L_{n,0})$  is a product of elements of  $h(e)_{e \in \mathbb{E}}$ . The key remark is that  $e_{n,0}^r$  appears only once in the reduced decomposition of  $L_{n,0}$  and in no other reduced decomposition of  $L_{m,0}$  with  $m \neq n$ . Applying Lemma 17, one has that under  $\bigotimes_{e \in \mathbb{E}^+} dh(e)$ ,  $(h(L_{n,0}))_{n \in \mathbb{N}}$  are independent and each of them is a Haar random variable. Besides, as  $Y_1$  is invariant by conjugation, for any  $n \in \mathbb{N}$ ,  $Q_1(h(\partial c_{n,0})) = Q_1(h(L_{n,0}))$ . Let  $f : G^{\mathbb{N}} \rightarrow \mathbb{R}$  be a measurable function, the following equality holds:

$$\begin{aligned} & \int_{Mult(P(\mathbb{G}), G)} f\left((h(L_{n,0}))_{n \in \mathbb{N}}\right) \prod_{n \in \mathbb{N}} Q_1(h(\partial c_{n,0})) \bigotimes_{e \in \mathbb{E}^+} dh(e) \\ &= \int_{Mult(P(\mathbb{G}), G)} f\left((h(L_{n,0}))_{n \in \mathbb{N}}\right) \prod_{n \in \mathbb{N}} Q_1(h(L_{n,0})) \bigotimes_{e \in \mathbb{E}^+} dh(e) \\ &= \int_{G^{\mathbb{N}}} f\left((g_n)_{n \in \mathbb{N}}\right) \bigotimes_{n \in \mathbb{N}} \left(Q_1(g_n) dg_n\right), \end{aligned}$$

which is the assertion we had to prove.  $\square$

**Lemma 17.** — Let  $(\alpha_i)_{i=1}^\infty$  be a sequence of independent  $G$ -valued random variables which are Haar distributed. Let  $(\beta_j)_{j=1}^\infty$  be a sequence of  $G$ -valued random variables such that for every  $j \in \mathbb{N}$ ,  $\beta_j$  is a product of elements of  $\{\alpha_i, \alpha_i^{-1}, i \in \mathbb{N}\}$ :

$$\beta_j = w_j((\alpha_i, \alpha_i^{-1})_{i \in \mathbb{N}}),$$

with  $w_j$  a word in  $n$  letters and their inverses.

Suppose that for any  $j \in \mathbb{N}$ , there exists an index  $i_j$  such that  $\alpha_{i_j}$  appears once in  $w_j$  and in no other word  $(w_{j'})_{j' \neq j}$ . Then  $(\beta_j)_{j=1}^\infty$  is a family of independent Haar distributed random variables.

*Proof.* — Let  $k$  be any integer. Let  $(j, j_1, \dots, j_k)$  be a  $k + 1$ -tuple of integers. Let  $i_j \in \mathbb{N}$  such that  $\alpha_{i_j}$  appears once in  $w_j$  and in no other word  $(w_{j'})_{j' \neq j}$ . Let  $F : G^k \rightarrow \mathbb{R}$  and  $f : G \rightarrow \mathbb{R}$  be two continuous functions. There exist  $w_1$  and  $w_2$  two words in  $(\alpha_i, \alpha_i^{-1})_{i \neq i_j}$ ,  $J$  a subset of  $\mathbb{N} \setminus \{i_j\}$  and  $\tilde{F}$  a continuous function from  $G^{\#J}$  to  $\mathbb{R}$  such that a.s.:

$$f(\beta_j)F(\beta_{j_1} \dots \beta_{j_k}) = f(w_1 \alpha_{i_j} w_2) \tilde{F}((\alpha_i)_{i \in J}).$$

Thus, the following sequence of equalities holds:

$$\begin{aligned} \mathbb{E}[f(\beta_j)F(\beta_{j_1} \dots \beta_{j_k})] &= \mathbb{E}\left[f(w_1 \alpha_{i_j} w_2) \tilde{F}((\alpha_i)_{i \in J})\right] \\ &= \int_G \mathbb{E}\left[f(w_1 x w_2) \tilde{F}((\alpha_i)_{i \in J})\right] dx \\ &= \int_G \mathbb{E}\left[f(x) \tilde{F}((\alpha_i)_{i \in J})\right] dx \\ &= \left(\int_G f(x) dx\right) \mathbb{E}\left[\tilde{F}((\alpha_i)_{i \in J})\right], \end{aligned}$$

where we used for the third equality the translation invariance of the Haar measure. Thus for any  $j \in \mathbb{N}$ ,  $\beta_j$  is a Haar random variable which is independent of  $(\beta_j, j \neq i)$ .  $\square$

**8.2. Construction of general planar Yang-Mills fields.** — In the last subsection we didn't create all the possible planar Yang-Mills fields. Indeed, we considered only processes which were invariant by conjugation by  $G$ . We are going to see in this subsection, that for any self-invariant by conjugation  $G$ -valued Lévy process  $Y$ , one can construct a Planar Markovian Holonomy Field.

Let us introduce the notion of support of a process.

**Definition 53.** — Let  $X = (X_t)_{t \in \mathbb{R}^+}$  be a random process. We define the support of  $X$  by the following equality:

$$H_X = \overline{\left\langle \bigcup_{t \in \mathbb{R}^+} H_{X_t} \right\rangle}.$$

**Remark 13.** — Let  $X$  be a Lévy process. Let us suppose that for any  $t \geq 0$ ,  $e \in H_{X_t}$ . Then for any  $t \in \mathbb{R}^+$ ,  $H_X = H_{X_t}$ . Indeed, using the property that  $e \in H_{X_t}$  for any  $t \geq 0$ ,  $H_{X_t}$  is increasing in  $t$ . Yet, as  $X$  is a Lévy process,  $H_{X_{2t}} = H_{X_t}$  for any  $t \geq 0$ .

The support of a process, say  $X$ , is the smallest closed group such that for any  $t \in \mathbb{R}^+$ ,  $\mathbb{P}(X_t \in H_X) = 1$ .

**Definition 54.** — Let  $(X_t)_{t \geq 0}$  be a  $G$ -valued process. It is self-invariant by conjugation if it is invariant by conjugation by  $H_X$ .

Let  $\eta$  be a finite Borel measure on  $G^n$ . For any  $g$  in  $G$ , we define a measure  $\eta^g$  on  $G^n$  by setting, for any continuous function  $f : G^n \rightarrow \mathbb{R}$ :

$$\eta^g(f) = \int_G f(g^{-1}g_1g, \dots, g^{-1}g_ng) \eta(dg_1, \dots, dg_n).$$

We can construct planar Yang-Mills measure for any self-invariant by conjugation Lévy process.

**Theorem 8.** — Every self-invariant by conjugation  $G$ -valued Lévy process  $Y$  is associated with a unique stochastically continuous Planar Markovian Holonomy Field  $(\mathbb{E}_{vol}^Y)_{vol}$  such that for any finite graph  $\mathbb{G}$ , for any rooted spanning tree  $T$  of  $\mathbb{G}$ , and any family of facial loops  $(c_F)_{F \in \mathbb{F}^b}$  oriented anti-clockwise, under  $\mathbb{E}_{vol}^Y$ , the law of  $(h(\mathbf{1}_{c_F, T}))_{F \in \mathbb{F}^b}$  is:

$$\int_G \left( \otimes_{F \in \mathbb{F}^b} m_{vol(F)} \right)^g dg,$$

where  $(m_t)_{t \geq 0}$  is the semi-group of convolution of measures associated with  $Y$ .

We have to be careful: on  $(\mathbb{E}_{vol}^Y, \mathcal{B})$ ,  $(h(\mathbf{1}_{c_F, T}))_{F \in \mathbb{F}^b}$  is not in general a sequence of independent variables.

*Proof.* — Let  $Y = (Y_t)_{t \geq 0}$  be a self-invariant by conjugation  $G$ -valued Lévy process. Let  $H$  be a group such that for any  $t \geq 0$ , almost surely  $Y_t \in H$ , and such that  $(Y_t)_{t \geq 0}$  is invariant by conjugation by  $H$ . The process  $Y$  can be seen as a  $H$ -valued Lévy process invariant by conjugation. Thus, applying Propositions 22 and 24, there exists a  $H$ -valued Planar Markovian Holonomy Field such that for any finite graph  $\mathbb{G}$ , for any rooted spanning tree  $T$  of  $\mathbb{G}$ , and any family of facial loops  $(c_F)_{F \in \mathbb{F}^b}$  oriented anti-clockwise, under  $\mathbb{E}_{vol}^Y$ :

1. the random variables  $(h(\mathbf{1}_{c_F, T}))_{F \in \mathbb{F}^b}$  are independent,
2. for any  $F \in \mathbb{F}^b$ ,  $h(\mathbf{1}_{c_F, T})$  has the same law as  $Y_{vol(F)}$ .

Using Proposition 12, we can extend the group on which  $\mathbb{E}_{vol}^Y$  is defined, from  $H$  to  $G$ . We will denote it by  $\mathbb{E}_{vol}^Y$ . It is a  $G$ -valued Planar Markovian Holonomy Field, and by definition, for any finite graph  $\mathbb{G}$ , for any rooted spanning tree  $T$  of

$\mathbb{G}$ , and any family of facial loops  $(c_F)_{F \in \mathbb{F}^b}$  oriented anti-clockwise, under  $\mathbb{E}_{vol}^Y$ , the law of  $(h(\mathbf{1}_{c_F, T}))_{F \in \mathbb{F}^b}$  is:

$$\int_G \left( \otimes_{F \in \mathbb{F}^b} m_{vol(F)} \right)^g dg.$$

This ends the proof. □

We are led to classify the planar Yang-Mills fields according to their degree of symmetry. In Section 11, we will prove equivalent conditions in order to classify planar Yang-Mills fields.

**Definition 55.** — *Let  $(\mathbb{E}_{vol}^Y)_{vol}$  be a  $G$ -valued planar Yang-Mills field associated with  $Y = (Y_t)_{t \in \mathbb{R}^+}$ . The Yang-Mills field  $(\mathbb{E}_{vol}^Y)_{vol}$  and the Lévy process  $Y$  are pure if  $(Y_t)_{t \geq 0}$  is invariant by conjugation in  $G$  and mixed if not pure. They are also non-degenerate if  $H_Y = G$  and degenerate if  $H_Y \neq G$ .*

According to this definition, any planar Yang-Mills field is either pure non-degenerate, pure degenerate or mixed degenerate.

### 9. Braids and probabilities II: a geometric point of view, infinite random sequences and random processes

We would like now to characterize the discrete Planar Markovian Holonomy Fields. For this, we will use intensively the invariance by area-preserving homeomorphism. The braid group will appear again in a geometric way as the diffeotopy group of the  $n$ -punctured disk. In the next subsection, we explain the general idea which will be behind the proof of Theorem 18. In order to characterize the Markovian Holonomy Fields, one will have to understand first the generalization of the next subsection to any surface and thus one will have to work with the mapping class groups of general surfaces.

**9.1. Braids as the diffeotopy group of the  $n$ -punctured disk.** — Let  $\mathbb{D}$  be the disk of center 0 and radius 1 and  $Q_n = \{q_k = \frac{2k-1-n}{n}, 1 \leq k \leq n\}$ . Let  $\text{Diff}(\mathbb{D}, Q_n, \partial D)$  be the group of diffeomorphisms of  $\mathbb{D}$  which fix the set  $Q_n$  and fix pointwise a neighborhood of  $\partial D$ . The class of isotopy of the identity mapping in  $\text{Diff}(\mathbb{D}, Q_n, \partial D)$  is a normal subgroup called  $\text{Diff}_0(\mathbb{D}, Q_n, \partial D)$ . We define the diffeotopy group of the disk with  $n$  points by:

$$\mathcal{M}_n(\mathbb{D}) = \text{Diff}(\mathbb{D}, Q_n, \partial D) / \text{Diff}_0(\mathbb{D}, Q_n, \partial D).$$

One important theorem is that:

$$(18) \quad \mathcal{M}_n(\mathbb{D}) \simeq \mathcal{B}_n.$$

This isomorphism is constructed by sending some special elements, the half-twists, on the canonical free family of generator of the braid groups. A half-twist permutes the points  $q_k$  and  $q_{k+1}$ , for some  $k$ , and does not move the other points  $(q_i)_{i \notin \{k-1, k\}}$ . For a precise definition, one considers for  $1 \leq k \leq n$ ,  $t_k$  the isotopy class of the diffeomorphism  $\tilde{t}_k$  equals to identity outside the disk of radius  $\frac{2}{n}$  centered at  $\frac{q_k + q_{k+1}}{2}$  and defined by  $\tilde{t}_k(x) = \psi \circ t \circ \psi^{-1}$ , where:

$$\psi : x \mapsto \frac{n}{2} \left( x - \frac{q_k + q_{k+1}}{2} \right),$$

$$t(re^{i\theta}) = re^{i2\pi(\theta + \alpha(r))},$$

and  $\alpha$  is a smooth function from  $[0, 1]$  to itself, which is equal to 0 on a neighborhood of 1, and to  $\frac{1}{2}$  at  $\frac{1}{2}$ .

This geometric construction of the braid group allows us to recover the action of the braid group which was given by Definition 48 and which will be the hidden idea in the proof of Theorem 18.

Indeed, the group  $\mathcal{M}_n(\mathbb{D})$  acts on the fundamental group of  $\mathbb{D} \setminus Q_n$  which is isomorphic to  $\mathbb{F}_n$  the free group of rank  $n$ . We will take  $-\frac{i}{2}$  the base point for the fundamental group of  $\mathbb{D} \setminus Q_n$ . Let  $x_k$  be the homotopy class of the loop based at  $-\frac{i}{2}$  which goes only around  $q_k$  anti-clockwise. One can verify that the action of  $\mathcal{M}_n(\mathbb{D})$  on  $\mathbb{F}_n$ , with the identification given in (18), is the action given by Definition 48.

Besides, given a finite graph  $\mathbb{G}$  on  $\mathbb{R}^2$ , the fundamental group of  $\mathbb{G}$  is isomorphic to the fundamental group of the disk without one point in each of the bounded faces, that is  $\pi_1(\mathbb{D} \setminus Q_{|\mathbb{F}_b|})$ . Thus, we will have a natural action of a braid group on  $\pi_1(\mathbb{G})$  which is isomorphic to the reduced group of loops on  $\mathbb{G}$  defined in Section 6.1. One consequence of the existence of such action is Proposition 35.

**9.2. Infinite sequence.** — Using Proposition 20, every finite sequence of random variables which are invariant by conjugation by their own support is invariant by braids. For finite sequence random variables, it is more difficult to characterize finite sequence of random variables which are invariant by braids. As for exchangeable sequences of random variables, it is easier to work with infinite sequence of random variables which are braidable.

**Definition 56.** — *An infinite random sequence  $\xi = (\xi_i)_{1 \leq i}$  in  $G$  is braidable (or braid-invariant or invariant by braid) if for any integer  $n$  and any braid  $\beta \in \mathcal{B}_n$ , the following equality in law holds:*

$$\beta \bullet (\xi_i)_{1 \leq i \leq n} = (\xi_i)_{1 \leq i \leq n}.$$

*We say that  $\xi$  is spreadable if for any increasing sequence of positive integers  $k_1 < k_2 < \dots$ , we have the equality in law:*

$$(\xi_{k_i})_{1 \leq i} = (\xi_i)_{1 \leq i}.$$

These properties seem to be quite different, yet we are going to prove that one condition is weaker than the other.

**Lemma 18.** — *Any braidable family of random variables is spreadable.*

*Proof.* — Let  $(\xi_i)_{1 \leq i}$  be a braidable infinite sequence of random variables. Let  $\mathbf{k} = (k_1 < k_2 < \dots < k_n)$  be a finite strictly increasing sequence of integers. We define the braid  $\beta_{\mathbf{k}}$  by:

$$\beta_{\mathbf{k}} = \beta_n^{-1} \dots \beta_{k_{n-1}}^{-1} \beta_2^{-1} \dots \beta_{k_2-1}^{-1} \beta_1^{-1} \dots \beta_{k_1-1}^{-1}$$

We have drawn in Figure 18 the braid  $\beta_{\mathbf{k}}$  with  $\mathbf{k} = (2, 3, 6)$ .

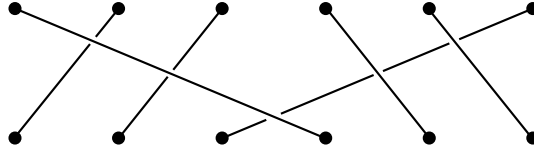


FIGURE 18. The braid  $\beta_{\mathbf{k}}$ ,  $\mathbf{k} = (2, 3, 6)$ .

As the lines attaching  $(i, 1)$  to  $(k_i, 0)$  for  $i = 1, \dots, n$  are on the top of the diagram, this braid verifies that for any element  $(g_1, \dots, g_{k_n})$  of  $G^{k_n}$ , for any integer  $i$  between 1 and  $n$ ,  $(\beta_{\mathbf{k}} \bullet (g_1, \dots, g_{k_n}))_i = g_{k_i}$ .

Thus, if  $\xi = (\xi_i)_{1 \leq i}$  is a braidable random sequence, and if  $\mathbf{k} = (k_0 < k_2 < \dots < k_{n-1})$  is a finite strictly increasing sequence of integers, the following equality in law holds:

$$\beta_{\mathbf{k}} \bullet (\xi_j)_{1 \leq j \leq k_n} = (\xi_j)_{1 \leq j \leq k_n},$$

By restricting it for  $i$  between 0 and  $n - 1$ , we get the desired equality in law:

$$(\xi_{k_1}, \dots, \xi_{k_n}) = (\xi_1, \dots, \xi_n),$$

from which one can conclude that  $\xi$  is spreadable. □

Let  $m$  be a probability measure on  $G$ . We denote by  $m^{\otimes \infty}$  the measure on  $G^{\mathbb{N}}$  such that the unidimensional projections are independent and identically distributed with law  $m$ .

**Definition 57.** — *Let  $\xi$  be an infinite random sequence in  $G$ . Let  $\mathcal{A}$  be a  $\sigma$ -field. We say that  $\xi$  is i.i.d. conditionally to  $\mathcal{A}$  if there exists a random measure  $\eta$  on  $G$ ,  $\mathcal{A}$ -measurable, such that the conditional distribution of  $\xi$  given  $\mathcal{A}$  is  $\eta^{\otimes \infty}$ , that is:  $\mathbb{P}[\xi \in \cdot \mid \mathcal{A}] = \eta^{\otimes \infty}$ . It is said conditionally i.i.d. if there exists a  $\sigma$ -field  $\mathcal{A}$  such that it is i.i.d. conditionally to  $\mathcal{A}$ .*



If  $\xi$  is i.i.d. conditionally to  $\mathcal{A}$ , its law is of the form:

$$\int_{\mathcal{M}_1(G)} m^{\otimes \infty} d\nu(m),$$

where  $\nu$  is the law of  $\eta$ . If we just want to keep in mind the form of the law of  $\xi$ , we will say that  $\xi$  is a *mixture of i.i.d. random sequences*. We have then an extension of de Finetti-Ryll-Nardzewski's theorem for the braid group.

**Theorem 9.** — *Let  $\xi = (\xi_n)_{n \in \mathbb{N}}$  be a sequence of  $G$ -valued random variables. The following conditions are equivalent:*

1. *the sequence  $\xi$  is braidable,*
2. *the sequence  $\xi$  is i.i.d. conditionally to the tail  $\sigma$ -field  $\mathcal{T}_\xi = \bigcap_{n \in \mathbb{N}} \sigma(\xi_k, k \geq n)$  and conditionally to  $\mathcal{T}_\xi$ , almost surely the law of  $\xi_1$  is invariant by conjugation by its own support.*

*If one to the two conditions holds, then  $\xi$  is of the form  $\int_{\mathcal{M}_1(G)} m^{\otimes \infty} d\nu(m)$ , where  $\nu$ -a.s.,  $m$  is almost surely invariant by its own support.*

*Proof.* — Let  $\xi = (\xi_n)_{n \in \mathbb{N}}$  be a sequence of  $G$ -valued random variables.

An application of Proposition 20 to any subsequence of the form  $(\xi_n)_{n=1}^N$  shows that condition 2 implies condition 1.

Now, let us suppose that  $\xi$  is braidable. As a consequence of Lemma 18, the infinite sequence  $\xi$  is spreadable. Using the de Finetti-Ryll-Nardzewski's Theorem (Theorem 1.1 of [16]), any spreadable infinite sequence of random variables in  $G$  is i.i.d. conditionally to the tail  $\sigma$ -field:  $\xi$  is conditionally i.i.d., conditionally to  $\mathcal{T}_\xi$ . Besides, conditionally to  $\mathcal{T}_\xi$ ,  $\xi$  is still braidable: an application of Proposition 20 shows that conditionally to  $\mathcal{T}_\xi$ ,  $(\xi_i)_{i=1}^\infty$  is an i.i.d. sequence of random variables invariant by conjugation by their own support: the condition 2 holds.  $\square$

In the next theorem, we give a condition under which one can characterize the mixture which appears in the last theorem. In order to do so, we consider the diagonal conjugation of  $G$  on  $G^\mathbb{N}$  defined for any  $g \in G$  and  $(x_n)_{n \in \mathbb{N}} \in G^\mathbb{N}$  by  $g \cdot (x_n)_{n \in \mathbb{N}} = (g^{-1}x_n g)_{n \in \mathbb{N}}$ . We need also to define the property ( $\mathcal{P}$ ):

( $\mathcal{P}$ ) For any integer  $n \in \mathbb{N}$ ,  $(\xi_k)_{k \leq n}$  and  $(\xi_k)_{k > n}$  are  $\mathcal{I}$ -independent.

**Theorem 10.** — *Let  $\xi = (\xi_n)_{n \in \mathbb{N}}$  be a braidable sequence of  $G$ -valued random variables. Suppose that  $\xi$  is invariant by diagonal conjugation: it means that for any  $g \in G$ ,  $g \cdot \xi$  has the same law as  $\xi$ . If  $\xi$  satisfies the property ( $\mathcal{P}$ ), there exists  $m_0$  a probability measure on  $G$  invariant by conjugation by its own support such that the law of  $\xi$  is:*

$$\int_G (m_0^{\otimes \infty})^g dg.$$

*Proof.* — Let  $\xi$  be a braidable sequence of  $G$ -valued random variables which is invariant by diagonal conjugation and satisfies the property  $(\mathcal{P})$ . As a consequence of Theorem 9, there exists a random measure  $\eta$  on  $G$ , which is almost surely invariant by conjugation by its own support, such that the conditional distribution of  $\xi$  given  $\mathcal{T}_\xi$  is  $\eta^{\otimes\infty}$ . Let  $\nu$  be the law of  $\eta$ , the law of  $\xi$  is

$$\int_{\mathcal{M}^1(G)} m^{\otimes\infty} d\nu(m).$$

As  $\xi$  is invariant by diagonal conjugation by  $G$ , we only have to show that there exists a probability measure  $m_0$  such that the law of  $\xi$  on the invariant  $\sigma$ -field  $\mathcal{I}$  is equal to  $m_0^{\otimes\infty}$ .

Let  $k$  be an integer. Let  $f : G^k \rightarrow \mathbb{R}$  be a continuous function invariant by diagonal conjugation. As  $G^k$  is compact,  $f$  is bounded. As the sequence  $\xi$  satisfies the property  $(\mathcal{P})$ ,  $(f(\xi_{ik+1}, \dots, \xi_{ik+k}))_{i \geq 0}$  is an i.i.d. sequence of bounded random variables. Thus, by the law of large numbers, there exists a real  $l^k(f)$  such that:

$$\frac{1}{n} \sum_{0 \leq i \leq n} f(\xi_{ik+1}, \dots, \xi_{ik+k}) \xrightarrow[n \rightarrow \infty]{} l^k(f) \text{ a.s..}$$

Yet, by disintegration and the law of large numbers, we get also that:

$$\frac{1}{n} \sum_{0 \leq i \leq n} f(\xi_{ik+1}, \dots, \xi_{ik+k}) \xrightarrow[n \rightarrow \infty]{} \eta^{\otimes k}(f), \text{ a.s..}$$

The random variable  $\eta^{\otimes k}(f)$  is thus almost surely constant. Let us define the set of measures:

$$\Omega_{f,k} = \{m \in \mathcal{M}^1(G), m^{\otimes k}(f) = l^k(f)\}.$$

We just proved that for any integer  $k$ , for any continuous function  $f : G^k \rightarrow \mathbb{R}$ , invariant by diagonal conjugation,

$$\nu(\Omega_{f,k}) = 1.$$

Let us consider  $\mathcal{F}_k$  a dense countable set of continuous functions which are invariant by diagonal conjugation of  $G$  on  $G^k$ . Our previous discussion allows us to write the following equality:

$$\nu \left( \bigcap_{k \in \mathbb{N}} \bigcap_{f \in \mathcal{F}_k} \Omega_{f,k} \right) = 1.$$

Let us take  $m_0 \in \bigcap_{k \in \mathbb{N}} \bigcap_{f \in \mathcal{F}_k} \Omega_{f,k}$ . This particular measure is a probability measure which is invariant by its own support. Besides, for any positive integer  $k$  and any continuous function  $f$  on  $G^k$  invariant by diagonal conjugation,

$$m^{\otimes k}(f) = m_0^{\otimes k}(f), \nu(dm) \text{ a.s.}$$

We have just managed to prove that on the invariant  $\sigma$ -field, the law of  $\xi$  is  $m_0^{\otimes \infty}$ : the law of  $\xi$  is thus  $\int_G (m_0^{\otimes \infty})^g dg$ .  $\square$

Let  $\xi$  be a sequence of  $G$ -valued random variables which satisfies the properties in Theorem 10. Let  $m_0$  be a probability measure on  $G$ , given by Theorem 10, such that the law of  $\xi$  is  $\int_G (m_0^{\otimes \infty})^g dg$ . What we aim to do next is to give conditions which ensure the fact that  $\xi$  is actually a random sequence of i.i.d. variables: for this, it is enough to show that  $m_0$  is invariant by conjugation by  $G$ .

**9.3. Degeneracy of the mixture.** — As  $m_0$  is invariant by conjugation by its own support, one particular possibility in order to prove that  $m_0$  is invariant by conjugation by  $G$  is to prove that  $\text{Supp}(m_0) = G$ . We will call this case the non-degeneracy case.

*9.3.1. Non-degeneracy case.* —

**Proposition 26.** — *Let  $G$  be a finite group. Let  $\xi$  be a sequence of  $G$ -valued random variables which is braidable, satisfies property  $(\mathcal{P})$  and such that  $e \in \text{Supp}(\xi_1)$ . The following assertions are equivalent:*

1.  $\xi$  is a sequence of i.i.d. random variables which support generates  $G$ ;
2. there exists an integer  $k$  such that  $\text{Supp}(\prod_{i=1}^k \xi_i) = G$ .

In order to prove this proposition, we will need the two following facts which hold only when  $G$  is finite. The first assertion is that for any measure  $m$  on  $G$  such that  $e \in \text{Supp}(m)$ , there exists an integer  $k$  such that for any  $k' \geq k$ ,  $\text{Supp}(m^{*k'}) = H_m$ . The second asserts that no subgroup of  $G$  can intersect every conjugacy classes of  $G$ : this is the following Jordan's theorem.

**Theorem 11.** — *Let  $G$  be a finite group, let  $H$  be a subgroup of  $G$ . Let us suppose that:*

$$G = \bigcup_{g \in G} g^{-1}Hg,$$

then  $G = H$ .

*Proof.* — The proof is taken from Serre's lesson [26] and can be summarized in a simple calculation:

$$\#G \leq \# \left( \bigcup_{g \in G} g^{-1}Hg \right) = \# \left( \bigcup_{g \in G} (g^{-1}Hg \setminus e) \cup \{e\} \right) \leq \frac{\#G}{\#H} (\#H - 1) + 1,$$

which can hold if and only if  $H = G$ .  $\square$

We can now handle the proof of Proposition 26.

*Proof of Proposition 26.* — It is quite obvious that the assertion 1 implies the assertion 2. It remains to prove the other implication. As a consequence of Theorem 10, there exists a probability measure  $m_0$  invariant by conjugation by its own support such that the law of  $\xi$  is  $\int_G (m_0^{\otimes \infty})^g dg$ . Let  $m_0$  be such a probability measure. As  $e \in \text{Supp}(\xi_1)$ ,  $e$  is in the support of  $m_0$ : the support of  $m_0^{*k}$  is increasing in  $k$  and so is the support of  $\prod_{i=1}^k \xi_i$ . Let  $k$  be an integer such that  $\text{Supp} \left( \prod_{i=1}^k \xi_i \right) = G$ . For any  $k' \geq k$ ,  $\text{Supp} \left( \prod_{i=1}^{k'} \xi_i \right) = G$ . Let  $N$  be an integer greater than  $k$  such that  $\text{Supp}(m_0^{*N}) = H_{m_0^{*N}}$ .

As the law of  $\prod_{i=1}^N \xi_i$  is  $\int_G (m_0^{*N})^g dg$ , its support is then equal to  $\cup_{g \in G} g g^{-1} H_{m_0^{*N}} g$ . On the other side, we have chosen  $N$  such that  $\text{Supp}(\prod_{i=1}^N \xi_i) = G$ . Thus, one has the equality:

$$G = \bigcup_{g \in G} g^{-1} H_{m_0^{*N}} g$$

which implies, thanks to Jordan's theorem (Theorem 11), that  $H_{m_0^{*N}} = G$ .

This implies that  $H_{m_0} = G$ : the support of  $m_0$  generates the group  $G$ . We recall that  $m_0$  was invariant by conjugation by its support, hence by the subgroup generated by its support  $H_{m_0}$ . The probability  $m_0$  is thus invariant by conjugation by  $G$  and the law of  $\xi$  is thus  $m_0^{\otimes \infty}$ . Thus, the assertion 2 implies the assertion 1.  $\square$

For an arbitrary compact Lie group, it is not true in general that for any measure  $m$  on  $G$  such that  $e \in \text{Supp}(m)$ , there exists  $k$  such that  $\text{Supp}(m^{*k}) = H_m$ . Thus, in order to deal with any compact Lie group, we will substitute this fact by the Ito-Kawada's theorem, Theorem 12. As for Jordan's theorem, it does not hold when  $G$  is infinite, as in every compact Lie group, any maximal torus intersects all the conjugacy classes. Thus, we have to impose that the subgroup  $H$  intersects every conjugacy class "as much as"  $G$  does, which is the meaning of the condition imposed in up coming Lemma 19 or Proposition 19. Doing so, we will be able to prove the following proposition which holds for any arbitrary compact Lie group.

**Proposition 27.** — *Let  $G$  be a compact Lie group. Let  $\xi$  be a sequence of  $G$ -valued random variables which is braidable, invariant by diagonal conjugation of  $G$  on  $G^{\mathbb{N}}$  and satisfies property  $(\mathcal{P})$ . Let us suppose that  $e \in \text{Supp}(\xi_1)$ . The following assertions are equivalent:*

1.  $\xi$  is a sequence of i.i.d. random variables which support generates  $G$ .
2. the random variables  $\prod_{k=1}^n \xi_k$  converge in law to a Haar random variable as  $n$  goes to infinity.

Let us introduce Itô-Kawada's theorem and a measurable version of the theorem of Jordan which holds for any compact Lie group.

**Definition 58.** — *Let  $m$  be a probability measure on  $G$ . It is:*

- aperiodic if its support  $\text{Supp}(m)$  is not contained in a left or right proper coset of a proper closed subgroup of  $G$ ,
- non-degenerate if  $H_m = G$ .

**Remark 14.** — It is obvious that  $m$  is non-degenerate if it is seen as a measure on  $H_m$ . Besides, if  $e \in \text{Supp}(m)$  then  $m$  is aperiodic.

Under the condition of aperiodicity and non-degeneracy, the Itô-Kawada's theorem (Theorem 3.3.5. of [27], first proved in [17]) explains the behavior of  $m^{*n}$  when  $n$  goes to infinity.

**Theorem 12 (Itô-Kawada's Theorem).** — Let  $G$  be a compact topological Hausdorff group. Let  $\mu$  be a non-degenerate and aperiodic probability measure on  $G$ . The sequence  $\mu^{*n}$  converges in distribution to the normalized Haar measure on  $G$  as  $n$  goes to infinity.

Our measured version of Jordan's theorem, Theorem 11, valid for any compact Lie group, is given in the following proposition. Recall that for any compact Lie group  $K$ , we denote by  $\lambda_K$  the Haar measure on  $K$ .

**Proposition 28.** — Let  $G$  be a compact Lie group, let  $H$  be a closed subgroup. Let  $\lambda_G$  (resp.  $\lambda_H$ ) be the Haar measure on  $G$  (resp.  $H$ ). If

$$\int_G \lambda_{g^{-1}Hg} dg = \lambda_G,$$

then  $G = H$ .

*Proof.* — Let  $G$  be a compact Lie group, let  $H$  be a closed subgroup. Suppose that  $\int_G \lambda_{g^{-1}Hg} = \lambda_G$ . Then, for any function  $f : G \rightarrow \mathbb{R}$  invariant by conjugation continuous,  $\lambda_H(f) = \lambda_G(f)$ . Let us suppose that  $H$  is a proper closed subgroup of  $G$ . The space  $H \backslash G$  of right cosets of  $H$  is a nice topological space: it is a differentiable manifold and there exists  $\tilde{f}$  a non-constant real continuous function on  $H \backslash G$ . Let  $p : G \rightarrow H \backslash G$  be the canonical projection, the function  $f = \tilde{f} \circ p$  is a real non-constant square-integrable function  $f$  on  $G$  invariant by left multiplication by  $H$ : for any  $g \in G$ , for any  $h \in H$ ,  $f(g) = f(hg)$ . One can also assume that  $f$  is of zero mean on  $G$ .

Let  $E = \{ \phi \in L^2(G), \int_G \phi(g) d\lambda_G(g) = 0 \}$  be the space of square-integrable zero mean functions on  $G$ . The group  $G$  acts on  $E$ , by left multiplication on the argument, and this representation has no non-zero fixed point. On the other hand, the restriction of this representation on  $H$  has at least one fixed point, namely  $f$ . We can decompose  $E$  as a sum of irreducible representations of  $G$ :

$$E = \bigoplus_{i=1}^{\infty} E_i.$$

None of the  $E_i$  is the trivial representation of  $G$  as we have restricted the action of  $G$  to zero mean functions. The action of  $H$  on  $E$  admits a fixed point  $f$ . We can decompose  $f$  on  $\bigoplus_{i=1}^{\infty} E_i$ . As, for any integer  $i$ , the space  $E_i$  is invariant under the action of  $H$ , there exists at least an integer  $i_0$  such that  $E_{i_0}$  seen as a  $H$ -module is not irreducible. We denote by  $\chi_{i_0}$  the character of the  $G$ -module  $E_{i_0}$ . By the classical theory of character,

$$\int_G \chi_{i_0} d\lambda_G = \dim(E_{i_0}^G) = 0,$$

whereas:

$$\int_G \chi_{i_0} d\lambda_H = \int_H \chi_{i_0} d\lambda_H = \dim(E_{i_0}^H) \geq 1.$$

Thus, we just found a central function  $\chi_{i_0}$  such that:

$$\int_G \chi_{i_0} d\lambda_G \neq \int_G \chi_{i_0} d\lambda_H.$$

This ends the proof. □

We give an other formulation of Proposition 28 which seems interesting to us.

**Lemma 19.** — *Let  $G$  be a compact Lie group, let  $H$  be a closed subgroup of  $G$ . Let  $\lambda_G$  (resp.  $\lambda_H$ ) be the Haar measure on  $G$  (resp.  $H$ ). If for any  $G$ -invariant function  $f$  the equality  $\lambda_G(f) = \lambda_H(f)$  holds, then  $G = H$ .*

We have now all the tools in order to prove Proposition 27.

*Proof of Proposition 27.* — Let  $\xi$  be a sequence of  $G$ -valued random variables which is braidable, invariant by diagonal conjugation of  $G$  on  $G^{\mathbb{N}}$  and which satisfies property  $(\mathcal{P})$ . Let us suppose that  $e \in \text{Supp}(\xi_1)$ . As a consequence of Theorem 10, there exists a probability measure  $m_0$  invariant by conjugation by its own support such that the law of  $\xi$  is  $\int_G (m_0^{\otimes \infty})^g dg$ . Let  $m_0$  be such a probability measure. Using the hypothesis on  $\xi_1$ , we deduce that  $e \in \text{Supp}(m_0)$ .

Let us suppose that  $\xi$  is a sequence of i.i.d. random variables whose support generates  $G$ , then we can take  $m_0$  equal to the law of  $\xi_1$ . Then  $m_0$  is non-degenerate and aperiodic: by Theorem 12,  $m_0^{*n}$  converges to  $\lambda_G$  as  $n$  goes to infinity. This is equivalent to the condition 2.

Now, let us suppose instead that  $\prod_{k=1}^n \xi_k$  converges in law to a Haar random variable. As we have seen in Remark 14, the measure  $m_0$  is aperiodic and non-degenerate if seen as a measure on  $H_{m_0}$ . Thanks to the Itô-Kawada's theorem,  $m_0^{*n}$  converges to the Haar probability measure on  $H_{m_0}$ , which we will denote by  $\lambda_{H_{m_0}}$ , when  $n$  goes to infinity. For any integer  $n$ , the law of  $\prod_{i=1}^n \xi_i$  is  $\int_G (m_0^{*k})^g dg$

and thus, using the hypothesis on the law of  $\prod_{i=1}^n \xi_i$ , and our previous discussion, one gets the equality:

$$\lambda_G = \int_G \lambda_{g^{-1}H_{m_0}g} dg.$$

By Proposition 28, it follows that  $H_{m_0} = G$ : the measure  $m_0$  is thus invariant by conjugation by  $H_{m_0}$  which is equal to  $G$ , and the law of  $\xi$  is  $m_0^{\otimes \infty}$ . This finishes the proof of Proposition 27.  $\square$

*9.3.2. General case.* — Actually, one would like weaker conditions on  $\xi$  in order to understand the case when  $\xi$  is a sequence of i.i.d. random variables such that  $H_{\xi_1} \neq G$ . The main result in the general case is Theorem 13.

**Theorem 13.** — *Let  $G$  be a compact Lie group. Let  $\xi$  be a sequence of  $G$ -valued random variables which is braidable, invariant by diagonal conjugation of  $G$  on  $G^{\mathbb{N}}$  and satisfies property  $(\mathcal{P})$ . The following assertions are equivalent:*

1. *the sequence  $\xi$  is a sequence of i.i.d. random variables invariant by conjugation by  $G$ ,*
2. *there exists  $\nu$  a probability measure on  $G$  such that for any integer  $n$ , the law of  $\prod_{k=1}^n \xi_k$  is  $\nu^{*n}$ .*

We recall basic, yet crucial, results about representations and integration. First of all, Peter-Weyl's theorem asserts that the set of matrix elements of irreducible representations

$$\left\{ g \mapsto v(\pi(g)w), \pi \in \hat{G}, v \in V^*, w \in W \right\},$$

where  $\hat{G}$  is the set of irreducible representations of  $G$ , is dense for the uniform norm in the set of continuous functions on  $G$ . Thus, any measure  $m$  on  $G$  is fully characterized by its Fourier coefficients defined as:

$$\forall \pi \in \hat{G}, \pi(m) = \int_G \pi(g)m(dg).$$

Secondly, let  $\pi : G \rightarrow Gl(V)$  be an irreducible representation of dimension  $d_\pi$ . Let  $A$  be a matrix acting on  $V$ . By the Schur's lemma,

$$\int_G \pi(g)A\pi(g)^{-1}dg = \frac{Tr(A)}{d_\pi} Id.$$

**Definition 59.** — *Let  $m$  be a probability measure on a compact Lie group  $G$ . It is quasi-invariant by conjugation if there exists  $\nu$  a probability measure on  $G$  such that for any  $n \in \mathbb{N}$*

$$\int_G (m^g)^{*n} dg = \nu^{*n}.$$

It is obvious that any conjugation-invariant probability measure is quasi-invariant by conjugation. We can state now the main characterization of quasi-invariant by conjugation probability measures.

**Proposition 29.** — *Let  $m$  be a probability measure on a compact Lie group  $G$ . It is quasi-invariant by conjugation probability measure if and only if for any irreducible representation  $\pi$  of  $G$ , the matrix  $\pi(m)$  has only one eigenvalue.*

*Proof.* — Let  $m$  be a probability measure on a compact Lie group  $G$ . Using the remark about Peter-Weyl's theorem, there exists  $\nu$  a probability measure on  $G$  such that for any  $n \in \mathbb{N}$ :

$$\left( \int_G (m^g)^{*n} dg \right) = \nu^{*n}$$

if and only if for any  $n \in \mathbb{N}$  and any irreducible representation  $\pi \in \hat{G}$ :

$$(19) \quad \pi \left( \int_G (m^{*n})^g dg \right) = \pi(\nu^{*n}).$$

Let us remark that, in this case,  $\nu = \int_G m^g dg$  and thus  $\nu$  is invariant by conjugation by  $G$ . Let  $n$  be an integer. Let us compute  $\pi \left( \int_G (m^{*n})^g dg \right)$ .

$$\begin{aligned} \pi \left( \int_G (m^{*n})^g dg \right) &= \int_{G^2} \pi(g^{-1}g'g) m^{*n}(dg') dg \\ &= \int_G \pi(g) \left( \int_G \pi(g') m^{*n}(dg') \right) \pi(g)^{-1} dg \\ &= \frac{1}{d_\pi} \text{Tr}(\pi(m)^n) \text{Id}. \end{aligned}$$

Since  $\nu$  is invariant by conjugation by  $G$ , by the Schur's lemma,  $\pi(\nu)$  is a multiple of the identity matrix. The condition (19) boils down to the fact that for any integer  $n$ ,

$$\text{Tr}(\pi(m)^n) = \text{Tr} \left( \left( \frac{\text{Tr}(\pi(\nu))}{d_\pi} \text{Id} \right)^n \right).$$

Let  $A$  and  $B$  two square matrices of same size. Suppose that for any integer  $k$ ,  $\text{Tr}(A^k) = \text{Tr}(B^k)$ . As the coefficient of the characteristic polynomial can be written as a sum of products of traces of powers of the matrix, the characteristic polynomial of  $A$  and  $B$  are equal. Thus, they have the same spectrum. Applying this result to our concern, we get that  $\pi(m)$  has only one eigenvalue, which is equal to  $\frac{\text{Tr}(\pi(\nu))}{d_\pi}$ .  $\square$

It is natural to ask if a quasi-invariant by conjugation probability measure is invariant by conjugation. The answer is no, and we will construct a counter-example in the symmetric group  $\mathfrak{S}_3$ .



**Proposition 30.** — Let  $(\mu_t)_{t \geq 0}$  (resp.  $(\eta_t)_{t \geq 0}$ ) be the continuous semi-group of convolution of measures starting from  $\delta_{id}$  on the symmetric group  $\mathfrak{S}_3$ , associated with the jump measure  $m$  (resp.  $m_0$ ):

$$\begin{aligned} m((12)) &= 0, & m_0((12)) &= 1, \\ m((13)) &= 1, & m_0((13)) &= 1, \\ m((23)) &= 2, & m_0((23)) &= 1, \\ m((123)) &= 2, & m_0((123)) &= 1, \\ m((132)) &= 0, & m_0((132)) &= 1. \end{aligned}$$

The measure  $\mu_1$  is quasi-invariant by conjugation and for any  $n \in \mathbb{N}$ ,

$$\int_G (\mu_1^g)^{*n} dg = \eta_1^{*n}.$$

*Proof.* — We have to check that the condition of Proposition 29 is fulfilled by  $\mu_1$ . Actually we only have to show that for any  $\pi \in \widehat{\mathfrak{S}_3}$ ,  $\pi(m)$  has only one eigenvalue, which is equal to the one of  $\pi(m_0)$ . The group  $\mathfrak{S}_3$  has only three irreducible representations two of which have dimension one. It remains to compute  $\pi(m)$  where  $\pi$  is the representation of  $\mathfrak{S}_3$  on  $\{(a, b, c) \in \mathbb{R}^3, a + b + c = 0\}$ . We leave this calculation as an exercise.  $\square$

The quasi-invariant by conjugation property does not imply the invariance by conjugation property. Yet, we have the following theorem.

**Theorem 14.** — Let  $m$  be a probability measure on  $G$ . Let us suppose that  $m$  is invariant by conjugation by its own support. Then,  $m$  is quasi-invariant by conjugation if and only if  $m$  is invariant by conjugation by  $G$ .

*Proof.* — As we have already seen, a probability measure which is invariant by conjugation is quasi-invariant by conjugation. It remains to prove the “only if” part of the theorem. Let  $m$  be a quasi-invariant by conjugation probability measure. We know, thanks to Proposition 29, that for any irreducible representation  $\pi$  of  $G$ ,  $\pi(m)$  has only one eigenvalue. Set  $H = H_m$ . Any irreducible representation  $\pi \in \widehat{G}$  determines by restriction a representation of  $H$ . Since  $H$  is a closed subgroup of  $G$ , it is a compact Lie group, thus we can apply Peter-Weyl’s theorem which allows us to decompose any representation of  $H$  as a direct sum of irreducible representations:

$$\pi = \bigoplus_{i=1}^n \pi_i,$$

with  $\pi_i \in \widehat{H}$ . As  $m$  is invariant by conjugation in  $H$ , thanks to Schur’s lemma, for any  $i \in \{1, \dots, n\}$ ,  $\pi_i(m)$  is a scalar matrix, hence  $\pi(m)$  is diagonal. As it has only one eigenvalue, it is a multiple of the identity, showing then that  $m$  is invariant by conjugation in  $G$ .  $\square$

We have now all the tools to prove Theorem 13.

*Proof of Theorem 13.* — Let  $\xi$  be a sequence of  $G$ -valued random variables which is braidable, invariant by diagonal conjugation of  $G$  on  $G^{\mathbb{N}}$  and which satisfies property  $(\mathcal{P})$ . As a consequence of Theorem 10, there exists a probability measure  $m_0$  invariant by conjugation by its own support such that the law of  $\xi$  is  $\int_G (m_0^{\otimes \infty})^g dg$ .

Let us suppose that  $\xi$  is a sequence of i.i.d. random variables, then one can take  $m_0$  equal to the law of  $\xi_1$ : the law of  $\xi$  is equal to  $m_0^{\otimes \infty}$ . Thus for any  $n$ , the law of  $\prod_{k=1}^n \xi_k$  is  $m_0^{*n}$ .

Instead of assuming that  $\xi$  is a sequence of i.i.d. random variables, let us suppose that there exists  $\nu$  a probability measure on  $G$  such that for any  $n$ , the law of  $\prod_{k=1}^n \xi_k$  is  $\nu^{*n}$ . The law of  $\prod_{k=1}^n \xi_k$  is  $\int_G (m^{*k})^g dg$ : it shows that the probability measure  $m_0$  is quasi-invariant by conjugation. Yet, it is also invariant by conjugation by its own support. By Theorem 14,  $m$  is invariant by conjugation by  $G$ , and thus  $\xi$  is a sequence of i.i.d. random variables.  $\square$

**9.4. Processes.** — In this subsection, we apply the results we proved in Section 9.2 and 9.3 in order to prove similar results for  $G$ -valued processes indexed by  $\mathbb{R}^+$ . We define the increments of a process as Kallenberg does in [16]: what is interesting for us are the rational increments as defined in Definition 60.

*9.4.1. Definitions.* —

**Definition 60.** — Let  $X$  be a  $G$ -valued random process indexed by  $\mathbb{R}^+$ . We define the (rational) increments of  $X$  for  $n \in \mathbb{N}^* \cup (\mathbb{N}^*)^{-1}$  and  $j \geq 1$  as:

$$(20) \quad X_{n,j} = X_{\frac{j}{n}} X_{\frac{j-1}{n}}^{-1}.$$

It has spreadable (resp. braidable) increments if for every  $n \in \mathbb{N}^* \cup (\mathbb{N}^*)^{-1}$ , the sequence  $(X_{n,i})_{0 < i}$  is spreadable (resp. braidable).

**Definition 61.** — Let  $(X_t)_{t \in \mathbb{R}^+}$  be a Lévy process. It has auto-invariant by conjugation increments if for any  $0 = t_0 \leq t_1 \leq t_2 \cdots \leq t_k$ , the sequence of increments  $(X_{t_i} X_{t_{i-1}}^{-1})_{i=1}^k$  is auto-invariant by conjugation.

**Proposition 31.** — Let  $X$  be a  $G$ -valued Lévy process. The three following conditions are equivalent:

1.  $X$  has auto-invariant by conjugation increments,
2. for any  $t \in \mathbb{R}^+$ ,  $X_t$  is invariant by conjugation by its own support,
3.  $X$  is invariant by conjugation by  $H_X$ , thus self-invariant by conjugation.

*Proof.* — We will show that 1 implies 2, 3 implies 1 and at last 2 implies 3. Let  $X$  be a  $G$ -valued Lévy process.

Let us show that 1 implies 2. Let us assume that  $X$  has auto-invariant by conjugation increments. Let  $t \in \mathbb{R}^+$ , then  $(X_t, X_{2t}X_t^{-1})$  is auto-invariant by conjugation:  $X_t$  is invariant by conjugation by the support of  $X_{2t}X_t^{-1}$ . As  $X$  is a Lévy process,  $X_{2t}X_t^{-1}$  and  $X_t$  has the same law, thus the same support:  $X_t$  is invariant by conjugation by its own support.

Now, let us show that 3 implies 1. Let us assume that  $X$  is invariant by conjugation by  $H_X$ . By definition for any  $t \in \mathbb{R}^+$ ,  $\text{Supp}(X_t) \subset H_X$ . Let  $t_1 < t_2$  and  $t_3 < t_4$  be four non negative reals. As the process  $X$  is invariant by conjugation by  $H_X$ ,  $X_{t_2}X_{t_1}^{-1}$  is invariant by  $\text{Supp}(X_{t_4-t_3})$ , and thus by  $\text{Supp}(X_{t_4}X_{t_3}^{-1})$ . This implies that  $X$  has auto-invariant by conjugation increments.

It remains to prove that 2 implies 3. Let us first remark that, if  $U$  and  $V$  are two random independent variables in  $G$ ,  $\text{Supp}(UV) = \overline{\text{Supp}(U)\text{Supp}(V)}$ . Besides, if they are both invariant in law by conjugation by a set  $S$ ,  $UV$  is also invariant by conjugation by  $S$ . Moreover, if  $U$  is invariant by conjugation by any element of  $S$ , it is invariant by conjugation by any element of the closure of the semi-group generated by  $S$ :  $\overline{\bigcup_{k=1}^{\infty} S^k}$ , which, in the case where  $G$  is compact, is a group.

Let  $n$  be an integer and let  $t$  be a positive real. Using the hypothesis of self-invariance of  $X$ ,  $X_{\frac{t}{n}}$  is invariant by conjugation by  $\text{Supp}(X_{\frac{t}{n}})$ . Taking  $n$  independent copies of  $X_{\frac{t}{n}}$  and applying the remarks above, we find that  $X_t$  is still invariant by conjugation by  $\text{Supp}(X_{\frac{t}{n}})$ , and thus also for any integer  $k \geq 1$ , by  $\overline{\text{Supp}(X_{\frac{t}{n}})^k} = \text{Supp}(X_{\frac{k}{n}t})$ , or by the semi-group generated by  $\text{Supp}(X_{\frac{k}{n}t})$ , which is nothing but  $H_{X_{\frac{k}{n}t}}$ . Thus,  $X_t$  is invariant by conjugation by:

$$\overline{\bigcup_{q \in \mathbb{Q}^+} H_{X_{qt}}}.$$

Since the laws of  $(X_t)_{t \geq 0}$  form a continuous semi-group of convolution of measures, for any  $s \geq 0$ ,  $X_s \in \overline{\bigcup_{q \in \mathbb{Q}^+} H_{X_{q,t}}}$  a.s. and thus  $H_{X_s} \subset \overline{\bigcup_{q \in \mathbb{Q}^+} H_{X_{qt}}}$ , hence the equality:

$$H_X = \overline{\bigcup_{q \in \mathbb{Q}^+} H_{X_{qt}}}.$$

The conclusion of the discussion is that, for any positive real  $t$ ,  $X_t$  is invariant by conjugation by  $H_X$ . Using the fact that  $X$  has independent and stationary left increments, it implies that the Lévy process  $X$  is invariant by conjugation in  $H_X$ .  $\square$

In the following we will need a weak version of the notion of independence of increments.

**Definition 62.** — Let  $(X_t)_{t \in \mathbb{R}^+}$  be a  $G$ -valued process. It has  $\mathcal{I}$ -independent increments if for any increasing sequence of real  $0 = t_0 < t_1 < \dots < t_n < \dots$  the sequence  $(X_{t_n} X_{t_{n-1}}^{-1})_{n \in \mathbb{N}^*}$  satisfies the property  $(\mathcal{P})$  defined page 72.

9.4.2. *Generalized Bühlmann's Theorem.* — We can now state the generalization of Bühlmann's Theorem (Theorem 1.19 of [16]) for the braid group.

**Theorem 15.** — Let  $X$  be a stochastically continuous  $G$ -valued process indexed by  $\mathbb{R}_+$  with  $X_0 = e$ . The following conditions are equivalent:

1.  $X$  has braidable increments,
2.  $X$  is a mixture of self-invariant by conjugation Lévy processes.

The  $\sigma$ -field which makes the rational increments, as defined in Definition 60, conditionally i.i.d. is the  $\sigma$ -field  $\mathcal{T} = \bigcap_{t \in \mathbb{Q}} \sigma(X_s X_t^{-1}, s > t)$ . Besides, the following conditions are equivalent:

1.  $X$  is invariant by conjugation by  $G$  and has braidable and  $\mathcal{I}$ -independent increments,
2. there exists a self-invariant by conjugation Lévy process  $Y$ , such that the law of  $X$  is  $UYU^{-1}$ , where  $U$  is a Haar variable on  $G$  independent of  $Y$ .

*Proof.* — Let us prove the first part of the theorem. Let us show that the condition 2 implies the first one. Let us suppose that  $X$  is a mixture of self-invariant by conjugation Lévy processes. In order to show that  $X$  has braidable increments, it is enough to show that any self-invariant by conjugation Lévy process has braidable increments. Let  $Z$  be any self-invariant by conjugation Lévy process. By Proposition 31, for any  $n \in \mathbb{N}^* \cup (\mathbb{N}^*)^{-1}$ , the sequence of increments  $(Z_{n,j})_j$  defined in Definition 60 is a sequence of i.i.d. random variables which are invariant by conjugation by their own support. Hence, by Theorem 9, it is braidable: the process  $Z$  has braidable increments.

Now, let us suppose that  $X$  has braidable increments. Using Lemma 18,  $X$  has spreadable increments. Let us use an argument taken from the second proof of Theorem 1.19 of [16]. Let introduce the processes:

$$Y_n^k(t) = X \left( t + \frac{k-1}{n} \right) X \left( \frac{k-1}{n} \right)^{-1}, \quad t \in \mathbb{Q} \cap [0, n^{-1}], \quad k, n \in \mathbb{N}.$$

We note that for any  $n \in \mathbb{N}$ , the sequence  $(Y_n^k)_{k \in \mathbb{N}}$  is spreadable. Applying deFinetti-Ryll-Nardzewski's Theorem (Theorem 1.1 and Corollary 1.6 in [16]) which is valid for Polish spaces to these sequences, we conclude that for any  $n \in \mathbb{N}$ , conditionally to the tail  $\sigma$ -field  $\mathcal{T}_n$ , the sequence  $(Y_n^k)_{k \in \mathbb{N}}$  is a sequence of i.i.d. random variables. We considered  $t \in \mathbb{Q}$  in the definition of  $Y_n^k$  as the product of a countable family of Polish spaces is still a Polish space. Yet the  $\sigma$ -field  $\mathcal{T}_n$  we are conditioning on, is a.s. independent of  $n$ : we call it  $\mathcal{T}$ . Given  $\mathcal{T}$ ,  $X$  has conditionally stationary independent (rational) increments. For any  $t \in \mathbb{Q}^+$ , let  $m_t$  be the law of  $X_t$  conditionally to  $\mathcal{T}$ : for any  $t \in \mathbb{Q}^+$  and any  $s \in \mathbb{Q}^+$ ,

almost surely  $m_t * m_s = m_{t+s}$ . Besides, using the stochastic continuity of  $X$  and the fact that  $X_0 = e$ , one has that almost surely  $(m_t)_{t \in \mathbb{Q}^+}$  is uniformly continuous. We can extend the semi group  $(m_t)_{t \in \mathbb{Q}^+}$  to a semi-group  $(m_t)_{t \in \mathbb{R}^+}$ : by stochastic continuity the process  $X$  is then a mixture of Lévy processes. A similar argument as in the proof of Theorem 9, applied to the sequence  $(X_{nq})_{n \in \mathbb{N}}$  allows us to show that conditionally on  $\mathcal{T}$ , for any  $q \in \mathbb{Q}^+$ ,  $X_q$  is invariant by conjugation by its own support. Using a continuity argument allows to extend the result for any  $q \in \mathbb{R}^+$ . The result follows from Proposition 31:  $X$  is a mixture of self-invariant by conjugation Lévy process.

The second part of the theorem is deduced from Theorem 10 applied to the increments of  $X$ .  $\square$

*9.4.3. Degeneracy of the mixture.* — In this subsection, we generalize the previous sections in the setting of processes. The proofs will be omitted as the theorems follow directly from their counterpart in the setting of sequences and from Theorem 15.

Let us state the consequence of Proposition 26 for processes.

**Proposition 32.** — *Let  $G$  be a finite group. Let  $X$  be a stochastically continuous  $G$ -valued process invariant by conjugation by  $G$  such that  $X_0 = e$  and which has braidable and  $\mathcal{I}$ -independent increments. The following assertions are equivalent:*

1.  $X$  is a pure non-degenerate Lévy process,
2. there exists  $t \in \mathbb{R}^+$  such that  $\text{Supp}(X_t) = G$ .

*If one of the two conditions holds then for any  $t \in \mathbb{R}^+$ ,  $\text{Supp}(X_t) = G$ .*

Let us remark that, in order to prove the last proposition, we replace the property of  $\text{Supp}(m^{*k})$ , we used in the proof of Proposition 26, by Lemma 20.

**Lemma 20.** — *Let  $G$  be a finite group. Let  $(Y_t)_{t \geq 0}$  be a Lévy process on  $G$ , we have for any real  $t \geq 0$ ,  $\text{Supp}(Y_t) = H_Y$ .*

We now state the consequence of Proposition 27.

**Proposition 33.** — *Let  $G$  be a compact Lie group. Let  $X$  be a stochastically continuous  $G$ -valued process invariant by conjugation by  $G$  such that  $X_0 = e$  and which has braidable and  $\mathcal{I}$ -independent increments. Let us suppose that  $e \in \text{Supp}(X_t)$  for any  $t \in \mathbb{R}^+$ . The following conditions are equivalent:*

1. the process  $X$  is a pure non-degenerate Lévy process,
2. the random variable  $X_t$  converges in law to a Haar random variable on  $G$  when  $t$  goes to infinity.

If one has a better understanding on  $\text{Supp}(X_t)$  when  $X_t$  is a Lévy process on a compact group, then one could remove the condition  $e \in \text{Supp}(X_t)$  in the last theorem. Indeed, we believe that for any invariant by conjugation Lévy process  $(X_t)_{t \in \mathbb{R}^+}$  on a compact Lie group, there exist  $H$  a group of  $G$  and  $a \in \mathfrak{g}$  such that

for any  $t \geq 0$ ,  $\text{Supp}(X_t) = e^{ta}H$ . Yet, what seems almost trivial at first view, does not seem easy to get, once one reads the work of Tortrat.

In order to conclude this section, it remains to state the consequence of Theorem 13.

**Theorem 16.** — *Let  $G$  be a compact Lie group. Let  $X$  be a stochastically continuous  $G$ -valued process invariant by conjugation by  $G$  such that  $X_0 = e$  and which has braidable and  $\mathcal{I}$ -independent increments. The following assertions are equivalent:*

1. *the process  $X$  is a pure (i.e. invariant by conjugation by  $G$ ) Lévy process,*
2. *there exists  $Z$ , a  $G$ -valued Lévy process such that for any  $t \in \mathbb{R}^+$ ,  $X_t$  has the same law as  $Z_t$ .*

Let us suppose that  $X$  satisfies the hypothesis of the last theorem. By Theorem 15 there exists a Lévy process  $Y$  which is self-invariant by conjugation and such that for any function  $f$  invariant by conjugation by  $G$ ,  $\mathbb{E}[f(X_t)] = \mathbb{E}[f(Y_t)]$ . Let us suppose that the condition 2 holds: there exists  $Z$ , a  $G$ -valued Lévy process, invariant by conjugation by  $G$ , such that for any  $t \in \mathbb{R}^+$ ,  $X_t$  has the same law as  $Z_t$ . This means that the Lévy process  $Y$  satisfies for any  $t \geq 0$  and any invariant by conjugation function  $f$  on  $G$ :

$$\mathbb{E}[f(Y_t)] = \mathbb{E}[f(Z_t)].$$

This leads us to define the notion of one marginal quasi-invariant by conjugation Lévy processes. These are Lévy processes  $Y = (Y_t)_{t \geq 0}$  such that there exists a conjugation-invariant Lévy process  $Z = (Z_t)_{t \geq 0}$  such that for any  $t \geq 0$  and any invariant by conjugation function  $f$  on  $G$ ,  $\mathbb{E}[f(Y_t)] = \mathbb{E}[f(Z_t)]$ .

The last theorem is now equivalent to the fact that any one marginal quasi-invariant by conjugation Lévy process which has auto-invariant by conjugation increments is a Lévy process invariant by conjugation by  $G$ . But what about a general one marginal quasi-invariant by conjugation Lévy process, is it necessarily a Lévy process invariant by conjugation by  $G$ ? The answer is negative, as one can see that the Lévy process associated with  $(\mu_t)_{t \geq 0}$  defined in Proposition 30 is a one marginal quasi-invariant by conjugation Lévy process. Indeed, if  $Z$  is the Lévy process associated with  $(\eta_t)_{t \geq 0}$  defined in the same proposition, for any  $t \geq 0$  and any invariant by conjugation function  $f$  on  $G$ ,  $\mathbb{E}[f(Y_t)] = \mathbb{E}[f(Z_t)]$ .

We have put the emphasis on Theorem 16 as it has the simplest formulation. Yet, a more general theorem is the following.

**Theorem 17.** — *Let  $G$  be a compact Lie group. Let  $X$  be a stochastically continuous  $G$ -valued process invariant by conjugation by  $G$  such that  $X_0 = e$  and which has braidable and  $\mathcal{I}$ -independent increments. The following assertions are equivalent:*

1. *the process  $X$  is a pure (i.e. invariant by conjugation by  $G$ ) Lévy process,*

2. there exists  $(m_t)_{t \in \mathbb{R}^+}$  a family of measures on  $G$  invariant by conjugation by  $G$  such that for any  $t \geq 0$  and any  $k \in \mathbb{N}$ , the law of  $X_{kt}$  is  $m_t^{*k}$ .

*Proof.* — Let  $X$  be a  $G$ -valued process invariant by conjugation by  $G$  such that  $X_0 = e$  and which has braidable and  $\mathcal{I}$ -independent increments. If it is a pure Lévy process, the condition 2 is satisfied by setting, for any  $t \in \mathbb{R}^+$ ,  $m_t$  equal to the law of  $X_t$ .

Now let us suppose condition 2. Let  $(m_t)_{t \in \mathbb{R}^+}$  be a family of measures on  $G$  invariant by conjugation by  $G$  such that for any positive integer  $k$  and any  $t \geq 0$ , the law of  $X_{kt}$  is  $m_t^{*k}$ . Let  $t \in \mathbb{R}^+$  and let us consider the sequence  $\xi = \left( X_{kt} X_{(k-1)t}^{-1} \right)_{k \geq 1}$ . Using the stochastic continuity, this is a sequence of  $G$ -valued random variables which is braidable, invariant by diagonal conjugation by  $G$  and satisfies property  $(\mathcal{P})$ . Besides, for any integer  $k$ , the law of  $\prod_{i=k}^1 \xi_i$  is the law of  $X_{kt}$  and thus is equal to  $m_t^{*k}$ . By Theorem 13,  $\xi$  is a sequence of i.i.d. random variables invariant by conjugation by  $G$ . The process  $X$  is a Lévy process which is also invariant by conjugation by  $G$ .  $\square$

## 10. Characterization of stochastically continuous discrete Planar Markovian Holonomy Fields

In this section we will show the following theorem.

**Theorem 18.** — *Let  $(\mathbb{E}_{vol}^G)_{\mathbb{G}, vol}$  be a  $G$ -valued stochastically continuous weak discrete Planar Markovian Holonomy Field. There exists a  $G$ -valued Lévy process,  $(Y_t)_{t \geq 0}$ , self-invariant by conjugation such that  $(\mathbb{E}_{vol}^G)_{\mathbb{G}, vol}$  is the discrete planar Yang-Mills field associated with  $(Y_t)_{t \geq 0}$ . This means that for any measure of area  $vol$  and any graph  $\mathbb{G}$ :  $\mathbb{E}_{vol}^G = \mathbb{E}_{vol}^{Y, \mathbb{G}}$ .*

*If  $G$  is Abelian, the Lévy process is unique, and is characterized by the fact that for any simple loop  $l$  in  $\mathbb{R}^2$ , under  $\mathbb{E}_{vol}^{\mathbb{G}(l)}$ ,  $h(l)$  has the law as  $Y_{vol(\text{Int}(l))}$ .*

It is important to note that the Lévy process  $(Y_t)_{t \geq 0}$  is not unique if  $G$  is non-Abelian: it is unique up to an equivalence. We say that  $(Y_t)_{t \geq 0}$  and  $(Y'_t)_{t \geq 0}$  are equivalent if they have the same law when we restrict their law to the invariant  $\sigma$ -field on  $G^{\mathbb{R}^+}$ . We believe, but did not manage to prove it, that two Lévy processes are equivalent if and only if there exists a non-random element  $g \in G$  such that the law of  $(Y_t)_{t \geq 0}$  and the law of  $(g^{-1}Y'_t g)_{t \geq 0}$  are equal.

The Theorem 18 can be now summarized as:

**Theorem 19.** — *There exists a one-to-one correspondence between the set of equivalence classes of  $G$ -valued self-invariant by conjugation Lévy processes and the set of  $G$ -valued stochastically continuous in law weak discrete Planar Markovian Holonomy Fields.*

Before proving this theorem, let us consider its consequences.

**Theorem 20.** — *For a discrete Planar Markovian Holonomy Field, the following conditions are equivalent:*

- *it is stochastically continuous,*
- *it is locally stochastically  $\frac{1}{2}$ -Hölder continuous and continuously area-dependent.*

*Proof.* — If a discrete Planar Markovian Holonomy Field is stochastically continuous, by Theorem 18, it is a planar Yang-Mills field. By the proof of Proposition 22, it is locally stochastically  $\frac{1}{2}$ -Hölder continuous and continuously area-dependent.

Besides, if a discrete Planar Markovian Holonomy Field is locally stochastically  $\frac{1}{2}$ -Hölder continuous and continuously area-dependent, by Theorem 5, it is the restriction of a stochastically continuous Planar Markovian Holonomy Field. Thus by Remark 11, it is a stochastically continuous discrete Planar Markovian Holonomy Field.  $\square$

We have defined, in Section 4, four different notions of Planar Markovian Holonomy Fields. By now, we know that, by restriction, a strong Planar Markovian Holonomy Field defines a weak continuous one. Using the results of Section 5, a weak Planar Markovian Holonomy Field is, also when restricted, a discrete Planar Markovian Holonomy Field. Theorem 18 now allows us to show that the four different notions are equal when one considers stochastically continuous processes. Indeed, a stochastically continuous in law weak discrete Planar Markovian Holonomy Field is the restriction of a planar Yang-Mills field, which by the results of Section 8 was shown to be a  $G$ -valued stochastically continuous strong Planar Markovian Holonomy Field. Besides any planar Yang-Mills field is constructible. This discussion allows us to state the following theorems.

**Theorem 21.** — *Any  $G$ -valued stochastically continuous in law strong Planar Markovian Holonomy Field is a planar Yang-Mills fields: it is constructible.*

**Proposition 34.** — *Any  $G$ -valued stochastically continuous in law weak discrete Planar Markovian Holonomy Field is the restriction of a  $G$ -valued stochastically continuous strong Planar Markovian Holonomy Field.*

It is, at first, easy to be lost with the four notions of Planar Markovian Holonomy Fields: we encourage the reader to have a look at the diagram page 116 where we drawn the different links between all the notions introduced or used in this paper.

The last consequence of Theorem 18 is the Proposition 13. The reader can found below the proof of this proposition, but before explaining it, we think it would be interesting to explain how to construct a  $G$ -valued stochastically continuous discrete Planar Markovian Holonomy Field ( $\mathbb{E}_{vol}^G$ ) for which the natural restriction defined in Section 4.2.2 is not a discrete Planar Markovian Holonomy Field.



For this, we only have to consider the symmetrical group  $G = \mathfrak{S}_3$ . Let  $H$  be the subgroup of  $G$  isomorphic to  $\mathbb{Z}/3\mathbb{Z}$  composed of the neutral element  $e$ , and the two 3-cycles  $c$  and  $c^2$ . Let  $X$  be a  $H$ -valued Lévy process which jumps only by multiplication by  $c$ . As  $H$  is abelian,  $X$  is a self-invariant by conjugation  $G$ -valued Lévy process, and because of the condition on the jumps, for any time  $t$ :

$$(21) \quad \mathbb{P}[X_t = c] \neq \mathbb{P}[X_t = c^2].$$

Let  $(\mathbb{E}_{vol}^X)_{vol}$  be the  $G$ -valued planar Yang-Mills field associated with  $X$  and let us consider the restriction  $(\mathbb{E}_{vol}^{X,G})_{vol,G}$  of  $(\mathbb{E}_{vol}^X)_{vol}$  to the planar graphs: it is a  $G$ -valued stochastically continuous discrete Planar Markovian Holonomy Field. As  $H$  is normal in  $G$ , for any measure of area  $vol$ , for any finite planar graph  $\mathbb{G}$  and any loop in  $\mathbb{L}(\mathbb{G})$ , under  $\mathbb{E}_{vol}^{X,G}$ ,  $h(l)$  is almost surely in  $H$ . The natural restriction of  $(\mathbb{E}_{vol}^{X,G})_{\mathbb{G},vol}$ , as defined in Section 4.2.2 is not a  $H$ -valued discrete planar discrete Planar Markovian Holonomy Field: it does not satisfy the weak independence property. This is due to the fact that if  $U, A$  and  $B$  are three random variables such that  $U$  is a Haar variable on  $G$ ,  $A$  and  $B$  have the same law as  $X_1$ , then the two random variables  $UAU^{-1}$ ,  $UBU^{-1}$  are not independent: indeed, (21) implies that  $\mathbb{P}[(UAU^{-1}, UBU^{-1}) = (c, c)] \neq \mathbb{P}[UAU^{-1} = c]\mathbb{P}[UBU^{-1} = c]$ .

We now give the proof of Proposition 13.

*Proof of Proposition 13.* — By Theorem 18,  $(\mathbb{E}_{vol}^G)_{vol,G}$  is a planar Yang-Mills field associated with a self-invariant by conjugation Lévy process  $(Y_t)_{t \geq 0}$ . Besides, the law of a simple loop  $l$  under  $\mathbb{E}_{vol}^{G(l)}$  is equal to  $\int_G m_{vol}^g(\text{Int}(l)) dg$  where  $m_{vol}^g(\text{Int}(l))$  is the law of  $Y_{vol(\text{Int}(l))}$ . Hence, under the hypothesis of Proposition 13,  $(Y_t)_{t \geq 0}$  is actually  $H$ -valued. We can thus define, thanks to Theorem 8, the  $H$ -valued Yang-Mills field associated with  $(Y_t)_{t \geq 0}$ . By Proposition 24, its restriction to graphs is a  $H$ -valued stochastically continuous discrete Planar Markovian Holonomy Field and it is a simple calculation to see that it satisfies the required conditions.  $\square$

Let us begin the proof of Theorem 18. In Section 10.1, we show that the “two-dimensional time” objects which are the Markovian Holonomy Fields are characterized by a “one-dimensional time” process. Then in Section 10.2, it is shown that this “one-dimensional” time process has  $\mathcal{I}$ -independent increments. This allows us to prove Theorem 18 when  $G$  is abelian. In general, the result follows from the braidability of the “one-dimensional time” process which is proved in Section 10.3.

**10.1. First correspondence for  $G$ -valued discrete Planar Markovian Holonomy Field.**— We can go further than Proposition 17 in the characterization of discrete planar stochastically continuous Markovian Holonomy Fields. We will need for that a continuous version of the loops  $L_{i,j}$ .

**Definition 63.** — For any  $0 \leq s < t$  we define  $\partial c_s^t = (s, 1) \rightarrow (s, 0) \rightarrow (t, 0) \rightarrow (t, 1) \rightarrow (s, 1)$ , and  $p_s = (0, 0) \rightarrow (0, 1) \rightarrow (s, 1)$ . In the following, we will focus on the loops  $L_s^t$  defined by:

$$L_s^t = p_s \partial c_s^t p_s^{-1}.$$

**Remark 15.** — These loops satisfy the following equalities:

$$\begin{aligned} L_r^t &= L_r^s L_s^t, \forall 0 < r < s < t, \\ L_i^{i+1} &= L_{i,0}, \forall i \in \mathbb{N}. \end{aligned}$$

In Theorem 22, we will show that the process  $h(L_0^t)$  gives all the information one needs in order to understand a Planar Markovian Holonomy Field. This theorem is motivated by the following lemma which is a straightforward application of Theorem 8.

**Lemma 21.** — Let  $Y$  be a self-invariant by conjugation  $G$ -valued Lévy process. Let  $U$  be a Haar variable on  $G$  which is independent from  $Y$ . Let  $(\mathbb{E}_{vol}^Y)_{vol}$  be the stochastically continuous Planar Markovian Holonomy Field associated with  $Y$ . Under  $\mathbb{E}_{dx}^Y$ ,  $(h(L_0^t))_{t \in \mathbb{R}^+}$  has the same law as  $(UY_t U^{-1})_{t \in \mathbb{R}^+}$ .

**Theorem 22.** — If  $(\mathbb{E}_{vol}^G)_{G, vol}$  and  $(\tilde{\mathbb{E}}_{vol}^G)_{G, vol}$  are two stochastically continuous in law  $G$ -valued weak discrete Planar Markovian Holonomy Fields, the three assertions are equivalent:

1.  $(\mathbb{E}_{vol}^G)_{G, vol}$  and  $(\tilde{\mathbb{E}}_{vol}^G)_{G, vol}$  are equal,
2.  $(h(L_0^t))_{t \in \mathbb{R}^+}$  has the same law under  $\mathbb{E}_{dx, \gamma_0}$  as under  $\tilde{\mathbb{E}}_{dx, \gamma_0}$ ,
3. for any positive real  $\alpha$ ,  $(h(L_{n,0}))_{n \in \mathbb{N}}$  has the same law under  $\mathbb{E}_{\alpha, dx}^{\mathbb{N}^2}$  as under  $\tilde{\mathbb{E}}_{\alpha, dx}^{\mathbb{N}^2}$ .

We remind the reader that  $L_{i,j}$  was defined in Definition 45,  $\gamma_0$  at the beginning of Section 2 and  $\mathbb{E}_{dx, \gamma_0}$  in Remark 10.

*Proof.* — We will prove the equivalence between point 1 and point 2, then between 2 and 3. As the condition 1 clearly implies condition 2, let us show that 2 implies 1. Let  $(\mathbb{E}_{vol}^G)_{G, vol}$  and  $(\tilde{\mathbb{E}}_{vol}^G)_{G, vol}$  be two stochastically continuous in law  $G$ -valued weak discrete Planar Markovian Holonomy Fields.

Let us suppose that  $(h(L_0^t))_{t \in \mathbb{R}^+}$  has the same law under  $\mathbb{E}_{dx, \gamma_0}$  as under  $\tilde{\mathbb{E}}_{dx, \gamma_0}$ . Let  $vol$  be a measure of area and let  $\mathbb{G}$  be a simple connected finite planar graph in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$ . We have to show that  $\mathbb{E}_{vol}^G = \tilde{\mathbb{E}}_{vol}^G$ . The proof will consist in a sequence of simplifications, changing the graph and the measure of area little by little. Let  $v$  be a vertex of  $\mathbb{G}$ . By Proposition 5, these measures are characterized by the way they integrate functions of the form:  $h \mapsto f(h(l_1), \dots, h(l_m))$ , where

$f$  is a continuous function invariant by conjugation and  $l_1, \dots, l_n$  are elements of  $L_v(\mathbb{G})$ . Thus, we have to show that:

$$(\mathbb{E}_{vol}^{\mathbb{G}})_{|L_v(\mathbb{G})} = (\tilde{\mathbb{E}}_{vol}^{\mathbb{G}})_{|L_v(\mathbb{G})}.$$

The goal now is to inject the graph  $\mathbb{G}$  in  $\mathbb{N}^2$ . Yet, in order to do so, we need to introduce generic graphs. By Lemma 9, let us consider  $(\mathbb{G}_n)_{n \geq 0}$ , a sequence of generic graphs such that  $v \in G_n$  for any integer  $n$  and let us consider for each integer  $n$ ,  $\mathcal{L}_n : L_v(\mathbb{G}) \rightarrow L_v(\mathbb{G}_n)$  an injective function such that for any loop  $l$  in  $L_v(\mathbb{G})$ ,  $\mathcal{L}_n(l)$  converges with fixed endpoints to  $l$ .

Using the stochastic continuity in law of  $(\mathbb{E}_{vol}^{\mathbb{G}})_{\mathbb{G}, vol}$  (resp.  $(\tilde{\mathbb{E}}_{vol}^{\mathbb{G}})_{\mathbb{G}, vol}$ ), the measure  $(\mathbb{E}_{vol}^{\mathbb{G}})_{|L_v(\mathbb{G})}$  (resp.  $(\tilde{\mathbb{E}}_{vol}^{\mathbb{G}})_{|L_v(\mathbb{G})}$ ) can be recovered using the sequence of measures  $\left( (\mathbb{E}_{vol}^{\mathbb{G}_n})_{|L_v(\mathbb{G}_n)} \right)_{n \in \mathbb{N}}$  (resp.  $\left( (\tilde{\mathbb{E}}_{vol}^{\mathbb{G}_n})_{|L_v(\mathbb{G}_n)} \right)_{n \in \mathbb{N}}$ ). Thus, we can suppose that  $\mathbb{G}$  is generic.

Using Corollary 1, there exists  $\mathbb{G}'$  a subgraph of the  $\mathbb{N}^2$  graph such that the set of  $\mathbb{G} - \mathbb{G}'$  piecewise diffeomorphisms is not empty. Let  $\psi$  be such a homeomorphism. Let  $vol'$  be a measure of area on  $\mathbb{R}^2$  such that for any face  $F$  of  $\mathbb{G}$ ,  $vol'(\psi(F)) = vol(F)$ . Then using axiom **wDP**<sub>1</sub>, we know that:

$$\mathbb{E}_{vol'}^{\mathbb{G}'} \circ \psi^{-1} = \mathbb{E}_{vol}^{\mathbb{G}}.$$

We recall also that, by definition of  $\mathbb{E}_{vol'}^{\mathbb{N}^2}$ ,  $\mathbb{E}_{vol'}^{\mathbb{G}'} = \left( \mathbb{E}_{vol'}^{\mathbb{N}^2} \right)_{|Mult(P(\mathbb{G}'), \mathbb{G})}$ . The same discussion holds for  $(\tilde{\mathbb{E}}_{vol}^{\mathbb{G}})_{\mathbb{G}, vol}$ . Thus, if we show that, for any measure of area  $vol'$ ,

$$\mathbb{E}_{vol'}^{\mathbb{N}^2} = \tilde{\mathbb{E}}_{vol'}^{\mathbb{N}^2},$$

we will get the conclusion that for any measure of area  $vol$  and any connected finite planar graph  $\mathbb{G}$  in  $\mathcal{G}(\text{Aff}(\mathbb{R}^2))$ :  $\mathbb{E}_{vol}^{\mathbb{G}} = \tilde{\mathbb{E}}_{vol}^{\mathbb{G}}$ .

Let  $vol'$  be a measure of area on  $\mathbb{R}^2$ . Since  $\{L_{i,j}, i, j \in \mathbb{N}^2\}$  is a family which generates the group of reduced loops based at 0 and as we are considering gauge-invariant measures, we only have to prove that  $(h(L_{i,j}))_{i,j}$  has the same law under  $\mathbb{E}_{vol'}^{\mathbb{N}^2}$  as under  $\tilde{\mathbb{E}}_{vol'}^{\mathbb{N}^2}$ . The goal now is to show that it is actually enough to know that  $(h(L_{n,0}))_{n \in \mathbb{N}}$  has the same law under  $\mathbb{E}_{vol'}^{\mathbb{N}^2}$  as under  $\tilde{\mathbb{E}}_{vol'}^{\mathbb{N}^2}$ . We will only consider  $(h(L_{i,j}))_{i,j \in \{0,1\}}$  and we will give a graphical proof.

We consider the two graphs  $\mathbb{G}$  and  $\tilde{\mathbb{G}}$  drawn in Figure 19. Let  $vol''$  be a measure of area such that  $vol''|_{F_\infty} = vol'|_{\tilde{F}_\infty}$ , where  $F_\infty$  (resp.  $\tilde{F}_\infty$ ) is the unbounded face of  $\mathbb{G}$  (resp.  $\tilde{\mathbb{G}}$ ). Besides, we impose that the following condition holds for  $vol''$ :

$$(22) \quad \forall i \in \{1, \dots, 5\}, vol''(F'_i) = vol'(F_i).$$

The loops  $(L_{i,j})_{i,j \in \{0,1\}}$  belong to  $L_{(0,0)}(\mathbb{G})$ , and the loops  $(L_{i,0})_{i \in [0,3]}$  are in  $L_{(0,0)}(\tilde{\mathbb{G}})$ . The goal now is to show that in some sense, one can send the graph  $\mathbb{G}$



as the intersection of the  $\mathbb{N}^2$  graph and  $[0, n+1] \times [0, 1]$ . We define the following homeomorphism:

$$\begin{aligned} \psi : \quad \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (x, y) &\mapsto (\operatorname{sgn}(x).vol''([0, x] \times [0, 1]), y). \end{aligned}$$

The image of  $\mathbb{S}$  by  $\psi$ ,  $\psi(\mathbb{S})$  is a simple graph in  $\mathcal{G}(\operatorname{Aff}(\mathbb{R}^2))$ . Besides, for any bounded face  $F$  of  $\mathbb{S}$ ,  $vol''(F) = dx(\psi(F))$ . Let us define for any  $i \in \{0, \dots, n+1\}$ ,  $t_i = vol''(L_0^i)$ . We can apply the Axiom **wDP**<sub>1</sub> to the two graphs  $\mathbb{S}$  and  $\psi(\mathbb{S})$ , to the two measures of area  $vol''$  and  $dx$  and to  $\psi$ . It shows that under  $\mathbb{E}_{vol}^{\mathbb{N}^2}$  (resp.  $\tilde{\mathbb{E}}_{vol}^{\mathbb{N}^2}$ ),  $\left(h(L_{i,0})\right)_{i=0}^n$  has the same law as  $\left(h(L_{t_i}^{t_{i+1}})\right)_{i=0}^n$  under  $\mathbb{E}_{dx, \gamma_0}$  (resp.  $\tilde{\mathbb{E}}_{dx, \gamma_0}$ ).

Thus it remains to show that for any integer  $n$ , any sequence of positive reals  $t_0 < \dots < t_n$ ,  $\left(h(L_{t_i}^{t_{i+1}})\right)_{i=0}^n$  has the same law under  $\mathbb{E}_{dx, \gamma_0}$  and under  $\tilde{\mathbb{E}}_{dx, \gamma_0}$ . Yet, we started with the fact that  $\left(h(L_0^t)\right)_{t \in \mathbb{R}^+}$  has the same law under  $\mathbb{E}_{dx, \gamma_0}$  as under  $\tilde{\mathbb{E}}_{dx, \gamma_0}$ . This allows us to conclude.

It remains to show the equivalence between the conditions 2. and 3. Suppose that for any positive real  $\alpha$ ,  $\left(h(L_{n,0})\right)_{n \in \mathbb{N}}$  has the same law under  $\mathbb{E}_{\alpha dx}^{\mathbb{N}^2}$  as under  $\tilde{\mathbb{E}}_{\alpha dx}^{\mathbb{N}^2}$ . By the Axiom **wDP**<sub>4</sub>, we can change  $\mathbb{E}_{\alpha dx}^{\mathbb{N}^2}$  (resp.  $\tilde{\mathbb{E}}_{\alpha dx}^{\mathbb{N}^2}$ ) by  $\mathbb{E}_{\alpha dx, \gamma_0}$  (resp.  $\tilde{\mathbb{E}}_{\alpha dx, \gamma_0}$ ). As an application of the Axiom **wDP**<sub>1</sub>, with  $\psi$  equal to:

$$\psi : (x, y) \mapsto (\alpha x, y),$$

the random vector  $\left(h(L_{\alpha n}^{\alpha(n+1)})\right)_{n \in \mathbb{N}}$  has the same law under  $\mathbb{E}_{dx, \gamma_0}$  as under  $\tilde{\mathbb{E}}_{dx, \gamma_0}$ . Using the fact that:

$$h\left(L_0^{\frac{p}{q}}\right) = \prod_{i=p-1}^0 h\left(L_{\frac{i}{q}}^{\frac{i+1}{q}}\right),$$

we can conclude that  $\left(h(L_t)\right)_{t \in \mathbb{Q}^+}$  has the same law under  $\mathbb{E}_{dx, \gamma_0}$  as under  $\tilde{\mathbb{E}}_{dx, \gamma_0}$  and by stochastic continuity the same assertion holds for  $\left(h(L_t)\right)_{t \in \mathbb{R}^+}$ . Actually each implication in our former proof is an equivalence, showing thus the equivalence between condition 2. and condition 3.  $\square$

**10.2. Law of the conjugacy classes, the Abelian case.** — Let  $(\mathbb{E}_{vol}^G)_{G, vol}$  be a stochastically continuous in law  $G$ -valued weak discrete Planar Markovian Holonomy Field and let  $\mathbb{E}_{dx, \gamma_0}$  be the usual expectation associated with.

**Definition 64.** — *Until the end of this section, we define for any  $0 \leq s < t$ ,*

$$Z_s^t = h(L_s^t),$$

and  $Z_t = Z_0^t$ .

**Remark 16.** — Thanks to the multiplicativity property of random holonomy fields and Remark 15, it satisfies for any  $0 \leq r < s < t$ ,  $Z_r^t = Z_s^t Z_r^s$ , hence for any  $0 < s < t$ ,

$$Z_s^t = Z_t(Z_s)^{-1}.$$

Besides,  $Z_s^t$  is equal to  $h(p_s \partial c_s^t p_s^{-1})$ . By Remark 7, under any gauge-invariant measure on a subset of paths, containing  $p_s$  and  $\partial c_s^t$  and satisfying the hypothesis of Proposition 5, it has the same law as  $h(\partial c_s^t)$ . The left translation by  $s$  sends  $dx$  on itself and  $\partial c_s^t$  on  $\partial c_0^{t-s}$ : applying the diffeomorphism invariance **wDP**<sub>1</sub> for discrete Planar Markovian Holonomy Fields (Definition 36), we get that under  $\mathbb{E}_{dx, \gamma_0}$ ,

$$Z_s^t \text{ has the same law as } Z_{t-s}.$$

A last remark which can be made is that, thanks to the stochastic continuity property, under  $\mathbb{E}_{dx, \gamma_0}$ ,  $(Z_t)_{t \geq 0}$  is continuous in law. Besides  $Z_t$  converges in law, to the Dirac measure on  $e$  at  $t = 0$ . Indeed, as  $L_0^t$  converges to  $(0, 0) \rightarrow (0, 1) \rightarrow (0, 0)$ ,  $Z_t$  converges in law to  $h((0, 0) \rightarrow (0, 1) \rightarrow (0, 0))$  which by multiplicativity of  $h$  is constant equal to the neutral element of  $G$ .

A simple, yet important lemma is the following.

**Lemma 22.** — Under  $\mathbb{E}_{dx, \gamma_0}$ , for any  $t_0 > 0$ , for any finite subset  $T$  of  $[0, t_0]$  and any finite subset  $T'$  of  $[t_0, \infty[$ ,  $(Z_t)_{t \in T}$  and  $(Z_t Z_{t_0}^{-1})_{t \in T'}$  are  $\mathcal{I}$ -independent. This means that for any continuous functions  $f : G^T \rightarrow \mathbb{R}$  and  $f' : G^{T'} \rightarrow \mathbb{R}$  invariant by diagonal conjugation by  $G$ ,

$$\mathbb{E}_{dx, \gamma_0} \left[ f \left( (Z_t)_{t \in T} \right) f' \left( (Z_t Z_{t_0}^{-1})_{t \in T'} \right) \right] = \mathbb{E}_{dx, \gamma_0} \left[ f \left( (Z_t)_{t \in T} \right) \right] \mathbb{E}_{dx, \gamma_0} \left[ f' \left( (Z_t Z_{t_0}^{-1})_{t \in T'} \right) \right].$$

*Proof.* — Let  $t_0 > 0$ , let  $T$  be a finite subset of  $[0, t_0]$  and let  $T'$  be a finite subset of  $[t_0, \infty[$ . Obviously we can suppose that  $T' \subset ]t_0, \infty[$ . Let  $t'_0$  be any real strictly greater than  $t_0$  such that  $T' \subset [t'_0, \infty[$ . We remind the reader that for any  $t \in T'$ ,

$$Z_t Z_{t'_0}^{-1} = h(p_{t'_0} \partial c_{t'_0}^t p_{t'_0}^{-1}),$$

thus for any continuous functions  $f : G^T \rightarrow \mathbb{R}$  and  $f' : G^{T'} \rightarrow \mathbb{R}$  invariant by diagonal conjugation by  $G$ :

$$\mathbb{E}_{dx, \gamma_0} \left[ f \left( (Z_t)_{t \in T} \right) f' \left( (Z_t Z_{t'_0}^{-1})_{t \in T'} \right) \right] = \mathbb{E}_{dx, \gamma_0} \left[ f \left( (h(L_0^t))_{t \in T} \right) f' \left( (h(\partial c_{t'_0}^t))_{t \in T'} \right) \right].$$

Let us denote by  $t_1$  the maximum of  $T'$ . The loop  $L_0^t$  is in  $\overline{\text{Int}(L_0^{t_0})}$  for any  $t \in T$ , and the loop  $\partial c_{t'_0}^t$  is in  $\overline{\text{Int}(\partial c_{t'_0}^{t_1})}$  for any  $t \in T'$ . Besides we have  $\overline{\text{Int}(L_0^{t_0})} \cap \overline{\text{Int}(\partial c_{t'_0}^{t_1})} = \emptyset$ . We can thus apply **wP**<sub>2</sub>:

$$\mathbb{E}_{dx, \gamma_0} \left[ f \left( (Z_t)_{t \in T} \right) f' \left( (Z_t Z_{t'_0}^{-1})_{t \in T'} \right) \right] = \mathbb{E}_{dx, \gamma_0} \left[ f \left( (Z_t)_{t \in T} \right) \right] \mathbb{E}_{dx, \gamma_0} \left[ f' \left( (Z_t Z_{t'_0}^{-1})_{t \in T'} \right) \right].$$

Using the stochastic continuity of  $\mathbb{E}_{dx}^{\gamma_0}$  and taking the limit  $t'_0 \rightarrow t_0$ , we get the Lemma 22.  $\square$

When  $G$  is Abelian, for any  $n$ -tuple  $(g_1, \dots, g_n)$  of elements of  $G$ , the diagonal conjugacy class of  $(g_1, \dots, g_n)$  is reduced to  $\{(g_1, \dots, g_n)\}$ . Thus, the last lemma asserts that  $\sigma(\{Z_t, t \leq t_0\})$  is independent of  $\sigma(\{Z_t(Z_{t_0})^{-1}, t \geq t_0\})$ . Thanks to Remark 16,  $(Z_t)_{t \in \mathbb{R}^+}$  is a Lévy process. Applying Theorem 22 and Lemma 21,  $(\mathbb{E}_{vol}^{\mathbb{G}})_{\mathbb{G}, vol}$  is the planar Yang-Mills field associated with the Lévy process  $(Z_t)_{t \geq 0}$ . The Abelian part of Theorem 18 is thus proved.

As we have seen in Remark 4, it is not enough to know the independence of the conjugacy classes of a couple of random variables in order to understand the law of this couple. Thus, when  $G$  is not Abelian, we have to get rid of the conjugacy classes in Lemma 22: it is what we intend to do in the following subsection.

**10.3. Braidability, the non-commutative case.** — The following property satisfied by the process  $(Z_t)_{t \in \mathbb{R}^+}$  under  $\mathbb{E}_{dx, \gamma_0}$  will be the key point for the proof of Theorem 18. Recall the notion and notations used for the (rational) increments in Definition 60.

**Proposition 35.** — *Under  $\mathbb{E}_{dx, \gamma_0}$ ,  $(Z_t)_{t \in \mathbb{R}^+}$  has braidable increments.*

*Proof.* — The proof will be essentially graphical. The braid group with  $m$  strands is generated by the elementary braids  $(\beta_i)_{i=1}^{m-1}$  defined in Section 7. This allows us to reduce the braidability condition to the fact that for any integer  $n$ , any integer  $m$  and any integer  $i < m$ , the following equality in law holds:

$$\beta_i \bullet (Z_{n,1}, \dots, Z_{n,m}) = (Z_{n,1}, \dots, Z_{n,m}).$$

The proof does not depend on the value of  $n$ , we will suppose it equal to 1. We remind the reader that  $Z_{1,i} = h(p_{i-1} \partial c_{i-1}^i p_{i-1})$ : we have to understand the law of the variable associated with  $m$  lassos. Using the stochastic continuity in law of  $\mathbb{E}_{dx, \gamma_0}$ , we can “shrink” the meander of these lassos and we can suppose that their intersection is reduced to the base point as we did in the proof of Theorem 22. Let  $i$  be an integer, we will focus only on what happens in the interior of  $\partial c_{i-1}^{i+1}$ . Let us consider the graphs  $\mathbb{G}_1$  and  $\mathbb{G}_2$  drawn in Figure 21. They represent what is happening in  $\partial c_{i-1}^{i+1}$ : the loops in bold represent the part of the  $i^{th}$  and  $i+1^{th}$  lassos inside  $\partial c_{i-1}^{i+1}$ , and we added to it two paths in dots in order to consider simple graphs. The two graphs satisfy the hypothesis of Theorem 2 thus, they are homeomorphic. Let us consider a homeomorphism  $\phi$  between  $\mathbb{G}_1$  and  $\mathbb{G}_2$  which sends  $F_i$  on  $F'_i$  for any  $i \in \{1, \dots, 5\}$ . By Proposition 10, there exists a  $\mathbb{G}_1 - \mathbb{G}_2$  piecewise diffeomorphism  $\psi$  which sends  $F_i$  on  $F'_i$  for any  $i \in \{1, \dots, 5\}$  as it has to be equivalent to  $\phi$  on  $\mathbb{G}_1$ . Besides, one can remark that  $\mathbb{G}_2$  is the horizontal flip of  $\mathbb{G}_1$ . For any integer  $i \in \{1, \dots, 5\}$ ,  $dx(F_i) = dx(F'_i)$ : the piecewise diffeomorphism  $\psi$  can be taken such that  $dx(\psi(F)) = dx(F)$  for any bounded face  $F$  of  $\mathbb{G}_1$  and  $\psi$  is the identity on the unbounded face of  $\mathbb{G}_1$ . Using the area

preserving homeomorphism invariance axiom  $\mathbf{wDP}_1$ ,  $\mathbb{E}_{dx}^{\mathbb{G}_2} \circ \psi^{-1} = \mathbb{E}_{dx}^{\mathbb{G}_1}$ . Letting the shrinking parameter to zero in this equality allows us to recover the following equality in law, under  $\mathbb{E}_{dx, \gamma_0}$ ,  $\beta_i \bullet (Z_{1,1}, \dots, Z_{1,m}) = (Z_{1,1}, \dots, Z_{1,m})$ .  $\square$

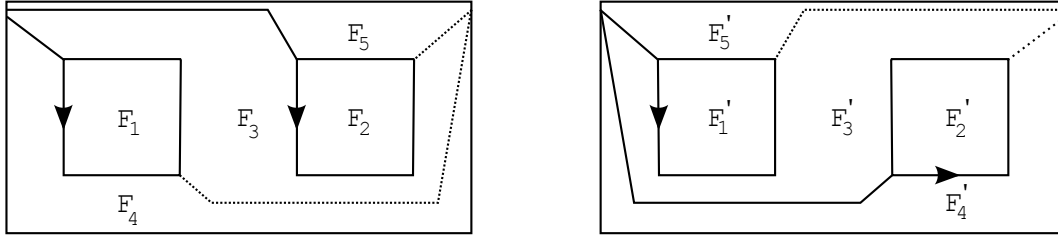


FIGURE 21. The graphs  $\mathbb{G}_1$  and  $\mathbb{G}_2$ .

Let remind the reader that we are working under  $\mathbb{E}_{dx, \gamma_0}$ . Using the results of Section 10.2 and 10.3, we already know that the process  $Z = (Z_t)_{t \geq \mathbb{R}^+}$  is invariant by conjugation by  $G$  and has braidable and  $\mathcal{I}$ -independent increments. By Theorem 15, there exists a self-invariant by conjugation Lévy process  $Y$ , such that the law of  $Z$  is  $UYU^{-1}$ , where  $U$  is a Haar variable on  $G$  independent of  $Y$ . Lemma 21, combined with Theorem 22 allows us to finish the proof of Theorem 18.

### 11. Classification of discrete Planar Markovian Holonomy Fields

We have seen in Theorem 18 that any stochastically continuous discrete Planar Markovian Holonomy Field is a planar Yang-Mills field. Let  $(\mathbb{E}_{vol}^{\mathbb{G}})_{\mathbb{G}, vol}$  be a stochastically continuous discrete Planar Markovian Holonomy Field, let  $Y = (Y_t)_{t \geq 0}$  be a Lévy process associated with  $(\mathbb{E}_{vol}^{\mathbb{G}})_{\mathbb{G}, vol}$ . We defined in Definition 55 the notions of pure non-degenerate/pure degenerate/ mixed degenerate Yang-Mills field, according to the degree of symmetry and the support of  $Y$ . In this section, we will see equivalent conditions to be in each of these categories. The theorems explained below will only be straightforward applications of Theorem 18 and the theorems of Section 9.4.3. Indeed, by definition,  $(\mathbb{E}_{vol}^{\mathbb{G}})_{\mathbb{G}, vol}$  is a pure non-degenerate (resp. pure) Yang-Mills planar field if and only if  $Y$  is pure non-degenerate (resp. pure). Applying Proposition 32 (resp. Proposition 33, resp. Theorem 16) to the process  $Z_t$  defined in Definition 64 allows us to prove Theorem 23 (resp. Theorem 24, resp. Theorem 25).

The first theorem gives an equivalent condition, when  $G$  is a finite group, for a stochastically continuous discrete Planar Markovian Holonomy Field to be a pure non-degenerate Yang-Mills planar field.

**Theorem 23.** — *Let  $G$  be a finite group, let  $(\mathbb{E}_{vol}^{\mathbb{G}})_{\mathbb{G}, vol}$  be a stochastically continuous discrete Planar Markovian Holonomy Field. It is a pure non-degenerate*



Yang-Mills planar field if and only if for any simple loop  $l$ , the support of  $h(l)$  under  $\mathbb{E}_{vol}^{\mathbb{G}(l)}$  is  $G$ .

The second theorem gives an equivalent condition, for any group  $G$ , for a stochastically continuous discrete Planar Markovian Holonomy Field to be a pure non-degenerate Yang-Mills planar field.

**Theorem 24.** — *Let  $(\mathbb{E}_{vol}^{\mathbb{G}})_{\mathbb{G},vol}$  be a stochastically continuous discrete Planar Markovian Holonomy Field. Let us suppose that for any loop  $l$  and any measure of area  $vol$ , under  $\mathbb{E}_{vol}^{\mathbb{G}(l)}$ ,  $e$  is in the support of  $h(l)$ . The Markovian Holonomy Field  $(\mathbb{E}_{vol}^{\mathbb{G}})_{\mathbb{G},vol}$  is a pure non-degenerate Yang-Mills planar field if and only if for any sequence of simple loops  $(l_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}^2$  satisfying  $vol(\text{Int}(l_n)) \xrightarrow[n \rightarrow \infty]{} \infty$ , one has:*

$$\mathbb{E}_{vol}^{\mathbb{G}(l_n)} \circ h(l_n)^{-1} \xrightarrow[n \rightarrow \infty]{} \lambda_G,$$

where  $\lambda_G$  is the Haar measure on  $G$ .

Again, one could remove the condition on the support of  $h(l)$  if one could understand the support of any Lévy process which is invariant by conjugation.

The third theorem gives an equivalent condition, for any compact Lie group  $G$ , for a stochastically continuous discrete Planar Markovian Holonomy Field to be a pure non-degenerate Yang-Mills planar field.

**Theorem 25.** — *Let  $(\mathbb{E}_{vol}^{\mathbb{G}})_{\mathbb{G},vol}$  be a stochastically continuous discrete Planar Markovian Holonomy Field. It is a pure Yang-Mills planar field if and only there exists a Lévy process  $(Z_t)_{t \geq 0}$  such that for any simple loop  $l$ , the law of  $h(l)$  under  $\mathbb{E}_{vol}^{\mathbb{G}(l)}$  is the law of  $Z_{vol(\text{Int}(l))}$ . In this case  $(Z_t)_{t \geq 0}$  is invariant by conjugation Lévy process and it is the Lévy associated with  $(\mathbb{E}_{vol}^{\mathbb{G}})_{\mathbb{G},vol}$ .*

## 12. Markovian Holonomy Fields

We now apply the theory we developed to the theory of Markovian Holonomy Fields defined in [21] by Lévy. The main result of this section is Theorem 28. After some basic definitions, taken from [21], and a resume of the results obtained in the same book, we will define the free boundary expectation on the plane associated with any Markovian Holonomy Field. By doing so, we obtain a Planar Markovian Holonomy Field: this will allow us to apply the results obtain previously in the theory of Planar Markovian Holonomy Fields.

### 12.1. Measured marked surfaces with $G$ -constraints. —

**Definition 65.** — Let  $M$  be a smooth compact surface. To any connected component of the boundary of  $M$  one can associate a non-oriented cycle (Definition 9). The union of these non-oriented cycles is denoted by  $\mathcal{B}(M)$ .

A collection of marks  $\mathcal{C}$  on  $M$  is a finite union of disjoint oriented smooth 1-dimensional submanifolds of the interior of  $M$  such that if  $c \in \mathcal{C}$  then the same curve with the opposite orientation belongs to  $\mathcal{C}$ . It can be seen as a collection of smooth non-oriented cycles on  $M$ . The couple  $(M, \mathcal{C})$  is called a marked surface. If  $M$  is oriented, the orientation of  $M$  induces an orientation on each connected component of the boundary. We call  $\mathcal{B}^+(M)$  the subset of  $\mathcal{B}(M)$  composed of the positively oriented representative of each non-oriented cycle. The marks do not carry a canonical orientation.

From now on, by a smooth surface one has to understand that we consider an oriented smooth surface.

**Definition 66.** — Let  $(M, \mathcal{C})$  be a marked surface. A graph on  $(M, \mathcal{C})$  is a graph on  $M$  such that each cycle of  $\mathcal{C}$  is represented by a loop in  $L(\mathbb{G})$ .

The Proposition 1.3.10. in [21] asserts also that for any graph  $\mathbb{G}$  on  $M$ , any cycle of  $\mathcal{B}(M)$  is represented by a loop in  $L(\mathbb{G})$ .

**Definition 67.** — Let  $(M, \mathcal{C})$  be a marked surface. Let  $\text{Conj}(G)$  be the set of conjugacy classes of  $G$ . A set of  $G$ -constraints on  $(M, \mathcal{C})$  is a mapping  $C : \mathcal{C} \cup \mathcal{B}(M) \rightarrow \text{Conj}(G)$  such that  $C(l^{-1}) = C(l)^{-1}$ . The set of  $G$ -constraints on  $(M, \mathcal{C})$  is denoted by  $\text{Conj}_G(M, \mathcal{C})$ .

**Notation 6.** — Let  $C$  be a set of  $G$ -constraints, let  $c$  be a cycle in  $\mathcal{C} \cup \mathcal{B}(M)$ , and let  $x$  be an element of  $G$ . We will denote by  $C_{c \rightarrow x}$  the unique set of  $G$ -constraints such that:

1. for any cycle  $c' \in \mathcal{C} \cup \mathcal{B}(M)$  different of  $c$  and  $c^{-1}$ ,  $C_{l \rightarrow x}(c') = C(c')$ ,
2.  $C_{c \rightarrow x}(c) = [x]$  and  $C_{c \rightarrow x}(c^{-1}) = [x^{-1}]$ .

**Definition 68.** — A measured marked surface with  $G$ -constraints is a quadruple  $(M, \text{vol}, \mathcal{C}, C)$  where  $(M, \mathcal{C})$  is a marked surface,  $\text{vol}$  is a measure of area on  $M$  and  $C$  is a set of  $G$ -constraints on  $(M, \mathcal{C})$ .

The isomorphism notion on the set of measured marked surfaces with  $G$ -constraints is the following:  $(M, \text{vol}, \mathcal{C}, C)$  and  $(M', \text{vol}', \mathcal{C}', C')$  are isomorphic if and only if there exists a diffeomorphism  $\psi : M \rightarrow M'$  such that:

- $\text{vol} \circ \psi^{-1} = \text{vol}'$ ,
- $\psi$  sends  $\mathcal{C}$  on  $\mathcal{C}'$ ,
- $\forall l \in \mathcal{C} \cup \mathcal{B}(M), C'(\psi(l)) = C(l)$ .

**12.2. Splitting of a surface.** — An important notion for the definition of Markovian Holonomy Fields is the operation of splitting. We will not define this notion properly in this paper but instead we refer the reader to Section 1.1.2 in [21] for a rigorous definition.

Let  $M$  be a smooth compact surface. A splitting of  $M$  is the data of a smooth compact surface  $M'$  and a gluing:  $M' \rightarrow M$ , which is an application which either glues two boundary components of  $M'$  (binary gluing), or glues a boundary of  $M'$  with its inverse (unary gluing). For example, performing a unary gluing on the boundary of a disk leads to  $\mathbb{R}P^2$ . The set which consists of the image of the boundary glued and its inverse is the joint of the gluing: we split according to this mark. Thus, we will consider a splitting as the inverse of the gluing: a splitting is the action to split a surface according to a mark drawn on it.

Besides there is uniqueness (modulo isomorphism) of the splitting according to a mark on a surface  $M$ : the split surface of  $M$  according to  $l$  is denoted by  $Spl_l(M)$ .

Let  $(M, \mathcal{C}, vol, C)$  be a measured marked surface with  $G$ -constraints. Let  $l$  be a mark in  $\mathcal{C}$  and  $f_l : Spl_l(M) \rightarrow M$  be the gluing associated with the splitting  $Spl_l(M)$ . Thanks to the empty intersection of the marks on  $M$ , we can transport the marks on  $Spl_l(M)$ . We will denote them  $Spl_l(\mathcal{C})$ . It is also possible to transport the measure of area on  $Spl_l(M)$  by setting  $Spl_l(vol) = vol \circ f_l$ .

In order to transport the  $G$ -constraints on  $Spl_l(M)$  one has to be careful. If the splitting is binary then we set  $Spl_l(C)(l') = C(f_l(l'))$  for any  $l' \in Spl_l(\mathcal{C})$ . If the splitting is unary then we define the same way  $Spl_l(C)(l')$  with one exception: if  $f(l') = l^{\pm 1}$ , then  $Spl_l(C)(l') = C(l)^{\pm 2}$ .

**12.3. Markovian Holonomy Fields.** — The definition of a Markovian Holonomy Field was first stated in Definition 3.1.2 of [21]. For us, by Markovian Holonomy Field, we understand oriented Markovian Holonomy Field.

**Definition 69.** — A  $G$ -valued Markovian Holonomy Field,  $HF$ , is the data, for each measured oriented marked surface with  $G$ -constraints  $(M, vol, \mathcal{C}, C)$ , of a finite measure  $HF_{(M, vol, \mathcal{C}, C)}$  on  $(\text{Mult}(P(M), G), \mathcal{I})$  such that:

- A<sub>1</sub>** : For any  $(M, vol, \mathcal{C}, C)$ ,  $HF_{(M, vol, \mathcal{C}, C)}(\exists l \in \mathcal{C} \cup \mathcal{B}(M), h(l) \notin C(l)) = 0$ .
- A<sub>2</sub>** : For any  $(M, vol, \mathcal{C})$  and any event  $\Gamma$  in  $\mathcal{I}$ , the function  $C \mapsto HF_{(M, vol, \mathcal{C}, C)}(\Gamma)$  is a measurable function on  $\text{Conj}_G(M, \mathcal{C})$ .
- A<sub>3</sub>** : For any  $(M, vol, \mathcal{C}, C)$  and any  $l \in \mathcal{C}$ ,

$$HF_{(M, vol, \mathcal{C} \setminus \{l, l^{-1}\}, C_{\mathcal{B}(M) \cup \mathcal{C} \setminus \{l, l^{-1}\}})} = \int_G HF_{(M, vol, \mathcal{C}, C_{l \rightarrow [x]})} dx,$$

where  $C_{l \rightarrow [x]}$  is defined in Notation 6.

- A<sub>4</sub>** : Let  $\psi : (M, vol, \mathcal{C}, C) \rightarrow (M', vol', \mathcal{C}', C')$  be a homeomorphism which preserves the orientation such that  $vol \circ \psi^{-1} = vol'$ ,  $\psi(\mathcal{C}) = \mathcal{C}'$  and  $C = C' \circ \psi$ . Let  $l_1, \dots, l_n$  be loops based at the same point of  $M$ . Assume that their images

$l'_1, \dots, l'_n$  by  $\psi$  are also rectifiable loops. Then, for any continuous function  $f : G^n \rightarrow \mathbb{R}$  invariant under the diagonal conjugation by  $G$ :

$$\int_{\text{Mult}(P(M), G)} f(h(l_1), \dots, h(l_n)) HF_{(M, \text{vol}, \mathcal{C}, C)}(dh) = \int_{\text{Mult}(P(M'), G)} f(h(l'_1), \dots, h(l'_n)) HF_{(M', \text{vol}', \mathcal{C}', C')}(dh).$$

**A<sub>5</sub>** : For any  $(M_1, \text{vol}_1, \mathcal{C}_1, C_1)$  and  $(M_2, \text{vol}_2, \mathcal{C}_2, C_2)$ , one has the identity:

$$HF_{(M_1 \sqcup M_2, \text{vol}_1 \sqcup \text{vol}_2, \mathcal{C}_1 \sqcup \mathcal{C}_2, C_1 \sqcup C_2)} = HF_{(M_1, \text{vol}_1, \mathcal{C}_1, C_1)} \otimes HF_{(M_2, \text{vol}_2, \mathcal{C}_2, C_2)}.$$

**A<sub>6</sub>** : For any  $(M, \text{vol}, \mathcal{C}, C)$ , any  $l \in \mathcal{C}$  and any gluing along  $l$ ,  $\psi : \text{Spl}_l(M) \rightarrow M$ , one has:

$$HF_{(\text{Spl}_l(M), \text{Spl}_l(\text{vol}), \text{Spl}_l(\mathcal{C}), \text{Spl}_l(C))} = HF_{(M, \text{vol}, \mathcal{C}, C)} \circ \psi^{-1}.$$

**A<sub>7</sub>** : For any  $(M, \text{vol}, \emptyset, C)$  and for any  $l$  in  $\mathcal{B}(M)$ ,

$$\int_G HF_{(M, \text{vol}, \emptyset, C_{l \rightarrow x})}(1) dx = 1.$$

Using Proposition 5, axiom **A<sub>4</sub>** has a simpler presentation if  $\psi$  is a diffeomorphism which preserves the orientation of  $M$ : the induced mapping from  $\text{Mult}(P(M'), G)$  to  $\text{Mult}(P(M), G)$  sends the measure  $HF_{(M', \text{vol}', \mathcal{C}', C')}$  on the measure  $HF_{(M, \text{vol}, \mathcal{C}, C)}$ .

The Markovian Holonomy Fields seem to be quite complicated objects. Actually it is easier to understand them when they are exposed in a less formal way. A Markovian Holonomy Fields is a **family of measures**. For each surface  $M$  with marks, constraints and measure of area, we are given a **gauge-invariant holonomy field on  $M$**  which satisfies the  **$G$ -constraints on the marks (A<sub>1</sub>)**. Moreover, the family of measures given by a Markovian Holonomy Field is **invariant under area-preserving diffeomorphisms, A<sub>4</sub>**, and satisfies a kind of **Markov property, A<sub>5</sub> and A<sub>6</sub>**. The measures associated with  $(M, \text{vol}, \mathcal{C}, C)$ , seen as a function of the  $G$ -constraints, provide a **regular disintegration of  $HF_{(M, \text{vol}, \emptyset, C_{|\mathcal{B}(M)})}$  (axioms A<sub>1</sub>, A<sub>2</sub> and A<sub>3</sub>)**. The last assumption is a **normalization axiom**.

In the definition of a Markovian Holonomy Field, we didn't specify any regularity condition on the field. In what follows, we will focus only on regular Markovian Holonomy Field in the following sense:

**Definition 70.** — Let  $HF$  be a  $G$ -valued Markovian Holonomy Field.

1. We say that  $HF$  is stochastically continuous if, for any  $(M, \text{vol}, \mathcal{C}, C)$ ,  $HF_{(M, \text{vol}, \mathcal{C}, C)}$  is stochastically continuous (Definition 14).

2. We say that  $HF$  is Fellerian if, for any  $(M, vol, C)$ , the function

$$(t, C) \mapsto HF_{(M, vol, C, C)}(1),$$

defined on  $\mathbb{R}_+^* \times \text{Conj}_G(M, C)$  is continuous.

3. We say that  $HF$  is regular if it is both stochastically continuous and Fellerian.

In the following, every Markovian Holonomy Field will be regular and we will forget by now on to specify the regularity.

**12.4. Partition functions for oriented surfaces.** — Let  $HF$  be a Markovian Holonomy Field.

For  $g$  an even positive integer and  $p$  a positive integer, let  $\Sigma_{p,g}^+$  be the connected sum of  $\frac{g}{2}$  tori with  $p$  holes. For  $g = 0$  we define  $\Sigma_{p,0}^+$  to be the sphere with  $p$  holes.

The classification of surfaces asserts that any connected oriented compact surface is diffeomorphic to one and exactly one of  $\{\Sigma_{p,g}^+, p \in \mathbb{N}, g \in 2\mathbb{N}\}$ . Besides, as a consequence of a theorem of Moser, if  $M$  and  $M'$  are oriented, if  $(M, vol, \emptyset, C)$  and  $(M', vol', \emptyset, C')$  are two measured marked surfaces with  $G$ -constraints, then they are isomorphic if and only if:

1.  $M$  and  $M'$  are diffeomorphic,
2.  $vol(M) = vol'(M')$ ,
3. there exists a bijection  $\psi = \mathcal{B}(M) \rightarrow \mathcal{B}^+(M')$  such that  $C = C' \circ \psi$  on  $\mathcal{B}^+(M)$ .

We define the partition functions of  $HF$ .

**Definition 71.** — Let  $g$  be an even positive integer,  $p$  be a positive integer and  $t$  be a positive real. Let  $vol$  be a measure of area on  $\Sigma_{p,g}^+$  of total mass  $t$ . Let  $\{b_1, b_2, \dots, b_p\}$  be an enumeration of  $\mathcal{B}^+(\Sigma_{p,g}^+)$ . We define the mapping:

$$\begin{aligned} Z_{p,g,t}^+(x_1, \dots, x_p) : \quad G^p &\longrightarrow \mathbb{R}_+^* \\ (x_1, \dots, x_p) &\mapsto Z_{p,g,t}^+(x_1, \dots, x_p) = HF_{(\Sigma_{p,g}^+, vol, \emptyset, C: \{b_i \mapsto [x_i], i\})}(1), \end{aligned}$$

It is called the partition function of the surface of genus  $g$  with  $p$  holes. Thanks to the diffeomorphism invariance, axiom **A<sub>4</sub>**, and Moser's theorem, it depends neither on the choice of  $vol$  nor on the choice of the enumeration:  $Z_{p,g,t}^+$  is a symmetric function.

**Remark 17.** — If  $p = 0$ , we define  $Z_{0,g,t}^+$  as the positive number equal to the mass of  $HF_{(\Sigma_{0,g}^+, vol, \emptyset, \emptyset)}$ .

Thanks to the discussion on the notion of isomorphism between measured marked surfaces with  $G$ -constraints, if  $(M, vol, \emptyset, C)$  is an oriented measured marked surface with  $G$ -constraints then there exist  $p, g$  such that  $M$  is diffeomorphic to  $\Sigma_{p,g}^+$  and then:

$$HF_{(M, vol, \emptyset, C)}(1) = Z_{p,g, vol(M)}^+(x_1, \dots, x_p),$$

where  $x_1, \dots, x_p$  are representatives of the  $p$  constraints put on  $\mathcal{B}^+(M)$ .

The Fellerian condition satisfied by regular Markovian Holonomy Fields implies that the partition functions are continuous in  $(t, x_1, \dots, x_p)$ . Besides we can reformulate the axiom of normalization **A<sub>7</sub>** (Definition 69) in terms of partition functions. If  $HF$  is a Markovian Holonomy Field, for any  $t > 0$ ,

$$\int_G Z_{1,0,t}^+(g) dg = 1,$$

that is to say:  $Z_{1,0,t}^+ dg$  is a probability measure on  $G$ . In one of the main theorems proved in Chapter 4 of [21], Lévy characterized the family of probability measures  $(Z_{1,0,t}^+ dg)_{t>0}$ .

**Theorem 26.** — *Let  $HF$  be a Markovian Holonomy Field. The probability measures  $(Z_{1,0,t}^+ dg)_{t>0}$  on  $G$  are the one dimensional distribution of a unique conjugation-invariant Lévy process  $(Y_t)_{t \geq 0}$  issued from the neutral element. Moreover, this Lévy process determines completely the partition functions of  $HF$ .*

We say that  $Y = (Y_t)_{t \geq 0}$  (resp.  $HF$ ) is the Lévy process (resp. a Markovian Holonomy Field) associated with  $HF$  (resp. to  $Y$ ). A natural question after this theorem is to ask if every Lévy process which is conjugation-invariant is associated with a Markovian Holonomy Field. Of course, some other conditions must hold such as the existence of a conjugation-invariant square-integrable density. Indeed, as the constraints on the boundary are conjugacy classes,  $Z_{p,g,t}^+(x)$  is a function of  $[x]$ . Besides, by the definition of regularity, it must be continuous in  $x$  thus square-integrable.

Hence the natural following definition:

**Definition 72.** — *Let  $(Y_t)_{t \geq 0}$  be a Lévy process on  $G$  issued from the neutral element. It is admissible if:*

- *it is invariant in law by conjugation,*
- *the distribution of  $Y_t$  admits a square-integrable density  $Q_t$  with respect to the Haar measure on  $G$  for any  $t > 0$ .*

**Remark 18.** — *Let  $H$  be the subgroup of  $G$  generated by  $G_0$ , the connected component of the neutral element, and the support of the jump measure  $\Pi$  of  $(Y_t)_{t \geq 0}$ . Using the results in [22], explained in Proposition 4.2.3 and Corollary 4.2.4 in [21], we can replace the square integrability by the fact that the function  $(t, x) \mapsto Q_t(x)$  is continuous on  $]0, \infty[ \times G$  and strictly positive on  $]0, \infty[ \times H$ . This way, we almost recover the Definition 4.2.6 exposed by Lévy, where it was assumed that  $H = G$ .*

The discussion we just had allows us to write:

**Proposition 36.** — *Let  $HF$  be a Markovian Holonomy Field, the Lévy process  $(Y_t)_{t \geq 0}$  associated with  $HF$  is an admissible Lévy process.*

In fact, we get all the admissible Lévy processes by studying the partition functions of the Markovian Holonomy Fields: this is given by Theorem 4.3.1 in the book [21].

**Theorem 27.** — *Every admissible  $G$ -valued Lévy process  $Y$ , such that for any  $t > 0$  the density of  $Y_t$  is strictly positive on  $G$ , is associated with a Markovian Holonomy Field.*

The proof of this assertion consists in constructing, just as we did for Planar Markovian Holonomy Fields, for every admissible  $G$ -valued Lévy, a special Markovian Holonomy Field  $YM$  which will be called a Yang-Mills measure. For this, Lévy used the edge paradigm of random holonomy fields. A Yang-Mills measure is a kind of deformation of the uniform measure. From now on, as Lévy did, we will suppose that for any  $t > 0$  the density of  $Y_t$  is strictly positive on  $G$ .

**12.5. Uniform measure and Yang-Mills measures.** — The uniform measure on  $\text{Mult}(P(\mathbb{G}), G)$  is almost a product of Haar measures as for any orientation  $\mathbb{E}^+$  of  $\mathbb{G}$ ,  $\text{Mult}(P(\mathbb{G}), G) \simeq G^{\mathbb{E}^+}$ . But one has to be careful because, given an oriented marked surface with  $G$ -constraints, the elements in  $\text{Mult}(P(\mathbb{G}), G)$  that we have to consider have to obey the constraints.

**Notation 7.** — *For any conjugacy class  $\mathcal{O} \subset G$ , and any integer  $n \geq 1$ , we denote by  $\delta_{\mathcal{O}(n)}$  the natural extension to  $G^n$  of the unique  $G^n$ -invariant probability measure on*

$$\mathcal{O}(n) = \{(x_1, \dots, x_n) \in G^n : x_1 \dots x_n \in \mathcal{O}\}$$

*under the  $G^n$  action  $(g_1, \dots, g_n) \bullet (x_1, \dots, x_n) = (g_1 x_1 g_2^{-1}, \dots, g_n x_n g_1^{-1})$ .*

Let  $(M, \text{vol}, \mathcal{C}, C)$  be an oriented measured marked surface with  $G$ -constraints, endowed with a graph  $\mathbb{G}$ . Let  $l_1, \dots, l_q$  be  $q$  disjoint simple loops in  $L(\mathbb{G})$  such that  $\mathcal{C} \cup \mathcal{B}(M) = \{l_1, l_1^{-1}, \dots, l_q, l_q^{-1}\}$ . For any  $i \in \{1, \dots, q\}$ , we can decompose  $l_i = e_{i,1} \dots e_{i,n_i}$ . We consider  $\mathbb{E}^+$ , an orientation of  $\mathbb{G}$ , such that for any  $i \in \{1, \dots, q\}$  and  $j \in \{1, \dots, n_i\}$ ,  $e_{i,j} \in \mathbb{E}^+$ . We label  $e_1, \dots, e_m$  the other edges of  $\mathbb{E}^+$ . Since  $\mathbb{E}^+$  is finite, it is equivalent to give a measure on  $G^{\mathbb{E}^+}$  or on  $\text{Mult}(P(\mathbb{G}), G)$ . The uniform measure  $U_{M,\mathcal{C},C}^{\mathbb{G}}$  is the measure provided by the following measure on  $G^{\mathbb{E}^+}$ :

$$dg_1 \otimes \dots \otimes dg_m \otimes \delta_{C(l_1)(n_1)}(dg_{1,n_1} \dots dg_{1,1}) \otimes \dots \otimes \delta_{C(l_q)(n_q)}(dg_{q,n_q} \dots dg_{q,1}).$$

This probability measure on  $\text{Mult}(P(\mathbb{G}), G)$  does not depend on any of the choices we made.

**Notation 8.** — *We define also a similar measure without constraints for any oriented smooth surface  $M$  (resp.  $\mathbb{R}^2$ ) endowed with a graph (resp. a planar*

graph)  $\mathbb{G}$ . Let  $\mathbb{E}^+$  be an orientation of  $\mathbb{G}$ . The measure on  $\text{Mult}(P(\mathbb{G}), G)$  seen on  $\mathbb{G}^{\mathbb{E}^+}$  as:

$$\bigotimes_{e \in \mathbb{E}^+} dg_e,$$

is denoted by  $U_M^{\mathbb{G}}$ .

We can give the definition of Yang-Mills measures.

**Definition 73.** — Let  $(Y_t)_{t \geq 0}$  be an admissible Lévy process on  $G$  issued from the neutral element. For any real  $t$ , let  $Q_t$  be the density of  $Y_t$ . A Markovian Holonomy Field  $YM$  is called a Yang-Mills measure (or sometimes a Yang-Mills field) associated with  $(Y_t)_{t \geq 0}$  if for any measured marked surface with  $G$ -constraints  $(M, \text{vol}, \mathcal{C}, C)$  and any graph  $\mathbb{G}$  on  $(M, \mathcal{C})$ ,

$$\left( YM_{(M, \text{vol}, \mathcal{C}, C)} \right)_{|\text{Mult}(P(\mathbb{G}), G)} = \prod_{F \in \mathbb{F}} Q_{\text{vol}(F)}(h(\partial F)) U_{M, \mathcal{C}, C}^{\mathbb{G}}(dh).$$

A Yang-Mills measure associated with  $(Y_t)_{t \geq 0}$  is a Markovian Holonomy Field associated with  $(Y_t)_{t > 0}$ . Thus, in order to prove Theorem 27, it is enough to prove the following.

**Proposition 37.** — For any admissible  $G$ -valued Lévy process  $Y$ , such that for any  $t > 0$  the density of  $Y_t$  is strictly positive on  $G$ , there exists a unique Yang-Mills measure  $YM$  associated with  $Y$ .

For this proposition, one has to introduce as we did for Planar Markovian Holonomy Fields, a discrete analog of Markovian Holonomy Fields: discrete Markovian Holonomy Fields. The definition of discrete Markovian Holonomy Field can be found in Section 3.2 of [21].

Then one can show that the family of measures:

$$\left( \left( YM_{(M, \text{vol}, \mathcal{C}, C)} \right)_{|\text{Mult}(P(\mathbb{G}), G)} \right)_{(M, \text{vol}, \mathcal{C}, C, \mathbb{G})}$$

is a Fellerian continuously area-dependent (Proposition 4.3.11 in [21]) and locally stochastically  $\frac{1}{2}$ -Hölder continuous (Proposition 4.3.15 in [21]) discrete (Proposition 4.3.10 in [21]) Markovian Holonomy Field associated with  $Y$ . Then it is shown, in Theorem 3.2.9 of [21], that under these regularity conditions, every discrete Markovian Holonomy Field can be extended to a continuous stochastically continuous Markovian Holonomy Field.

The definition of discrete holonomy field follows closely the definition of a Markovian Holonomy Field except for the invariance by diffeomorphism which becomes almost a combinatorial condition. It is the same difference between the axioms  $\mathbf{P}_1$  and  $\mathbf{DP}_1$  of Definitions 33 and 36 of Section 4.1.



**Remark 19.** — *The difference between the assumption  $\mathbf{A}_4$  in definition 3.1.2 in [21] and  $\mathbf{D}_4$  in definition 3.2.1 in [21] makes the proof of Lemma 3.2.2. in the same book incomplete. Thus, it is not clear that any Markovian holonomy field determines a discrete Markovian holonomy field.*

This remark leads us to the following definition.

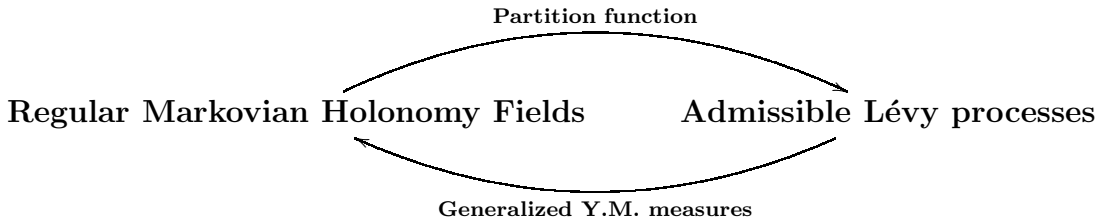
**Definition 74.** — *Let  $HF$  be a Markovian Holonomy Field. It is constructible if the family of measures:*

$$\left( (HF_{(M,vol,C,C)})_{|_{\text{Mult}(P(\mathbb{G}),G)}} \right)_{(M,vol,C,C,\mathbb{G})}$$

*is a discrete Markovian Holonomy Field.*

It is still an open question to know if any Markovian Holonomy Field is constructible.

**12.6. Conjecture and main theorem.** — We can resume the results of Proposition 36 and Theorem 27 by the following diagram.



Besides, it was shown that the left arrow goes into the constructible regular Markovian Holonomy Fields and the composition of the two arrows is equal to the identity on the set of admissible Lévy processes.

It is natural to wonder if the two arrows are each other inverse, which lead us to the following conjecture:

**Conjecture 1.** — *Every  $G$ -valued regular Markovian Holonomy Field is a Yang-Mills field.*

From this conjecture, it would be true that every  $G$ -valued regular Markovian Holonomy Field is constructible.

Let  $m$  be a point on  $M$ , a smooth oriented surface with boundary. We define  $L_m^0(M)$  the set of loops in  $L_m(M)$  which are contractible after filling all the holes of  $M$  by discs. Let us state the main result of the section.

**Theorem 28.** — *Let  $G$  be a compact Lie group. Let  $(HF_{(M,vol,C,C)})_{(M,vol,C,C)}$  a  $G$ -valued regular Markovian Holonomy Field and  $(Y_t)_{t \in \mathbb{R}}$  its associated  $G$ -valued*

*Lévy process.* Let  $(YM_{(M,vol,C,C)})_{M,vol,C,C}$  be the Yang-Mills field associated with  $(Y_t)_{t \in \mathbb{R}}$ .

For any oriented surface  $M$ , any measure of area  $vol$ , any  $G$ -constraints  $C$  on  $\mathcal{B}(M)$ , the following equality holds:

$$\left( HF_{(M,vol,\emptyset,C)} \right)_{|\mathcal{M}ult(L_m^0(M),G)} = \left( YM_{(M,vol,\emptyset,C)} \right)_{|\mathcal{M}ult(L_m^0(M),G)}.$$

Let  $\mathcal{C}$  be a set of marks on  $M$ . Let us choose an orientation of  $\mathcal{C}$  denoted by  $\mathcal{C}^+$ . Let  $C$  be a set of constraints on  $\mathcal{B}(M)$ . We endow the set of  $G$ -constraints on  $\mathcal{C} \cup \mathcal{B}(M)$  with the measure  $d\lambda_{\mathcal{C}|\mathcal{B}(M)}$  coming from:

$$\bigotimes_{c \in \mathcal{C}^+} dg_c \otimes \bigotimes_{b \in \mathcal{B}(M)^+} \delta_{C(b)}.$$

By disintegration, for any set of marks  $\mathcal{C}$  on  $M$ , for any set of constraints on  $\mathcal{B}(M)$ ,  $d\lambda_{\mathcal{C}|\mathcal{B}(M)}$  almost surely:

$$\left( HF_{(M,vol,C,C)} \right)_{|\mathcal{M}ult(L_m^0(M),G)} = \left( YM_{(M,vol,C,C)} \right)_{|\mathcal{M}ult(L_m^0(M),G)}.$$

**Remark 20.** — To get the full Conjecture 1, one would have to generalize Theorem 28 in order to include all the remaining loops, including the generators of the fundamental group of the surface without holes, as known in the physics literature the Polyakov loops.

**12.7. Markovian Holonomy Fields and free boundary conditions on the plane.** — Given a Markovian Holonomy Field  $HF$ , we have noticed that the measures  $(HF_{(M,vol,C,C)})$  are not probability measures. One way to deal with probability measures would be to normalize them by their mass. Yet, a better way to get a probability measure in our case is to consider the free boundary condition measure.

12.7.1. *Free boundary condition on a surface.* —

**Definition 75.** — Let  $HF$  be any Markovian Holonomy Field. For any surface  $M$  homomorphic to a disk  $\Sigma_{0,1}^+$  endowed with a measure of area  $vol$ , we define the free boundary condition expectation on  $M$  associated with  $HF$  as the probability measure on  $(\mathcal{M}ult(P(M),G), \mathcal{B})$  such that for any measurable function  $f : G^n \rightarrow \mathbb{R}^+$  and any finite family,  $c_1, \dots, c_n$ , of elements of  $P(M)$ :

$$\mathbb{E}_{M,vol}^{HF} \left( f(h(c_1), \dots, h(c_n)) \right) = \int_G \widehat{HF}_{(M,vol,\emptyset,\partial M \rightarrow [x])} \left( f(h(c_1), \dots, h(c_n)) \right) dx,$$

where  $\widehat{HF}_{M,vol,\emptyset,\partial M \rightarrow [x]}$  is the extension of  $HF_{M,vol,\emptyset,\partial M \rightarrow [x]}$  to the Borel  $\sigma$ -field given by Proposition 3.

**Remark 21.** — In this definition we have extended the  $\sigma$ -field to the Borel  $\sigma$ -field, in a way such that the new measure becomes invariant by the gauge group. In order for the definition of  $\mathbb{E}_{M,vol}^{HF}$  to be consistent with the way we named it, one has to verify that it is indeed a probability measure. But the constant function 1 being gauge-invariant,  $\hat{1}_{J_{c_1, \dots, c_n}} = 1$  which leads to:

$$\mathbb{E}_{M,vol}^{HF}(1) = \int_G HF_{(M,vol,\emptyset,\partial M \rightarrow [x])}(1) dx = 1,$$

the last equality coming from the normalization axiom **A<sub>7</sub>** in Definition 69.

We can calculate the free boundary condition expectation on  $M$  of a Yang-Mills field.

**Lemma 23.** — Let  $YM$  be a Yang-Mills field associated with  $(Y_t)_{t \in \mathbb{R}}$ . Let  $Q_t$  be the density of  $Y_t$  for any  $t > 0$ . Then for any surface  $M$  homeomorphic to a disk endowed with a measure of area  $vol$  and a graph  $\mathbb{G}$ ,

$$\left(\mathbb{E}_{M,vol}^{YM}\right)_{|_{\text{Mult}(P(\mathbb{G}),G)}} = \prod_{F \in \mathbb{F}} Q_{vol(F)}(h(\partial F)) U_M^{\mathbb{G}}(dh),$$

$U_M^{\mathbb{G}}$  being defined in Notation 8.

*Proof.* — This follows from the fact that  $\int_G U_{M,\emptyset,\partial M \rightarrow x}^{\mathbb{G}} dx = U_M^{\mathbb{G}}$  which is a consequence of equality (26) of Lemma 2.3.4 in [21]:

$$\int_G \left[ \int_{G^n} f \delta_{[y](n)}(dx_1, \dots, dx_n) \right] dy = \int_{G^n} f dx_1 \dots dx_n.$$

□

Actually, we can define the free boundary condition expectation on  $M$  for any surface having at least one hole by a definition like Definition 75. This is the one which should be used to understand the law of loops on torus.

*12.7.2. Free boundary condition on the plane.* — In this section, we will define the free boundary expectation on the plane. This expectation will allow us to forget about the dependence on  $M$  and to define a discrete Planar Markovian Holonomy Field. The free boundary expectation satisfies a projectivity property exposed below.

**Proposition 38.** — Let  $HF$  be a Markovian Holonomy Field. Let  $(M, vol)$  and  $(M', vol')$  be two measured oriented connected compact two-dimensional sub-manifolds of  $\mathbb{R}^2$  which are homeomorphic to the unit disk. Let us suppose that  $M \subset M'$ . Let us assume that  $vol'_M = vol$ . The free boundary condition expectations on  $M$  and  $M'$  are related by:

$$\mathbb{E}_{M,vol}^{HF} = \mathbb{E}_{M',vol'}^{HF} \circ \rho_{M,M'}^{-1},$$

where we remind the reader that  $\rho_{M,M'}$  was defined in Notation 2. Thus, for any measure of area  $vol$  on  $\mathbb{R}^2$ , the family:

$$\left\{ \left( \mathcal{M}ult(P(M), G), \mathcal{B}, \mathbb{E}_{M, vol|_M}^{HF} \right)_{M \subset \mathbb{R}^2}, (\rho_{M, M'})_{M \subset \mathring{M}'} \right\},$$

is a projective family of probability spaces.

*Proof.* — Let  $(M, vol)$  and  $(M', vol')$  be two measured connected compact two-dimensional sub-manifolds of  $\mathbb{R}^2$  such that  $M \subset \mathring{M}'$  and  $vol'|_M = vol$ . Let  $m$  be a point of  $M$ . Since  $\mathbb{E}_{M, vol}^{HF}$  and  $\mathbb{E}_{M', vol'}^{HF} \circ \rho_{M, M'}^{-1}$  are gauge-invariant, it is enough, by Proposition 5, to show that, for any continuous conjugation-invariant function  $f$  on  $G^n$  and any  $n$ -tuple of loops  $l_1, \dots, l_n$  in  $M$  based in  $m$ , we have:

$$\mathbb{E}_{M', vol'}^{HF} \left[ f(h(l_1), \dots, h(l_n)) \right] = \mathbb{E}_{M, vol}^{HF} \left[ f(h(l_1), \dots, h(l_n)) \right].$$

The following sequence of equalities gives the desired result:

$$\begin{aligned} & \mathbb{E}_{M', vol'}^{HF} \left[ f(h(l_1), \dots, h(l_n)) \right] \\ &= \int_G \int_{\mathcal{M}(P(M'), G)} f(h(l_1), \dots, h(l_n)) HF_{(M', vol', \emptyset, \partial M' \rightarrow [x])} (dh) dx \\ &= \int_G \int_G \int_{\mathcal{M}(P(M'), G)} f(h(l_1), \dots, h(l_n)) HF_{(M', vol', \partial M, \{\partial M' \rightarrow [x], \partial M \rightarrow [y]\})} (dh) dy dx \quad \mathbf{A_3} \\ &= \int_G \int_G \int_{\mathcal{M}(P((M' \setminus \mathring{M}) \sqcup M), G)} f(h(l_1), \dots, h(l_n)) \\ & \quad HF_{((M' \setminus \mathring{M}) \sqcup M, vol'_{|(M' \setminus \mathring{M})} \sqcup vol, \emptyset, \{\partial M' \rightarrow [x], \partial M \rightarrow [y]\})} (dh) dy dx \quad \mathbf{A_6} \\ &= \int_G \int_G \int_{\mathcal{M}(P(M' \setminus \mathring{M}), G)} \int_{\mathcal{M}(P(M), G)} f(h(l_1), \dots, h(l_n)) HF_{(M, vol, \emptyset, \{\partial M \rightarrow [y]\})} (dh) \\ & \quad HF_{(M' \setminus \mathring{M}, vol'_{|M' \setminus \mathring{M}}, \emptyset, \{\partial M' \rightarrow [x], \partial M \rightarrow [y]\})} (dh') dy dx \quad \mathbf{A_5} \\ &= \int_G \int_G \int_{\mathcal{M}(P(M), G)} f(h(l_1), \dots, h(l_n)) HF_{(M, vol, \emptyset, \{\partial M \rightarrow [y]\})} (dh) \\ & \quad HF_{(M' \setminus \mathring{M}, vol'_{|M' \setminus \mathring{M}}, \emptyset, \{\partial M' \rightarrow [x], \partial M \rightarrow [y]\})} (1) dy dx \\ &= \int_G \int_{\mathcal{M}(P(M), G)} f(h(l_1), \dots, h(l_n)) HF_{(M, vol, \emptyset, \{\partial M \rightarrow [y]\})} (dh) \\ & \quad \left( \int_G HF_{(M' \setminus \mathring{M}, vol'_{|M' \setminus \mathring{M}}, \emptyset, \{\partial M' \rightarrow [x], \partial M \rightarrow [y]\})} (1) dx \right) dy \end{aligned}$$

$$\begin{aligned}
&= \int_G \int_{\mathcal{M}(P(M), G)} f(h(l_1), \dots, h(l_n)) HF_{(M, vol, \emptyset, \{\partial M \rightarrow [y]\})}(dh) dy \quad \mathbf{A}_7 \\
&= \mathbb{E}_{M, vol}^{HF} \left[ f(h(l_1), \dots, h(l_n)) \right],
\end{aligned}$$

where we applied successively the definition of  $\mathbb{E}_{M', vol'}^{HF}$ , the axioms  $\mathbf{A}_3$ ,  $\mathbf{A}_6$  and  $\mathbf{A}_5$ . Then after a change of notation and a Fubini exchange of integrals, the normalization axiom  $\mathbf{A}_7$  with the definition of  $\mathbb{E}_{M, vol}^{HF}$  lead us to the result.  $\square$

The free boundary expectation for the plane is the projective limit of this family of measured spaces.

**Definition 76.** — *Let  $HF$  be a Markovian Holonomy Field. Let  $vol$  be a measure of area on  $\mathbb{R}^2$ . The free boundary condition expectation on  $\mathbb{R}^2$  defined on  $(\text{Mult}(P(\mathbb{R}^2), G), \mathcal{B})$  is the projective limit of:*

$$\left\{ \left( \text{Mult}(P(M), G), \mathcal{B}, \mathbb{E}_{M, vol|_M}^{HF} \right)_{M \subset \mathbb{R}^2}, (\rho_{M, M'})_{M \subset M'} \right\}.$$

We will denote it by  $\mathbb{E}_{vol}^{HF}$ . This holonomy field is gauge-invariant.

In Lemma 23, the free boundary condition expectation on a surface associated with a Yang-Mills measure was computed. It gives us the value of the restriction to the set of paths drawn on an embedded graph of the free boundary condition expectation on the plane associated with a Yang-Mills measure. Using Proposition 9, any connected finite planar graph  $\mathbb{G}$  can be extended as an embedded graph. This allows us to compute easily the free boundary condition expectation on the plane associated with a Yang-Mills measure.

**Proposition 39.** — *Suppose that  $\mathbb{R}^2$  is endowed with a measure of area  $vol$  and a planar graph  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ . Let  $Y = (Y_t)_{t \geq 0}$  be an admissible  $G$ -valued Lévy process with associated semigroup of densities  $(Q_t)_{t \geq 0}$  such that for any  $t > 0$   $Q_t$  is strictly positive on  $G$ . Let  $YM$  be the Yang-Mills measure associated with  $Y$ . The free boundary condition expectation on  $\mathbb{R}^2$  restricted on  $\text{Mult}(P(\mathbb{G}), G)$  satisfies:*

$$\left( \mathbb{E}_{vol}^{YM} \right)_{|_{\text{Mult}(P(\mathbb{G}), G)}}(dh) = \prod_{F \in \mathbb{F}^b} Q_{vol(F)}(h(\partial F)) U_{\mathbb{R}^2}^G(dh).$$

In order to simplify the proof, we will use the up coming Theorem 29.

*Proof.* — We have already seen in the proof of Proposition 25 that

$$\left( \prod_{F \in \mathbb{F}^b} Q_{vol(F)}(h(\partial F)) U_{\mathbb{R}^2}^G(dh) \right)_{vol, \mathbb{G}}$$

is a stochastically continuous discrete weak Planar Markovian Holonomy Field. Applying Theorem 29 and the constructibility result of Section 5,

$\left( (\mathbb{E}_{vol}^{YM})_{|\mathcal{M}ult(P(\mathbb{G}),G)} \right)_{\mathbb{G},vol}$  is also such a stochastically continuous weak discrete Planar Markovian Holonomy Field. Using Theorem 22, we only have to check that  $(h(L_{n,0}))_{n \in \mathbb{N}}$  has the same law under  $\mathbb{E}_{dx}^{YM}$  as under  $\prod_{F \in \mathbb{F}^b} Q_1(h(\partial F))U_{\mathbb{R}^2}^{N^2}(dh)$ .

By Proposition 25 and Proposition 22, we already know that under the measure  $\prod_{F \in \mathbb{F}^b} Q_1(h(\partial F))U_{\mathbb{R}^2}^{N^2}(dh)$ ,  $(h(L_{n,0}))_{n \in \mathbb{N}}$  is a sequence of i.i.d. random variables with common law  $Q_1(g)dg$ .

Let us show that the same result holds under  $(\mathbb{E}_{vol}^{YM})_{|\mathcal{M}ult(P(\mathbb{G}),G)}$ . We would like to apply Lemma 23 to  $\mathbb{N}^2$  in order to understand the law of  $(h(L_{n,0}))_{n \in \mathbb{N}}$  under  $\mathbb{E}_{dx}^{YM}$ , yet,  $\mathbb{N}^2$  is not an embedded graph. For any  $n \in \mathbb{N}$ , let  $\mathbb{G}_n$  be the finite planar graph  $\mathbb{G} \cap ([0, n] \times [0, 1])$ . Let  $\partial D(0, n + 1)$  be the loop based at  $(n + 1, 0)$  turning anti-clockwise, representing the cycle bounding the disk of radius  $n + 1$  centered at  $(0, 0)$ . We define  $\mathbb{G}'_n$  by:

- $\mathbb{E}'_n = \mathbb{E}_n \cup \{ \partial D(0, n + 1), \partial D(0, n + 1)^{-1}, e_{n,0}^r, (e_{n,0}^r)^{-1} \}$ ,
- $\mathbb{V}'_n = \mathbb{V}_n \cup \{ (n + 1, 0) \}$ .

The finite planar graph  $\mathbb{G}'_n$  is an embedded graph and  $\mathbb{G}_n$  is a subgraph of  $\mathbb{G}$ . Using Lemma 23, we know that:

$$(\mathbb{E}_{vol}^{YM})_{|\mathcal{M}ult(P(\mathbb{G}'_n),G)}(dh) = \prod_{F \in \mathbb{F}'_n} Q_{vol(F)}(h(\partial F))U_{\mathbb{R}^2}^{\mathbb{G}'_n}(dh).$$

Using the same arguments as in the proof of Proposition 25, under the measure  $\prod_{F \in \mathbb{F}'_n} Q_{vol(F)}(h(\partial F))U_{\mathbb{R}^2}^{\mathbb{G}'_n}(dh)$ , the sequence  $(h(L_{k,0}))_{0 \leq k \leq n-1}$  is a sequence of i.i.d. variables with common law  $Q_1(g)dg$ . Thus, under  $(\mathbb{E}_{vol}^{YM})_{|\mathcal{M}ult(P(\mathbb{G}),G)}$ ,  $(h(L_{n,0}))_{n \in \mathbb{N}}$  is a sequence of i.i.d. random variables with common law  $Q_1(g)dg$ . This allows us to conclude the proof.  $\square$

This last proposition and Proposition 25 show that we have two ways of constructing a planar Yang-Mills field associated with an admissible Lévy process. Either we can define it directly as in Proposition 21, or we can take the free boundary condition expectation on  $\mathbb{R}^2$  of a Yang-Mills field. We have thus the following result.

**Proposition 40.** — *For any planar graph  $\mathbb{G}$ , any measure of area  $vol$ , any family of facial loops  $(c_F)_{F \in \mathbb{F}^b}$  oriented anti-clockwise, and any rooted spanning tree  $T$  of  $\mathbb{G}$ , under the free boundary condition on the plane of any Yang-Mills measure  $YM$ ,  $\mathbb{E}_{vol}^{YM}$ , the random variables  $(h(\mathbf{1}_{c_F,T}))_{F \in \mathbb{F}^b}$  are independent, and for any  $F \in \mathbb{F}^b$ ,  $h(\mathbf{1}_{c_F,T})$  has the same law as  $Y_{vol(F)}$ .*

12.7.3. *Free boundary condition expectation on  $\mathbb{R}^2$  as a Planar Markovian Holonomy Field.* — The free boundary condition expectation on  $\mathbb{R}^2$  is interesting as it allows us to link the theory of Markovian Holonomy Fields with the one of Planar Markovian Holonomy Fields. Consider  $HF$  a regular Markovian Holonomy Field and let  $(\mathbb{E}_{vol}^{HF})_{vol}$  be the free boundary condition expectation on the plane associated with  $HF$ .

**Theorem 29.** — *The family  $(\mathbb{E}_{vol}^{HF})_{vol}$  is a stochastically continuous strong Planar Markovian Holonomy Field.*

Using the theory of Planar Markovian Holonomy Fields we developed, it is enough to show that for any  $vol$ ,  $\mathbb{E}_{vol}^{HF}$  is stochastically continuous and that its restriction to  $P(\text{Aff}(\mathbb{R}^2))$  is a  $G$ -valued weak stochastically continuous Planar Markovian Holonomy Field. As we have already checked the weight condition and as we have noticed the gauge-invariance of the free boundary condition expectation in Definition 76, it is enough to show that it is stochastically continuous and that the axioms  $\mathbf{wP}_1$ ,  $\mathbf{wP}_2$  and  $\mathbf{wP}_3$  in Definition 34 hold. These are proved in Lemmas 24, 25, 26 and 27.

**Lemma 24.** — *For any measure of area  $vol$ ,  $\mathbb{E}_{vol}^{HF}$  is a stochastically continuous random holonomy field.*

*Proof.* — Let  $vol$  be a measure of area. Let  $p_n$  be a sequence of paths which converges, as  $n$  goes to  $\infty$ , to a path  $p$  for the convergence with fixed endpoints. Let  $\mathbb{D}$  be a disc centered in  $(0, 0)$  such that for any integer  $n$ ,  $p_n \in \mathbb{D}$ . We remind the reader that  $\widehat{HF}_{(\mathbb{D}, vol|_{\mathbb{D}}, \emptyset, \partial\mathbb{D} \rightarrow [x])}$  is the extension given by Proposition 3 of  $HF_{(\mathbb{D}, vol|_{\mathbb{D}}, \emptyset, \partial\mathbb{D} \rightarrow [x])}$  to the Borel  $\sigma$ -field. By definition,

$$\begin{aligned} \mathbb{E}_{vol}^{HF} [d_G(h(p_n), h(p))] &= \int_G \widehat{HF}_{(\mathbb{D}, vol|_{\mathbb{D}}, \emptyset, \partial\mathbb{D} \rightarrow [x])} [d_G(h(p_n), h(p))] dx \\ &= \int_G HF_{(\mathbb{D}, vol|_{\mathbb{D}}, \emptyset, \partial\mathbb{D} \rightarrow [x])} \left[ (\widehat{d_G})_{J_{\{p_n, p\}}} (h(p_n), h(p)) \right] dx. \end{aligned}$$

As  $p_n$  and  $p$  have the same endpoints,  $J_{\{p_n, p\}}$  is equal to  $G^2$  and acts on  $G^2$  as the following:  $(k_1, k_2) \bullet (g_1, g_2) = (k_2^{-1}g_1k_1, k_2^{-1}g_2k_1)$ . The invariance of  $d_G$ , by right and left translations, implies that  $(\widehat{d_G})_{J_{\{p_n, p\}}} = d_G$ . This leads to:

$$\mathbb{E}_{vol}^{HF} [d_G(h(p_n), h(p))] = \int_G HF_{(\mathbb{D}, vol|_{\mathbb{D}}, \emptyset, \partial\mathbb{D} \rightarrow [x])} [d_G(h(p_n), h(p))] dx.$$

As  $HF$  is regular it is stochastically continuous, thus we have:

$$HF_{(\mathbb{D}, vol|_{\mathbb{D}}, \emptyset, \partial\mathbb{D} \rightarrow [x])} [d_G(h(p_n), h(p))] \xrightarrow[n \rightarrow \infty]{} 0.$$

Thus, with an argument of dominated convergence,  $\mathbb{E}_{vol}^{HF} [d_G(h(p_n), h(p))]$  converges to zero as  $n$  goes to infinity.  $\square$

**Lemma 25.** — *The family of random holonomy fields  $(\mathbb{E}_{vol}^{HF})_{vol}$  satisfies the area-preserving diffeomorphisms at infinity invariance  $\mathbf{wP}_1$ .*

*Proof.* — Consider  $vol$  and  $vol'$  two measures of area on  $\mathbb{R}^2$ . Let  $\psi$  be a diffeomorphism at infinity which preserves the orientation and let  $R$  be a positive real such that:

1.  $vol' = vol \circ \psi^{-1}$ ,
2.  $\psi : \mathbb{D}(0, R)^c \rightarrow \psi(\mathbb{D}(0, R)^c)$  is a diffeomorphism.

Using the gauge-invariance of  $\mathbb{E}_{vol}^{HF}$ , as we did in order to show the Axiom  $\mathbf{wDP}_2$  in the proof of Proposition 24, we can prove the invariance by orientation and area preserving homeomorphisms by considering loops based at the same point. Let  $l_1, \dots, l_n$  be loops in  $\text{Aff}(\mathbb{R}^2)$  based at the same point such that for any  $i \in \{1, \dots, n\}$ ,  $l'_i = \psi(l_i)$  is in  $\text{Aff}(\mathbb{R}^2)$ . Let  $R'$  be a positive real such that  $R'$  is greater than  $R$  and such that for any  $i \in \{1, \dots, n\}$ ,  $l_i$  is in  $M_{R'} = \mathbb{D}(0, R')$ . The set  $M' = \psi(M_{R'})$  is a connected compact oriented two-dimensional sub-manifold of  $\mathbb{R}^2$ . Let us consider  $f : G^n \rightarrow \mathbb{R}$ , a continuous function invariant by diagonal conjugation. We have the following equalities:

$$\begin{aligned} \mathbb{E}_{vol}^{HF} \left[ f((h(l_i)_{i=1}^n)) \right] &= \mathbb{E}_{M_{R'}, vol|_{M_{R'}}}^{HF} \left[ f((h(l_i)_{i=1}^n)) \right] \\ &= \int_G \int_{\text{Mult}(P(M_{R'}), G)} f((h(l_i)_{i=1}^n)) HF_{(M_{R'}, vol|_{M_{R'}}, \emptyset, \partial M_{R'} \rightarrow [x])} dx \\ &= \int_G \int_{\text{Mult}(P(M'), G)} f((h(l'_i)_{i=1}^n)) HF_{(M', vol'|_{M'}, \emptyset, \partial M' \rightarrow [x])} dx \quad \mathbf{A}_4 \\ &= \mathbb{E}_{M', vol'|_{M'}}^{HF} \left[ f((h(l'_i)_{i=1}^n)) \right] = \mathbb{E}_{vol'}^{HF} \left[ f((h(l'_i)_{i=1}^n)) \right]. \end{aligned}$$

Thus  $\mathbf{wP}_1$  is satisfied by  $(\mathbb{E}_{vol}^{HF})_{vol}$ . □

**Lemma 26.** — *The family of random holonomy fields  $(\mathbb{E}_{vol}^{HF})_{vol}$  satisfies the weak independence property  $\mathbf{wP}_2$ .*

*Proof.* — Let  $vol$  be a measure of area on  $\mathbb{R}^2$ . Let  $l$  and  $l'$  be two loops in  $\text{Aff}(\mathbb{R}^2)$  such that  $\overline{\text{Int}(l)} \cap \overline{\text{Int}(l')} = \emptyset$ . We can always consider  $\tilde{l}$  and  $\tilde{l}'$  two smooth simple loops in  $\mathbb{R}^2$  such that the closure of their interiors are also disjoint and such that  $l \subset \text{Int}(\tilde{l})$  and  $l' \subset \text{Int}(\tilde{l}')$ . Thanks to this remark we can suppose that  $l$  and  $l'$  are smooth. Using the gauge-invariance of  $\mathbb{E}_{vol}^{HF}$ , as we did in order to show the Axiom  $\mathbf{wDP}_2$  in the proof of Proposition 24, we can work with loops. Let us consider  $l_1, \dots, l_n$  some loops in  $\overline{\text{Int}(l)}$  and  $l'_1, \dots, l'_m$  some loops in  $\overline{\text{Int}(l')}$ . The aim is to prove that for any continuous functions  $f$  and  $g$ , from  $G^n$ , respectively  $G^m$ , to  $\mathbb{R}$ , we have:

$$\mathbb{E}_{vol}^{HF} \left[ f((h(l_i)_{i=1}^n)) g((h(l'_i)_{i=1}^m)) \right] = \mathbb{E}_{vol}^{HF} \left[ f((h(l_i)_{i=1}^n)) \right] \mathbb{E}_{vol}^{HF} \left[ g((h(l'_i)_{i=1}^m)) \right].$$



We will use the notation and the result in Remark 6. Let  $\mathcal{L}_0$  be a smooth loop such that  $\mathcal{L}_0$  surrounds  $l$  and  $l'$ . We will make the following abuse of notation: by  $\mathcal{L}_0$  we will also denote  $\overline{\text{Int}(\mathcal{L}_0)}$ . The same notation will hold for  $l$  and  $l'$ . Using the different axioms in Definition 69, we have:

$$\begin{aligned}
& \mathbb{E}_{dx}^{HF} \left[ f((h(l_i))_{i=1}^n) g((h(l'_i))_{i=1}^m) \right] = \mathbb{E}_{\mathcal{L}_0, dx}^{HF} \left[ f((h(l_i))_{i=1}^n) g((h(l'_i))_{i=1}^m) \right] \\
&= \int_G \int_{\text{Mult}(P(\mathcal{L}_0), G)} \widehat{(f \otimes g)}_{J_{(l_i)_{i=1}^n, (l'_i)_{i=1}^m}} ((h(l_i))_{i=1}^n, (h(l'_i))_{i=1}^m) HF_{(\mathcal{L}_0, dx, \emptyset, \mathcal{L}_0 \rightarrow [y])} (dh) dy \\
&= \int_G \int_{\text{Mult}(P(\mathcal{L}_0), G)} \widehat{f}_{J_{(l_i)_{i=1}^n}} ((h(l_i))_{i=1}^n) \widehat{g}_{J_{(l'_i)_{i=1}^m}} ((h(l'_i))_{i=1}^m) HF_{(\mathcal{L}_0, dx, \emptyset, \mathcal{L}_0 \rightarrow [y])} (dh) dy \\
&= \int_{G^3} \int_{\text{Mult}(P(\mathcal{L}_0), G)} \widehat{f}_{J_{(l_i)_{i=1}^n}} ((h(l_i))_{i=1}^n) \widehat{g}_{J_{(l'_i)_{i=1}^m}} ((h(l'_i))_{i=1}^m) \tag{A3} \\
& \quad HF_{(\mathcal{L}_0, dx, \emptyset, \{\mathcal{L}_0 \rightarrow [y], l' \rightarrow [z], l \rightarrow [w]\})} (dh) dy dz dw \\
&= \int_{G^3} \int_{\text{Mult}(P(l), G)} \widehat{f}_{J_{(l_i)_{i=1}^n}} ((h(l_i))_{i=1}^n) HF_{(l, dx, \emptyset, l \rightarrow [w])} (dh) \tag{A6} \\
& \quad \int_{\text{Mult}(P(l'), G)} \widehat{g}_{J_{(l'_i)_{i=1}^m}} ((h(l'_i))_{i=1}^m) HF_{(l', dx, \emptyset, l' \rightarrow [z])} (dh) \tag{A5} \\
& \quad \int_{\text{Mult}(P(\overline{\mathcal{L}_0 \setminus (l \cup l')}), G)} HF_{(\mathcal{L}_0, dx, \emptyset, \{\mathcal{L}_0 \rightarrow [y], l' \rightarrow [z], l \rightarrow [w]\})} (dh) dy dz dw.
\end{aligned}$$

Thanks to **A7**,  $\int_G \int_{\text{Mult}(P(\overline{\mathcal{L}_0 \setminus (l \cup l')}), G)} HF_{(\mathcal{L}_0, dx, \emptyset, \{\mathcal{L}_0 \rightarrow [y], l' \rightarrow [z], l \rightarrow [w]\})} (dh) dy = 1$ . Thus  $\mathbb{E}_{dx}^{HF} \left[ f((h(l_i))_{i=1}^n) g((h(l'_i))_{i=1}^m) \right]$  is equal to

$$\begin{aligned}
& \int_G \int_{\text{Mult}(P(l), G)} \widehat{f}_{J_{l_1, \dots, l_n}} ((h(l_i))_{i=1}^n) HF_{(\overline{\text{Int}(l)}, dx, \emptyset, l \rightarrow [w])} (dh) dw \\
& \quad \int_G \int_{\text{Mult}(P(l'), G)} \widehat{g}_{J_{l'_1, \dots, l'_m}} ((h(l'_i))_{i=1}^m) HF_{(l', dx, \emptyset, l' \rightarrow [z])} (dh) dz,
\end{aligned}$$

which is equal to  $\mathbb{E}_{dx}^{HF} \left[ f((h(l_i))_{i=1}^n) \right] \mathbb{E}_{dx}^{HF} \left[ g((h(l'_i))_{i=1}^m) \right]$ .  $\square$

**Remark 22.** — Using the same kind of calculation, and using the stochastic continuity, we can show that for any simple loop  $l$ , the law of  $h(l)$  under  $\mathbb{E}_{vol}^{HF}$  is the law of  $Y_{vol(\overline{\text{Int}(l)})}$  where  $(Y_t)_{t \in \mathbb{R}^+}$  is the Lévy process associated with  $HF$ .

**Lemma 27.** — The family of random holonomy fields  $(\mathbb{E}_{vol}^{HF})_{vol}$  satisfies the locality property **wP3**.

*Proof.* — Let  $l$  be a simple loop, let  $vol$  and  $vol'$  be two measures of area whose restrictions to the closure of the interior of  $l$  are equal. The random holonomy fields  $\mathbb{E}_{vol}^{HF}$  and  $\mathbb{E}_{vol'}^{HF}$  being stochastically continuous, by Proposition 5, we only

gave to prove, for any loops  $l_1, \dots, l_n$  in the interior of  $l$ , and for any continuous function  $f : G^n \rightarrow \mathbb{R}$  invariant under the diagonal action of  $G$ , that:

$$\mathbb{E}_{vol}^{HF} \left[ f(h(l_1), \dots, h(l_n)) \right] = \mathbb{E}_{vol'}^{HF} \left[ f(h(l_1), \dots, h(l_n)) \right].$$

Using Riemann's uniformization theorem, we can find a smooth curve  $\tilde{l}$  in the interior of  $l$  such that  $l_1, \dots, l_n$  are in the interior of  $\tilde{l}$ . Let  $M$  be the closure of the interior of  $\tilde{l}$ , we can write the following equalities:

$$\begin{aligned} \mathbb{E}_{vol}^{HF} \left[ f(h(l_1), \dots, h(l_n)) \right] &= \mathbb{E}_{M, vol|_M}^{HF} \left[ f(h(l_1), \dots, h(l_n)) \right] \\ &= \mathbb{E}_{M, vol'_M}^{HF} \left[ f(h(l_1), \dots, h(l_n)) \right] \\ &= \mathbb{E}_{vol'}^{HF} \left[ f(h(l_1), \dots, h(l_n)) \right]. \end{aligned}$$

This allows us to conclude. □

**12.8. Proof of Theorem 28.** — We now give the proof of Theorem 28. The second part about marks is a consequence of the first part by conditioning. Besides, because of the splitting property of Markovian Holonomy Fields, we only have to show the result for a sphere with  $p$  holes without marks.

*Proof of Theorem 28.* — Let  $(HF_{(M, vol, \mathcal{C}, \mathcal{C})})_{(M, vol, \mathcal{C}, \mathcal{C})}$  be a  $G$ -valued regular Markovian Holonomy Field and let  $(Y_t)_{t \in \mathbb{R}^+}$  be its associated  $G$ -valued Lévy process. Let  $(\mathbb{E}_{vol}^{HF})_{vol}$  be its associated free boundary condition expectation on the plane defined in Definition 76. It is a  $G$ -valued stochastically continuous strong Planar Markovian Holonomy Field as shown in Theorem 29. Hence, by Theorem 6, it induces a stochastically continuous in law  $G$ -valued weak discrete Planar Markovian Holonomy Field  $(\mathbb{E}_{vol}^{HF, G})_{\mathbb{G}, vol}$ . Theorem 26 and Remark 22 ensure that the condition in order to apply Theorem 25 is satisfied by  $(\mathbb{E}_{vol}^{HF, G})_{\mathbb{G}, vol}$ : it is a pure discrete planar Yang-Mills field, denoted by  $(\mathbb{E}_{vol}^{Y, G})_{\mathbb{G}, vol}$ , associated with the Lévy process  $(Y_t)_{t \in \mathbb{R}^+}$  whose associated semi-group of convolution is  $(Z_{1,0,t}^+ dg)_{t > 0}$ . By stochastic continuity, for any measure of area  $vol$ ,  $\mathbb{E}_{vol}^{HF} = \mathbb{E}_{vol}^Y$ .

Let  $YM$  be the Yang-Mills field associated with  $(Y_t)_{t \in \mathbb{R}^+}$ . Let  $(\mathbb{E}_{vol}^{YM})_{vol}$  be the associated free boundary condition expectation on the plane. Using Proposition 40, for any measure of area,  $\mathbb{E}_{vol}^{YM} = \mathbb{E}_{vol}^Y$ . For any disk-shaped surface  $M$  endowed with a measure of area  $vol$ :

$$\mathbb{E}_{M, vol}^{HF} = \mathbb{E}_{M, vol}^{YM}.$$

With this equality in mind, we need to show that for any surface  $M$  homeomorphic to  $\Sigma_{p,1}^+$  (a sphere with  $p$  holes), any measure of area  $vol$ , and any  $x \in G$ , the following equality holds:

$$HF_{(M, vol, \emptyset, \{\partial M \rightarrow [x]\})} = YM_{(M, vol, \emptyset, \{\partial M \rightarrow [x]\})},$$

or the equivalent one:  $\widehat{HF}_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})} = \widehat{YM}_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})}$ . Only the equality for a disk-shaped surface will be explained as one can follow the same proof for the general case. Let  $t$  be equal to  $vol(M)$ , using Definition 75:

$$\mathbb{E}_{M,vol}^{HF} = \int_G \widehat{HF}_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})} dx,$$

and the same holds for  $YM$ . Thus, if we set  $Z_t(x) = HF_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})}(1)$ , which is also equal to  $YM_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})}(1)$ , we can write:

$$\mathbb{E}_{M,vol}^{HF} = \int_G \frac{\widehat{HF}_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})}}{Z_t(x)} Z_t(x) dx.$$

Besides, the law of  $h(\partial M)$  is  $Z_t(g)dg$ : it implies that  $\frac{\widehat{HF}_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})}}{Z_t(x)}$  is a disintegration of  $\mathbb{E}_{M,vol}^{HF}$  with respect to  $h(\partial M)$ . The same holds for  $YM$ . By almost sure uniqueness of the disintegration we have:

$$\frac{\widehat{HF}_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})}}{Z_t(x)} = \frac{\widehat{YM}_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})}}{Z_t(x)}, \text{ a.s. in } x,$$

thus:

$$\widehat{HF}_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})} = \widehat{YM}_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})}, \text{ a.s. in } x.$$

In order to finish we have to remove the *a.s.* part. Thanks to Proposition 5 and Lemma 1, we need to show that, for any continuous function  $f$  invariant by diagonal conjugation from  $G^n$  to  $G$ , and any piecewise affine loops  $l_1, \dots, l_n$  in the interior of  $M$  based at the same point:

$$\widehat{HF}_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})}(f(h(l_1), \dots, h(l_n))) = \widehat{YM}_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})}(f(h(l_1), \dots, h(l_n))).$$

We can find a mark  $l$  which surrounds the  $n$  loops  $l_1, \dots, l_n$ . By  $M'$  we denote the closure of the interior of  $l$ . Applying the Axioms **A<sub>6</sub>** and **A<sub>5</sub>**, we get that for any continuous function  $f$  invariant by diagonal conjugation from  $G^n$  to  $G$ :

$$\begin{aligned} & \widehat{HF}_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})}(f(h(l_1), \dots, h(l_n))) \\ &= \int f(h(l_1), \dots, h(l_n)) HF_{(M,vol,\emptyset,\{\partial M \rightarrow [x]\})}(dh) \\ &= \int_G \int f(h(l_1), \dots, h(l_n)) HF_{(M,vol,l,\{\partial M \rightarrow [x], l \rightarrow [y]\})}(dh) dy \\ &= \int_G \int f(h(l_1), \dots, h(l_n)) \mathbf{A}_6, \mathbf{A}_5 \\ & \quad HF_{(M',vol|_{M'},\emptyset,\{\partial M' \rightarrow [y]\})} \otimes HF_{(M \setminus M',vol|_{M \setminus M'},\emptyset,\{\partial M \rightarrow [x], \partial M' \rightarrow [y]\})}(dh) dy \\ &= \int_G \int f(h(l_1), \dots, h(l_n)) HF_{(M',vol|_{M'},\emptyset,\{\partial M' \rightarrow [y]\})}(dh) Z_{vol(M \setminus M')}(x, y) dy \end{aligned}$$

Thus, if we only consider the restriction on  $\mathcal{M}ult(P(M'), G)$ :

$$\widehat{HF}_{(M, vol, \emptyset, \{\partial M \rightarrow [x]\})} = \int_G \widehat{HF}_{(M', vol|_{M'}, \emptyset, \{\partial M' \rightarrow [y]\})} Z_{vol(M \setminus \dot{M}')} (x, y) dy,$$

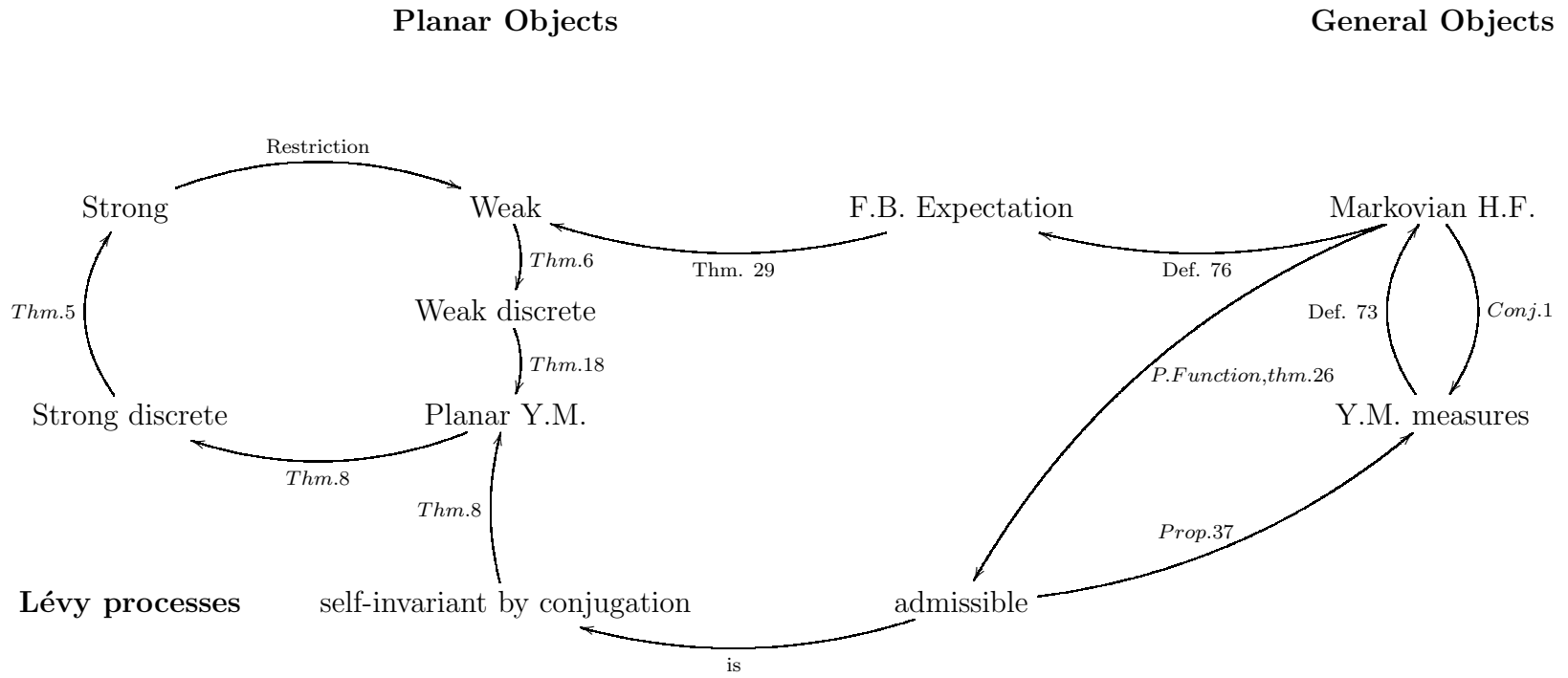
and as the partition function, for the disk, of  $HF$  and  $YM$  are equal, the partition functions for any surface of  $HF$  and  $YM$  are equal and thus one has:

$$\widehat{YM}_{(M, vol, \emptyset, \{\partial M \rightarrow [x]\})} = \int_G \widehat{YM}_{(M', vol|_{M'}, \emptyset, \{\partial M' \rightarrow [y]\})} Z_{vol(M \setminus \dot{M}')} (x, y) dy,$$

Once we restrain the measures on  $\mathcal{M}ult(P(M'), G)$ , for any  $x \in G$ :

$$\begin{aligned} \widehat{HF}_{(M, vol, \emptyset, \{\partial M \rightarrow [x]\})} &= \int_G \widehat{HF}_{(M', vol|_{M'}, \emptyset, \{\partial M' \rightarrow [y]\})} Z_{vol(M \setminus \dot{M}')} (x, y) dy \\ &= \int_G \widehat{YM}_{(M', vol|_{M'}, \emptyset, \{\partial M' \rightarrow [y]\})} Z_{vol(M \setminus \dot{M}')} (x, y) dy \\ &= \widehat{YM}_{(M, vol, \emptyset, \{\partial M \rightarrow [x]\})}. \end{aligned}$$

The equality now holds for any  $x$  and not just for almost every  $x$ .  $\square$



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