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Variability of travel time, congestion and the cost of travel

Nicolas Coulombel [*] André de Palma[†]

Abstract

The variability of travel time modifies the rush hour traffic and the cost of commuting. The bottleneck model of road congestion with fixed peak-load demand is augmented of an additive random delay. When individuals have (α-β-γ) preferences, there exists a unique Nash equilibrium. The variability of travel time leads to departure rates that are spread more evenly over the rush hour than when travel times are deterministic. This equilibrium mechanism mitigates both congestion and the cost of unreliability. This implies that "single-traveler models," which treat congestion as an exogenous phenomenon, overestimate the value of reliability for the rush hour. Application with two distributions, uniform and exponential, provides an appraisal of the extent of the overestimation.

Key words: Value of reliability; Nash equilibrium; Bottleneck model; Scheduling; Random delay.

JEL codes: C61, R41

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1. Introduction

Travel time reliability matters at various levels, from route choice (Abdel-Aty et al., 1995) to mode choice and to the choice of departure time (Chorus et al., 2006). For Bates et al. (2001), first, travel time unreliability can be detrimental to scheduling or even doing one's activities by causing early or late arrivals at destination. Missing a flight because of delays in public transit illustrates this point. Second, people could distaste travel time variability for the stress it generates, or because it complicates the planning of activities. The user's risk aversion reflects this second phenomenon. Palma and Picard (1995) have estimated the risk aversion of drivers using a CARA representation of preferences, and have found that slightly less than one third of the drivers are "risk- lovers," meaning that they have negative risk aversion parameters.

Assessing the value of travel time reliability is a relatively recent topic in transportation literature. In their review, Bates et al. (2001) underlined the lack of research and the absence of this issue in most national guidelines for the evaluation of road and public transport schemes. In line with the anterior but less general work of Noland and Small (1995), Fosgerau and Karlström (2010) have studied the effects of the variability of travel time on the choice of departure time and on the cost of the trip. They consider the case of a single individual with (α-β-γ) preferences (Vickrey, 1969; Small, 1982) and facing a fixed curve of travel time, travel time being a function of departure time. Fosgerau and Engelson (2011) apply the same setting with a different utility function. In both works, the authors derive

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the optimal departure time as well as the minimum cost of the trip; moreover, they derive a formal expression for the value of reliability (VoR), which provides a sound theoretical foundation. The "single traveler" analytical framework, however, does not consider the effect of the variability of travel time on congestion, treating travel time as an exogenous variable. As each individual adapts its departure time in response to higher or lower variability of travel time, the congestion time profile is likely to change. The single traveler framework, while valid when congestion is mild, is therefore unfit to assess the value of travel time reliability for the rush hour period.

In order to address this shortcoming, we develop a theoretical framework to study the effects of the variability of travel time on the morning commute when congestion is endogenous and reflects departure time decisions. The most salient issues are the cost for users (and the VoR) and how the unreliability of travel time changes the time profile of departures, and subsequently the congestion time profile. Our model builds on the standard bottleneck model of road congestion with peak-load demand. The variability of travel time is modelled by an additive random delay, which varies according to departure time. We are thereby in line with works which applied the bottleneck model to phenomena contributing to the variability of travel time: stochastic demand (Palma et al., 1983; Arnott et al., 1991), stochastic capacity (Arnott et al., 1996, 1999; Li et al., 2008) and incidents (Fosgerau and Lindsey, 2012). Our work is closely related to Siu and Lo (2009), who study a bottleneck model with a bounded random delay. We generalize their work by lifting their assumptions of boundedness and constancy of the delays over the rush

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hour. We also find that under some broad assumptions on the random delay, there exists a unique Nash equilibrium for the bottleneck model with an additive random delay and (α-β-γ) preferences. Compared to the case where travel times are deterministic, the variability of travel time leads to spreading of the peak: departures are spread more evenly over the rush hour, a feature which reduces congestion. This also mitigates the effect of travel time unreliability on the expected cost of the trip, leading to a lower VoR than when travel times are exogenous (as in single traveler models). When the maximum random delay is bounded and small enough, individuals' adjustments completely offsett the cost of unreliability, hence a VoR of 0. This result stems from a specific feature of (α-β-γ) preferences, which is that individuals who are sure to be early or late whatever the random delay are risk neutral. Lastly, the effect of the variability of travel time on the timing of the rush hour is non-trivial and depends on the distribution of the random delay.

Our results entail two caveats as far as cost-benefit analysis is concerned. First, non-equilibrium models strongly overestimate the VoR for the rush hour period because they fail to capture equilibrium mechanisms, which mitigate the cost of unreliability. Second, models based on (α-β-γ) preferences likely underestimate the VoR (except in the case of "risk-lover" drivers), because utility is piece-wise linear rather than strictly convex. Utility functions featuring risk aversion, such as the CARA and CRRA functions, should be preferred when deriving the VoR.

2. Modeling Background

We present the standard bottleneck model of road congestion with fixed peak-load demand, from which our model derives. The bottleneck model of road congestion was firstly introduced by Vickrey (1963, 1969). A thorough economic analysis of the standard model with inelastic demand can be found in Arnott et al. (1990).

2.1. The standard bottleneck model of road congestion with fixed peak-load demand

Every morning, N individuals considered as a continuum commute from point A (home) to point B (workplace). Each individual travels with his own car. A single road joins A and B, which starts with a bottleneck with fixed capacity (or service rate) s. Whenever the departure rate $r(t)$ is greater than s, a queue develops at the bottleneck entrance if it does not exist yet. The commuting time is:

$$T(t) = T_0 + \frac{Q(t)}{s}. \tag{1}$$

T_0 is the free flow travel time between the bottleneck exit A and B. $Q(t)/s$ is the congestion delay, equal to the ratio between the length of the queue $Q(t)$ and the service rate. If t_s is the time at which the queue starts to develop, as long as the right-hand side term remains positive, the length of the queue is given by:

$$Q(t) = \int_{t_s}^{t} r(u)\,du - s(t - t_s). \tag{2}$$

$Q(t)$ is the difference between the total number of departures from t_s to t and the bottleneck outflow over the same period. The bottleneck operates at full capacity s

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until the queue is cleared.

In the remainder of the text, we assume that any departure rate function is right-continuous. Two functions r_1 and r_2 equal to one another everywhere except on a set of measure zero generate the same time profiles for both the total number of departures and for the queue. They are indistinguishable for all practical purposes, hence the assumption of right continuity. This also ensures that the function Q is continuous and that its derivative is continuous everywhere except possibly at points where r is discontinuous. At these points, Q admits at least a right derivative. The derivative of Q is given by:

$$\begin{cases} Q'(t) = 0 & \text{if } Q(t) = 0 \text{ and } r(t) \le s \\ Q'_r(t) = r(t) - s & \text{if } Q(t) = 0 \text{ and } r(t) > s \text{ or } Q(t) > 0 \text{ and } r \text{ is discontinuous in } t \\ Q'(t) = r(t) - s & \text{otherwise,} \end{cases} \quad (3)$$

where subscript r stands for the right-hand side derivative. Considering Eq. (1), all previous statements about Q hold true for T except for formulae of the derivatives, which should be divided by s.

2.2. Choice of departure time and cost of the trip

Individuals choose their departure time with the twofold objective of being on time at their destination and of minimizing their travel time. Vickrey (1969) and later Small (1982) have formalized this problem using what is now commonly called (α-β-γ) preferences. The overall cost of the trip values both the time spent on

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travelling and the "schedule delay," meaning that being early or late at one's destination compared to one's preferred arrival time is costly:

$$C(t) = \alpha T(t) + \beta\left(t^* - t - T(t)\right)^+ + \gamma\left(t + T(t) - t^*\right)^+, \tag{4}$$

where $(x)^+$ is the positive part of x. $C(t)$ is the cost of the trip when leaving at t, $T(t)$ the travel time, and t^*-t-$T(t)$ the schedule delay. The schedule delay is said to be early if it is positive, late if it is negative. It is measured relatively to a preferred arrival time t^*, which here represents the work starting time. For simplicity of exposition, we assume that all individuals have the same t^*.

The cost of one minute of travel time is α; the cost of being one minute early at your destination is β and the cost of being one minute late is γ. These fixed parameters set the terms of the trade-off between travel time and schedule delay when choosing the departure time. As usual, we assume $\beta < \alpha$ (the case $\beta \geq \alpha$ is discussed in the Appendix of Arnott et al., 1985).

Each individual seeks to minimize the cost of his trip by adjusting his departure time. A Nash equilibrium is reached when no individual has an incentive to change it. The equilibrium is fully characterized by the departure rate function $r(t)$, with the condition that the cost of the trip $C(t)$ must be minimized (and thus constant) on the set $\{t \,/\, r(t)>0\}$.

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2.3. Results

There exists a Nash equilibrium for the standard bottleneck model (Smith, 1984) and it is unique (Daganzo, 1985; Lindsey, 2004). At this equilibrium:

- departures are spread from t_q to $t_{q'}$. The departure rate is strictly positive on this interval.
- there is congestion throughout the whole rush hour (except in t_q and $t_{q'}$) and the bottleneck always operates at full capacity. Consequently, the length of the rush hour is: $t_{q'} - t_q = N/s$.
- congestion varies so as to equalize the cost of the trip throughout the rush hour.

This entails the conditions regarding the travel time:

$$\begin{cases} T'(t) = \dfrac{\beta}{\alpha - \beta} & \text{for } t \in \left]t_q, t_p\right[\\ T'(t) = -\dfrac{\gamma}{\alpha + \gamma} & \text{for } t \in \left]t_p, t_{q'}\right[\end{cases} \tag{5}$$

where t_p is the congestion peak: leaving at t_p means facing the biggest queue at the bottleneck.

Using Eq. (2) and (5) implies:

$$\begin{cases} r(t) = \dfrac{\alpha}{\alpha - \beta}\, s & \text{for } t \in \left[t_q, t_p\right[\\ r(t) = \dfrac{\alpha}{\alpha + \gamma}\, s & \text{for } t \in \left[t_p, t_{q'}\right[\end{cases} \tag{6}$$

for the departure rate. Out of the rush hour, the cost of the trip is greater despite enjoying no congestion because the schedule delay is too high. During the first part $[t_q,t_p[$ of the rush hour, the departure rate $\alpha s/(\alpha\text{-}\beta)$ is greater than s. The length of the queue increases and is maximal at the peak t_p. The waiting time is also the

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longest at t_p and, conversely, the schedule delay is the shortest and null. People leaving before the peak are thereby early, while those leaving after the peak are late. After t_p, the departure rate $\alpha s/(\alpha+\gamma)$ is lower than s. The queue recedes until disappearing at time $t_{q'}$, which is the end of the rush hour.

Using the notation $\delta = \beta\gamma/(\beta+\gamma)$, we obtain:

$$\begin{cases} t_q = t^* - T_0 - \dfrac{\delta}{\beta}\dfrac{N}{s} \\ t_p = t_s + \dfrac{\alpha-\beta}{\alpha}\dfrac{\delta}{\beta}\dfrac{N}{s} \\ t_{q'} = t^* - T_0 + \dfrac{\delta}{\gamma}\dfrac{N}{s} \end{cases} \quad \text{and} \quad \begin{cases} T(t_p) = T_{\max} = T_0 + \dfrac{\delta}{\alpha}\dfrac{N}{s} \\ C_{eq} = \alpha T_{\max} = \alpha T_0 + \delta\dfrac{N}{s} \end{cases} \tag{7}$$

3. The Bottleneck Model with Stochastic Travel Times

After presenting the bottleneck model with stochastic travel times, we show that a user equilibrium exists and is unique.

3.1. The stochastic bottleneck model

The bottleneck model with stochastic travel times is based on the same setup as the standard model, with the difference that the travel time includes a stochastic component which represents day-to-day variability. This term takes the form of a random delay, noted ε_t, which may vary over the course of the day. Accordingly, the travel time $\tilde{T}$ is now given by:

$$\tilde{T}(t) = T_0 + \frac{Q(t)}{s} + \varepsilon_t, \tag{8}$$

where a tilde emphasizes the stochastic nature of a variable.

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The random delay being additive, the congestion level does not influence the scale of the delay. ε_t captures phenomena contributing to the variability of T_0, including weather conditions and variations in the speed of drivers, after exiting the bottleneck. A multiplicative form where $Q(t)/s$ would be multiplied by one plus a random noise would allow us to model the variability of $Q(t)/s$ resulting from fluctuations in the level N of travel demand or in the bottleneck capacity s. We focus on the variability of T_0 because, while the two cases are relatively similar, the algebra is much simpler with an additive random delay than with a multiplicative one.

We do not consider the role of the provision of dynamic information regarding travel times. This excludes dynamic strategies based on such information, such as leaving home when ε_t appears to be low. It also implies that the dynamics of ε_t has no importance here, and we can assume any correlation structure for the $(\varepsilon_t)_{t\in\Re}$ as long as the basic assumptions mentioned below hold.[1]

As pointed out by Bates et al. (2001), the most general case where the set of the distributions of $(\varepsilon_t)_{t\in\Re}$ has no specific properties does not provide any significant result. We assume that:

[1] If one wants to maintain some basic flow properties (such as First In First Out), some constraints must be imposed on the time dynamics of ε_t. In particular, assuming that the error terms are independent could lead to overtakings (if $\varepsilon_{t+\delta}$ is much lower than ε_t), in contradiction with the First In First Out principle.

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- all $(\varepsilon_t)_{t\in\Re}$ follow the same law;
- they have the same mean μ and the same standard deviation σ.

The distribution function of the normalized random variable $\hat{\varepsilon}_t = (\varepsilon_t - \mu)/\sigma$ is denoted f and its cumulative distribution function F. Again for the sake of clarity we assume that f is piecewise continuous. In particular, there is no punctual mass in the distribution of f. This assumption is supported empirically (Richardson and Taylor, 1978; Giulano, 1989).

3.2. Choice of departure time and expected cost of the trip

Travel time being stochastic, the cost of the trip becomes a random variable. As the realized values of the random delays $(\varepsilon_t)_{t\in\Re}$ are not known beforehand, the choice of departure time is based on the ex-ante cost. It is given by:

$$\bar{C}(t) = \mathrm{E}\left(\tilde{C}(t)\right) = \alpha\bar{T}(t) + \beta\sigma\int_{-\infty}^{m(t)}\left(m(t)-u\right)f(u)\,du + \lambda\sigma\int_{m(t)}^{+\infty}\left(m(t)-u\right)f(u)\,du \qquad (9)$$

$$\text{with } m(t) = \frac{t^* - t - \bar{T}(t)}{\sigma} \qquad (10)$$

where $\bar{T}(t) = T_0 + Q(t)/s + \mu$ is the expected travel time when leaving at t and $m(t)$ is often referred to as the normalized safety margin (Fosgerau and Karlström, 2010). The ex-ante cost of the trip is the sum of $\alpha\bar{T}(t)$, the value of the average travel time, and of the expected cost of the schedule delay. Compared to the deterministic case, the variability of travel time mainly affects the latter term. The first term is affected only through the average delay μ, which modifies the average travel time.

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It is unchanged if the random error has a zero mean ($\mu = 0$).

Using $\int_{-\infty}^{+\infty} uf(u)du = 0$, Eq. (9) is rewritten as:

$$\bar{C}(t) = \alpha\bar{T}(t) + (\beta+\gamma)\sigma\left(m(t)F(m(t)) + G(m(t))\right) - \gamma\sigma m(t) \tag{11}$$

where $G(x) = \int_{x}^{+\infty} uf(u)du$ is a continuous and positive function.

3.3. Equilibrium: existence and uniqueness

We establish several lemmas which are then used to prove the existence and the uniqueness of the Nash equilibrium for the bottleneck model with an additive random delay (BARD).

The function $H(t,T)$ is:

$$H(t,T) = \alpha T + (\beta+\gamma)\sigma\left(\hat{m}(t,T)F(\hat{m}(t,T)) + G(\hat{m}(t,T))\right) - \gamma\sigma\hat{m}(t,T) \tag{12}$$

with $\hat{m}(t,T) = \dfrac{t^* - t - T}{\sigma}$ (13)

The expected cost of the trip is related to H by:

$$\bar{C}(t) = H(t,\bar{T}(t)) \tag{14}$$

H and $\bar{T}$ being continuous functions, the expected cost of the trip is a continuous function of t. Lemma 1 provides additional properties of the expected trip cost.

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Lemma 1 *When the expected travel time is constant, the expected cost of the trip is first decreasing then increasing in t. It decreases and increases strictly with t if $F^{-1}(\delta/\beta)$ is a singleton, in which case there exists a unique minimum for the departure time $t = t^* - \bar{T} - \sigma F^{-1}(\delta/\beta)$.*

For any given departure time, the expected cost of the trip increases strictly with the expected travel time.

Proof These properties stem from the two partial derivatives of *H*. The derivative with respect to *t* is:

$$\frac{\partial H(t,T)}{\partial t} = -(\beta+\gamma)F\left(\hat{m}(t,T)\right)+\gamma = \gamma-(\beta+\gamma)F\left(\frac{t^*-t-T}{\sigma}\right) \tag{15}$$

The First Order Condition (FOC) is:

$$\frac{\partial H(t,T)}{\partial t} = 0 \Leftrightarrow F\left(\frac{t^*-t-T}{\sigma}\right) = \frac{\delta}{\beta}. \tag{16}$$

The first point is then straightforward. The second partial derivative is:

$$\frac{\partial H(t,T)}{\partial T} = \alpha+\gamma-(\beta+\gamma)F\left(\frac{t^*-t-T}{\sigma}\right). \tag{17}$$

As $\beta < \alpha$, $\frac{\partial H(t,T)}{\partial T} > 0$, hence the second point. ■

For the remainder of the text, we assume that the set $\{x / f(x) > 0\}$ is always convex. This ensures that the equation $F(x) = \delta/\beta$ admits a unique solution. Not making this assumption leads to multiple equilibriums, but our results still hold

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true. Moreover, for any departure function *r*, we refer to the set $\{t / r(t) > 0\}$ as the rush hour. Equilibrium solutions are denoted by a superscript *v*.

Lemma 2 *For any Nash equilibrium of the BARD, the rush hour is a right-open segment of length N/s, denoted by* $\left[t_q^v, t_{q'}^v\right[$ *with* $t_{q'}^v - t_q^v = N/s$. *All users face congestion except for the first one, who leaves at* t_q^v, *and for the last one, who leaves at* $t_{q'}^v$.

Proof In Appendix A. ■

At any equilibrium solution, there is no queue at the beginning and at the end of the rush hour, hence $\bar{T}\left(t_q^v\right) = \bar{T}\left(t_{q'}^v\right) = T_0 + \mu$. Moreover, the length of the rush hour is the same as in the deterministic case: $t_{q'}^v - t_q^v = t_{q'} - t_q = N/s$.

We can characterize the solutions further by providing another condition relative to the dynamics of the expected travel time.

Lemma 3 *During the rush hour, the expected travel time solves:*

$$\bar{T}'(t) = \frac{\beta F\left(m(t)\right) - \gamma\left(1 - F\left(m(t)\right)\right)}{\alpha - \left(\beta F\left(m(t)\right) - \gamma\left(1 - F\left(m(t)\right)\right)\right)} \quad \forall t \in]t_q^v, t_{q'}^v[. \tag{18}$$

Proof The derivation of Eq. (11) with respect to *t* yields:

$$\bar{C}'(t) = \alpha \bar{T}'(t) + (\beta + \gamma)\sigma m'(t) F\left(m(t)\right) - \gamma \sigma m'(t), \tag{19}$$

where $m'(t) = -\left(1 + \bar{T}'(t)\right)/\sigma$. This is rewritten as:

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$$\bar{C}'(t) = \bar{T}'(t)\Big(\alpha - (\beta+\gamma)F\big(m(t)\big)\Big) - (\beta+\gamma)F\big(m(t)\big) + \gamma. \tag{20}$$

The equilibrium condition $\bar{C}(t) = C_{eq}^{v}\ \forall t \in I$ implying that $\bar{C}'(t) = 0\ \forall t \in]t_q^v, t_{q'}^v[$, Lemma 3 follows. ■

As in the deterministic case, during the rush hour, congestion first increases then decreases. The peak however occurs at t_p^v instead of t_p, where t_p^v is defined by $m\left(t_p^v\right) = F^{-1}(\delta/\beta)$. The increase and decrease in congestion are smoother than in the deterministic case. When congestion increases $\bar{T}'(t) \le \beta/(\alpha-\beta)$; conversely, when congestion recedes, $\left|\bar{T}'(t)\right| \le \gamma/(\alpha+\gamma)$. Lastly, when σ tends toward zero, F tends toward the Heaviside step function and we find again the results of the deterministic case: $\bar{T}'(t) = \beta/(\alpha-\beta)$ for $t < t_p$ and $\bar{T}'(t) = -\gamma/(\alpha+\gamma)$ for $t > t_p$.

We establish the existence and uniqueness of a Nash equilibrium in the bottleneck model with additive random delay.

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Proposition 1 *There exists a unique Nash equilibrium in the BARD. It is given by:*

$$\begin{cases} r^v(t) = \dfrac{\alpha s}{\alpha - \left(\beta F\left(m(t)\right) - \gamma\left(1 - F\left(m(t)\right)\right)\right)} & \text{if } t \in \left[t_q^v, t_{q'}^v\right[\\ r^v(t) = 0 & \text{if } t \notin \left[t_q^v, t_{q'}^v\right[\end{cases} \tag{21}$$

$$\text{where} \begin{cases} m(t) \text{ is given by Eq. (X)} \\ T(t) = T_0 + \mu + \int_{t_q^v}^{t} \left(r^v(u) - s\right) du \\ t_q^v \text{ satisfies } H\left(t_q^v + N/s, T_0 + \mu\right) = H\left(t_q^v, T_0 + \mu\right) \text{ and } t_{q'}^v = t_q^v + N/s \end{cases}$$

Proof In Appendix B. ■

Again, when σ tends toward zero, $r^v(t)$ tends toward the deterministic solution given by Eq. (6).

4. Comparative Statics

The variability of travel time changes the Nash equilibrium. To better understand why, we plot (α-β-γ) preferences as a function of travel time T (and not as a function of departure time t as it is usually done).

The cost function of the trip is piecewise linear in T with a kink in $T = t^* - t$ (Figure 1). The slope is equal to α-β before the kink, then to $\alpha+\gamma$. As $\alpha+\gamma > \alpha-\beta$, the cost function of the trip is convex. Individuals with (α-β-γ) preferences are risk-averse (to some extent) and thus seek to insure themselves against travel time unreliability, modifying the equilibrium. Notwithstanding, if one remains on either side of the curve (he leaves at such a time that he is sure to be either early or late at

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his destination whatever the realized value of the random delay), the linearity of the cost of travel implies that this individual is risk-neutral. The random delay must be bounded on at least one side for this case to arise. If not, the possibility of an extreme delay means that one is never sure to be early or late, whatever the departure time.

We now study how travel time unreliability changes the main equilibrium features.

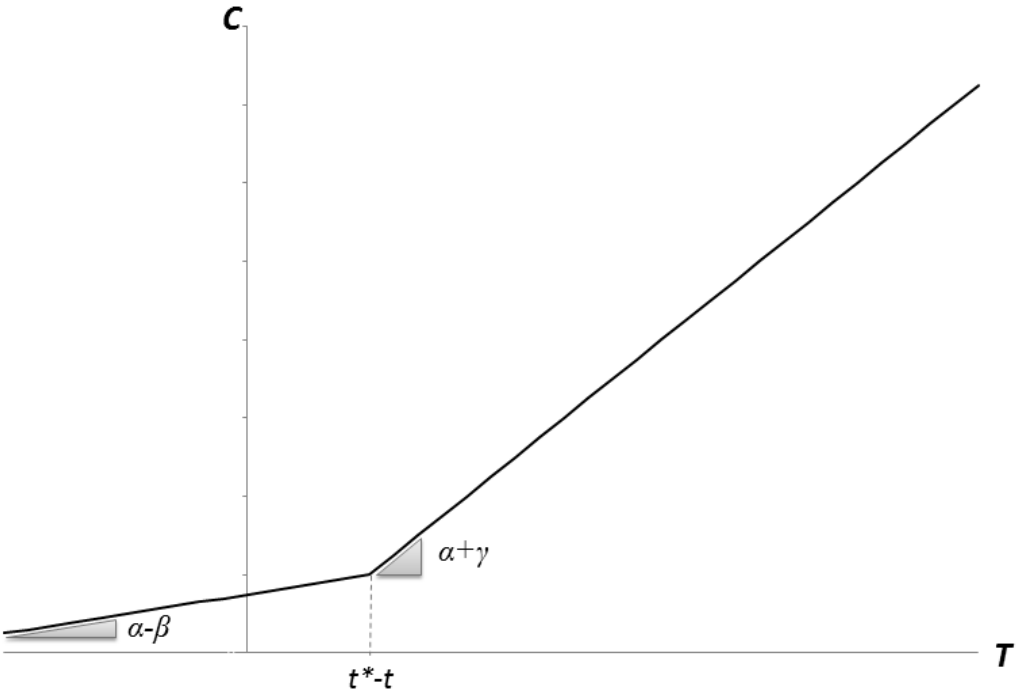

Figure 1: Relationship between trip cost (C) and travel time (T) under (α-β-γ) preferences

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4.1. Peak congestion level

The variability of travel time leads to reduction of the peak.

Proposition 2 *For the bottleneck model of road congestion with fixed demand, the maximum level of congestion is lower with stochastic travel times than with deterministic travel times.*

Proof As $\bar{T}'(t) \leq \beta/(\alpha-\beta)$ on $\left[t_q^v, t_p^v\right]$, $\bar{T}\left(t_p^v\right) > T\left(t_p\right) \Rightarrow t_p^v - t_q^v > t_p - t_q$. Similarly, $\bar{T}'(t) \geq -\gamma/(\alpha+\gamma)$ on $\left[t_p^v, t_{q'}^v\right]$, hence $\bar{T}\left(t_p^v\right) > T\left(t_p\right) \Rightarrow t_{q'}^v - t_p^v > t_{q'} - t_p$. Combining the two points yields $\bar{T}\left(t_p^v\right) > T\left(t_p\right) \Rightarrow t_{q'}^v - t_q^v > t_{q'} - t_q$, which is a contradiction given that $t_{q'}^v - t_q^v = t_{q'} - t_q = N/s$. ■

Under (α-β-γ) preferences, convexity of the cost function is strongest around the kink. The cost of unreliability is thus highest for people leaving at the peak, who would arrive at their preferred arrival time were it not for the random delay. Considering that people leaving at the beginning or at the end of the rush hour are less affected by the variability of travel time by being far from the kink of the cost function, peak reduction occurs as an equilibrium mechanism.

Travel time unreliability also leads to spreading of the peak. While in the deterministic case, the peak corresponds to a kink of the curve of travel time, in the stochastic case this curve is flat at the peak: $\bar{T}'\left(t_p^v\right) = 0$.

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4.2. Timing of the rush hour

The effect of the variability of travel time on the timing of the rush hour is not straightforward. Application in Section 5 will show that depending on the distribution, an increase in σ can shift the rush hour earlier or later. For a given distribution, the direction of the shift can even change depending on the value of σ. Given the notations $J_L(x) = \int_{-\infty}^{x} [x-u] f(u) du$ and $J_R(x) = \int_{x}^{+\infty} [u-x] f(u) du$, Proposition 3 provides some indication regarding the direction in which the shift occurs.

Proposition 3 *If* $J_L\left(-\frac{\delta}{\gamma}\frac{N}{s\sigma}\right) > J_R\left(\frac{\delta}{\beta}\frac{N}{s\sigma}\right)$, *the rush hour starts earlier in the BARD than in the deterministic case, and conversely.*

Proof In Appendix C. ■

Proposition 3 provides a condition to determine whether the rush hour occurs earlier or later than in the deterministic case. When f is symmetric, $J_R(x) = J_L(x)$; the rush hour starts earlier if $\beta < \gamma$, later if $\beta > \gamma$. Empirical studies providing support for $\beta < \gamma$ (Small, 1982, among others), our model indicates that people leave earlier for work when they face variable travel times (provided that the underlying distribution is symmetric). This result is conform to intuition.

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4.3. Equilibrium cost of the trip

Proposition 4 *In the BARD, the equilibrium expected cost of the trip increases with σ. It increases strictly with σ if the support of the random delay is unbounded on both sides.*

Proof In Appendix D. ■

The value of reliability (VoR), which is equal to $\frac{dC_e(\sigma)}{d\sigma}$, is always positive. This result is also conform to intuition: individuals with (α-β-γ) preferences are risk averse, at least to some extent. The VoR however is null if $dt_q^v / d\sigma$ and $\partial H / \partial \sigma$ are both equal to 0. This situation arises for low values of σ when the random delay is bounded, as established by the following proposition.

Proposition 5 *When the support of the random delay is bounded and when the maximum delay (in absolute value) is small enough, the equilibrium cost of the trip is independent of σ. The timing of the rush hour also remains unchanged.*

Proof In Appendix E. ■

If the random delay is bounded, low levels of variability change neither the equilibrium cost of the trip nor the timing of the rush hour, but they do reduce congestion. When the maximum delay is small compared to the length of the rush hour, the first person to leave is sure to be early and the last one to leave is sure to be late. These two persons are thus risk-neutral (see comments made at the

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beginning of this section regarding (α-β-γ) preferences) and act as "anchor points". As long as σ remains under the threshold, they leave at the same time and their cost of commute does not change. This keeps the equilibrium cost of the trip at the same level and the timing of the rush hour unchanged. The other individuals, starting with those leaving at the peak, adjust their departure time, however, hence a decrease in congestion.

Formal derivation of the VoR is not feasible in the general case as it is for the single traveler version of this model (Fosgerau and Karlström, 2010). It may be achieved for specific distribution functions, including the uniform case (Coulombel and Palma, 2012).

5. Application

An application illustrates the previous findings and provides an appraisal of the overestimation of the VoR for the rush hour when using the single-traveler model.

The preferred arrival time is set to 9.30 am, with an expected commute time of half an hour when there is no congestion (and including the mean random delay). As travel demand and transportation supply are of equal levels, the rush hour is one hour long. The cost parameters α, β, and γ are in the ratio 1.0:0.8:3.0 indicated by Bates et al. (2001) as typical from the literature.

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Table 1: Parameter values for the BARD

Type of parameter	Value
Schedule preferences	$\alpha = 1.2$, $\beta = 1$, $\gamma = 3$
Preferred arrival time	$t^* = 9.5$
Expected free flow travel time	$T_0 = 0.5$
Mean random delay	$\mu = 0$
Travel demand	$N = 1000$
Transportation supply	$s = 1000$

We consider two distributions for the random delay, uniform and exponential, and study in each case the influence of the variability of travel time on departure rates, congestion (through the expected travel time), and the equilibrium cost of the trip. A comparison of the two distributions concludes this section.

5.1. Uniform distribution

In the case of a uniform distribution, the distribution function *f*, the cumulative distribution function *F*, and function *G* are given by: [2]

$$f(x) = \frac{1}{2\sqrt{3}},\ F(x) = \frac{1}{2} + \frac{x}{2\sqrt{3}},\ \text{and } G(x) = \frac{3 - x^2}{4\sqrt{3}} \quad \text{if } x \in \left[-\sqrt{3}; \sqrt{3}\right]. \tag{22}$$

Starting from $\sigma = 0$ (deterministic case), an increase in σ causes people leaving

[2] Coulombel and Palma (2012) study the case of a uniform distribution extensively, including a computation of closed-form solutions for the equilibrium.

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around the peak to change their departure time (Figure 2). Instead of the two plateaus for the deterministic case, the normalized departure rate $r(t)/s$ decreases smoothly from $\alpha/(\alpha-\beta)$ to $\alpha/(\alpha+\gamma)$. As discussed previously, the unreliability of travel time is most costly for people leaving at the peak, for whom schedule delay disutility was null otherwise. For the others, the choice not to be on time partly insures them against the random delay. For instance, people leaving after the peak and who are on average late at their destination will benefit from situations where the travel time is shorter than usual. Far from the peak, people are perfectly insured against travel time unreliability, hence stable departure rates at the beginning and at the end of the rush hour (see discussion following Proposition 5).

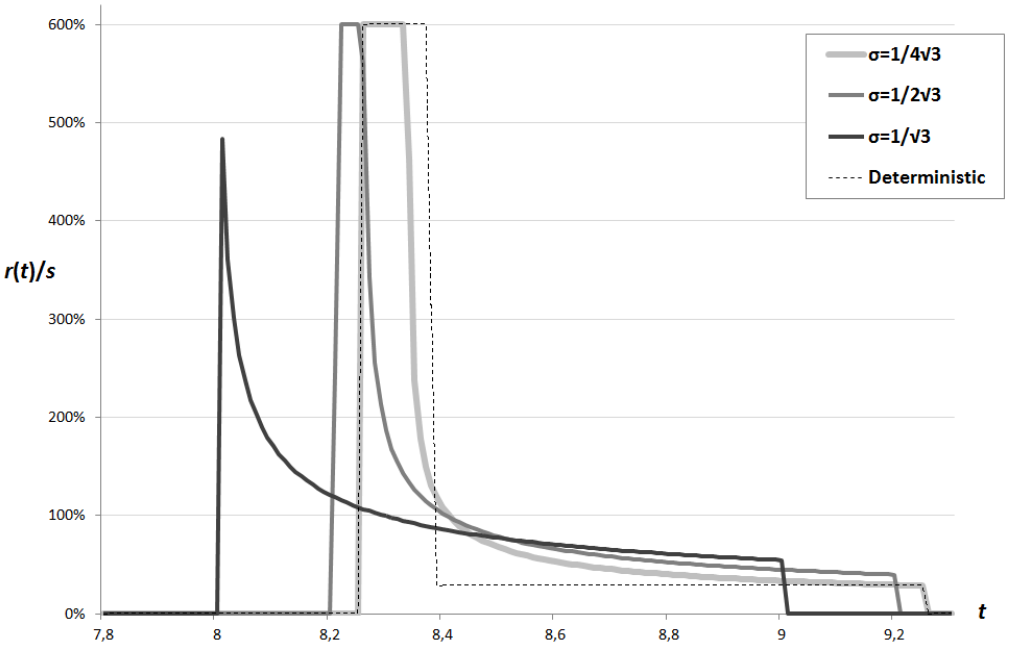

Figure 2: Effect of the variability of travel time on the normalized departure rates (uniform distribution)

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When σ exceeds a certain threshold (equal to $1/4\sqrt{3}$ in the application), changes get more important. In our example $\gamma > \beta$, so drivers leave home earlier and earlier as variability increases in order to insure themselves against the risk of being late (Proposition 3 with f symmetrical). The departure rate is homogenized further. While the level at the beginning of the rush hour remains the same at first, eventually the whole curve is flattened.

The congestion time profiles reflect the three main effects of the variability of travel time: reduction, spreading, and shifting of the peak (Figure 3). In the case of a uniform distribution, the random error is bounded. As indicated by Proposition 5, the peak shifts only when σ exceeds the threshold mentioned above ($\sigma > 1/4\sqrt{3}$). Again, here $\gamma > \beta$ so the rush hour starts earlier and earlier as σ grows; the shift would be in the opposite direction were γ to be lower than β. Contrary to the shift phenomenon, reduction and spreading of the peak occur even for low values of σ.

We break down the equilibrium cost of the trip as the sum of the mean costs of uncongested travel, congestion and schedule delay. The cost of uncongested travel is a constant equal to $\alpha(T_0+\mu)$ and is independent of σ. The mean cost of congestion decreases strictly with σ as a result of the decrease in congestion (Figure 4). On the other hand, travel time unreliability puts a burden on the travelers' schedules, causing the mean cost of schedule delay to increase strictly with σ.

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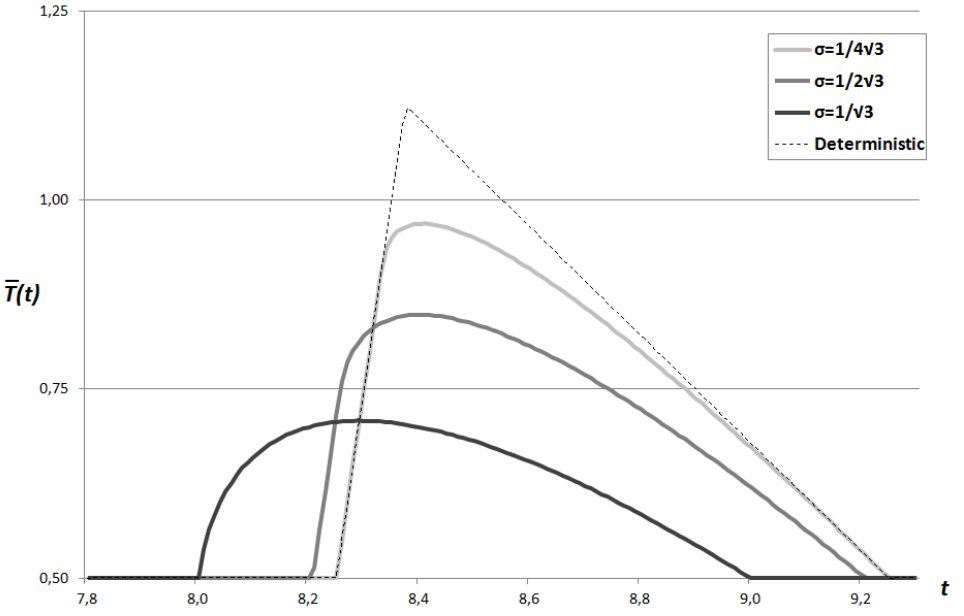

Figure 3: Effect of travel time variability on the expected travel time profile (uniform distribution)

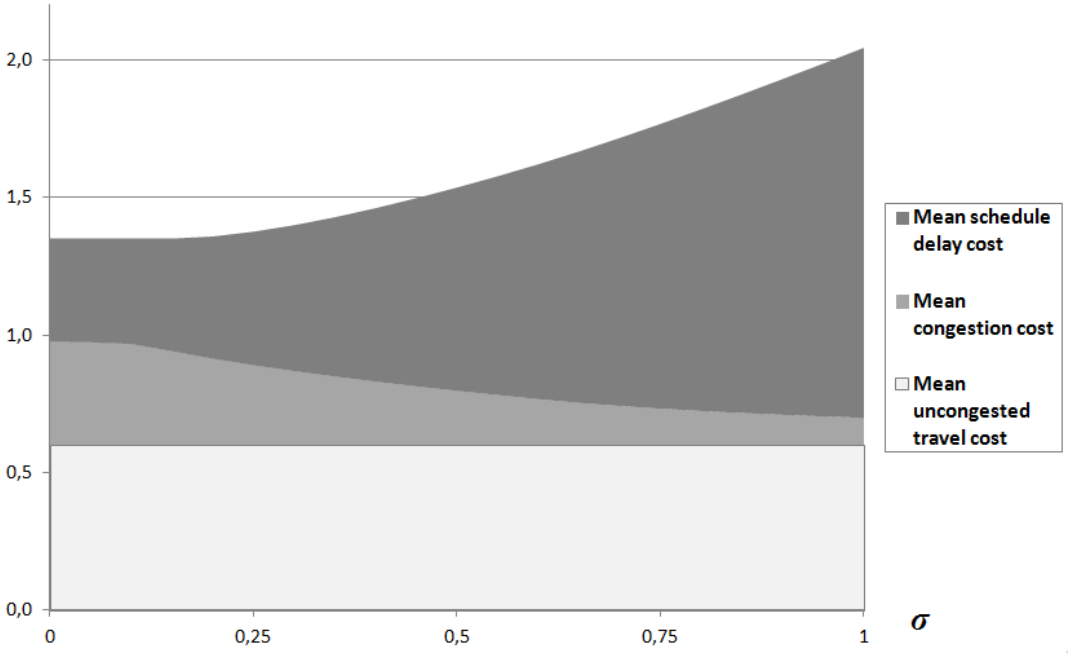

Figure 4: Effect of travel time variability on the three cost items (uniform distribution)

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All in all, the equilibrium cost of the trip remains unchanged at first, in accordance with Proposition 5. Individual adjustments perfectly offsett the cost of unreliability. When $\sigma \geq 1/4\sqrt{3}$, the equilibrium cost of the trip increases strictly with σ, in accordance with Proposition 4.

5.2. Exponential distribution

In the case of an exponential distribution, f, F, and G are equal to (recall that they correspond to a reduced variable):

$$f(x) = e^{-(x+1)},\ F(x) = 1 - e^{-(x+1)},\ \text{and}\ G(x) = (x+1)e^{-(x+1)}\ \text{for}\ x \geq -1. \qquad (23)$$

The application leads to the same comments as for the uniform distribution, except for three points:

- shifting of the peak occurs as soon as $\sigma > 0$ (Figure 5, Figure 6);
- the whole departure rate curve is altered and becomes smoother as soon as $\sigma > 0$ (Figure 5);
- the equilibrium cost of the trip increases strictly with σ whatever the value of σ (Figure 7).

Contrary to the uniform distribution, the support of the exponential distribution is unbounded. Even for very low values of σ, everyone may be late given that the probability of sustaining an extremely high delay is not zero. It ensues that all individuals are risk averse, while in the uniform case, only a fraction of them are

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risk averse when σ is small. Despite this difference, for low values of σ, the equilibrium cost of the trip barely increases with σ (Figure 7). Again, individual adjustments almost completely offset the cost of unreliability.

The effect of the variability of travel time on the timing of the rush hour is not univocal. As σ increases, the rush hour starts slightly later at first, but eventually it starts earlier and earlier (Figure 5, Figure 6). This point probably stems from the asymmetry of the exponential distribution, although we could not properly establish it.

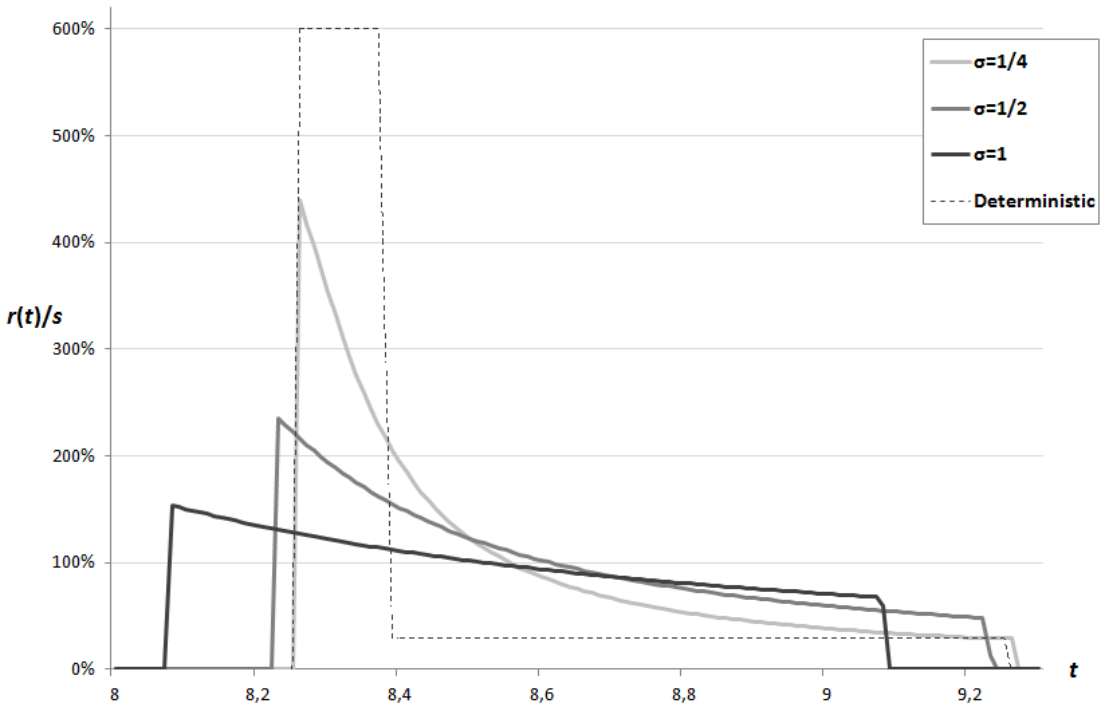

Figure 5: Effect of the variability of travel time on the normalized departure rates (exponential distribution)

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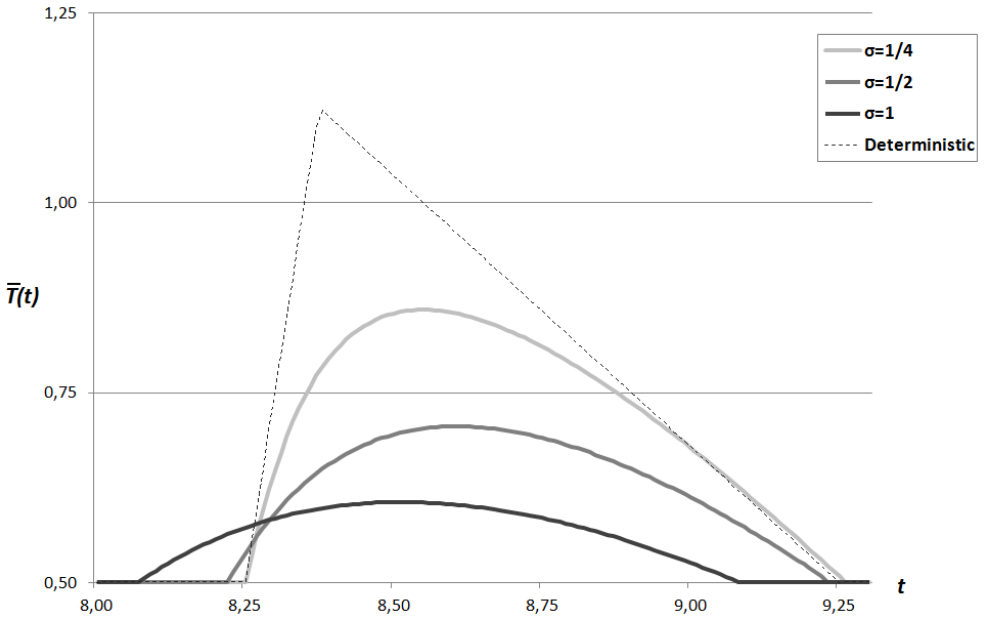

Figure 6: Effect of the variability of travel time on the expected travel time profile (exponential distribution)

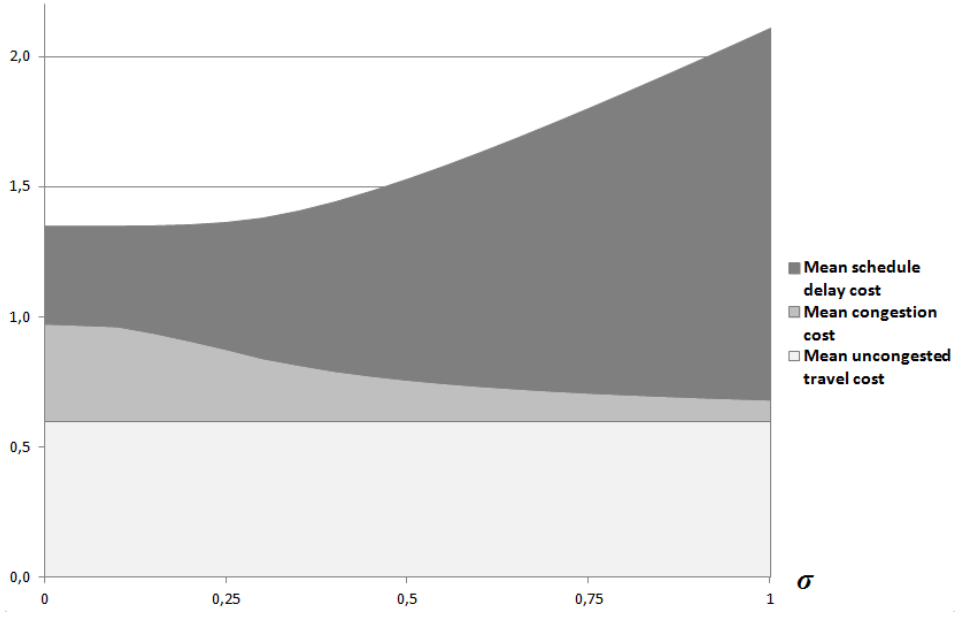

Figure 7: Effect of the variability of travel time on the three cost items (exponential distribution)

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5.3. Comparison of the two distributions

To conclude, we compare for the two distributions:

- the equilibrium and the "disequilibrium" costs, the latter being the mean expected cost of travel were people to choose the same departure time as in the deterministic case;
- the value of reliability (VoR) given by the stochastic bottleneck model with the one given by the single-traveler model exposed in Fosgerau and Karlström (2010).

The gap between the disequilibrium and the equilibrium costs of travel measures the extent to which individual adjustments mitigate the cost of the variability of travel time. The gap between the "equilibrium VoR" and the "F-K VoR" provides a similar indicator for the VoR (which is for reminder the derivative of the equilibrium cost of travel).

The mean expected cost of travel is lower at equilibrium than in disequilibrium for both distributions of the random delay (Figure 8). Individual adjustments are beneficial indeed, a point which was not completely obvious, even more so when uncertainty is limited. Beyond a certain threshold, the mean equilibrium and disequilibrium costs of travel both rise steadily.

For low and high values of σ, the equilibrium cost of the trip is higher for the exponential distribution, while for intermediate values of σ, it is higher for the

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uniform distribution. Again, this peculiar behavior could be linked to the asymmetry of the exponential distribution. Be it as it may, the spread between the two distributions remains limited, except for very high values of σ.

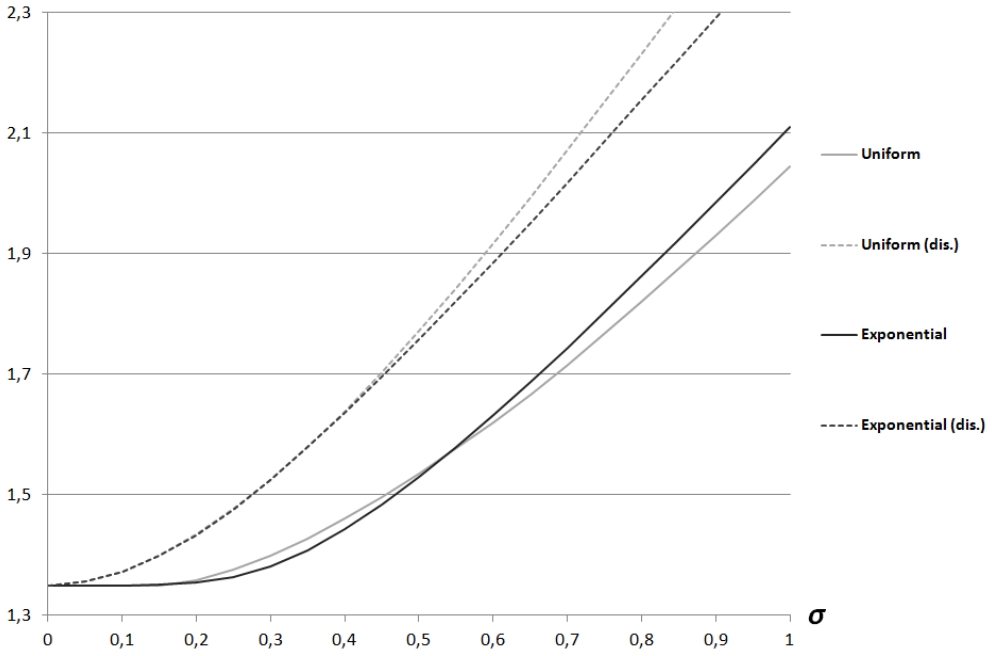

Figure 8: Comparison of the equilibrium and disequilibrium (dis.) costs of the trip for the two distributions

The "equilibrium VoR" is always positive, in accordance with Proposition 4 (Figure 9). It increases with σ for both distributions.[3] Like for the equilibrium cost of the trip, the two VoR are in the same order of magnitude when the variability of travel time remains moderate. The VoR is higher for the exponential distribution when variability becomes strong.

[3] Coulombel and Palma (2012) show this point in the uniform case. It has yet to be established in the general case.

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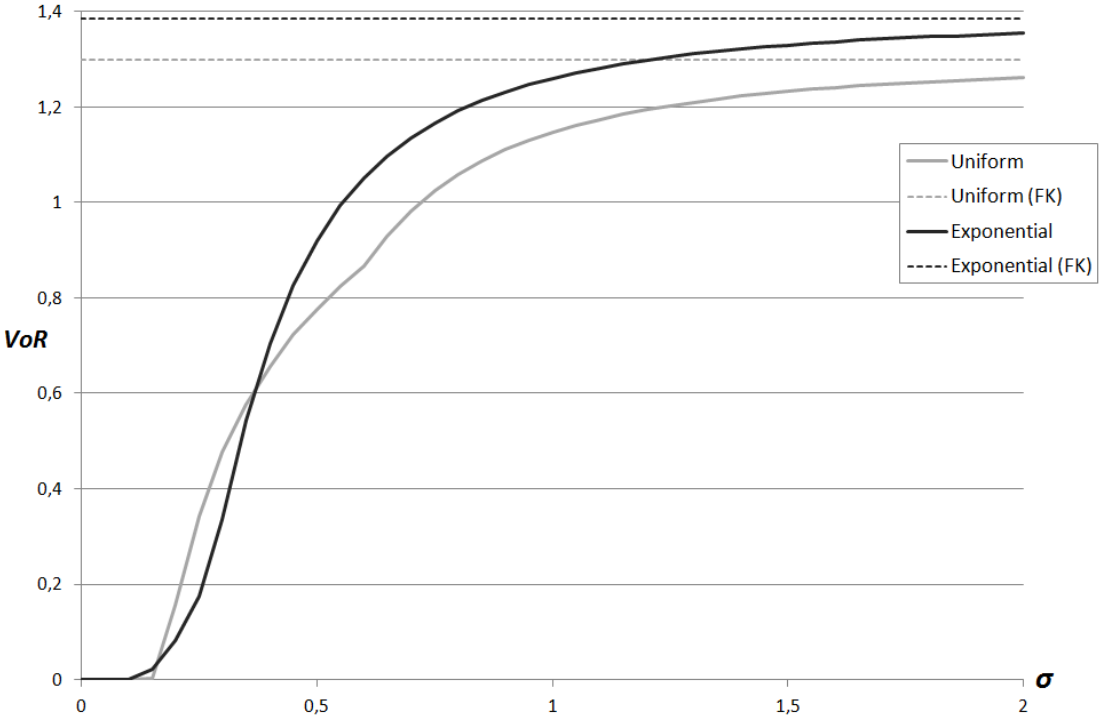

Figure 9: Comparison of the VoR for the stochastic bottleneck model and the single-traveler model

The "equilibrium VoR" asymptotically converges toward the "F-K VoR" for both distributions. This point, which is not demonstrated, is intuitive: when σ is high, congestion becomes close to null and the expected travel time is quasi-constant. On the other hand, for reasonable values of σ the VoR given by the single-traveler model is significantly higher than the VoR given by the equilibrium model, whatever the distribution. The single-traveler model substantially overestimates the value of reliability for the rush hour period.

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6. Conclusion

The bottleneck model with an additive random delay predicts that the variability of travel time has three main effects on travel conditions at the rush hour: (1) departures are spread more evenly; (2) congestion decreases; (3) the peak will occur sooner or later depending on individual preferences, which are captured by the (α-β-γ) parameters, and depending on the distribution of the random delay.

The risk associated with the unreliability of travel time has a cost, which is captured by the value of reliability. When all individuals adjust their departure time, the decrease in congestion greatly mitigates this cost. By treating congestion as an exogenous process, single-traveler models significantly overestimate the value of reliability for the rush hour - at least for reasonable levels of variability of travel time. On the other hand, for the two distributions (uniform and exponential) considered in our application, our model predicts a very low cost of unreliability, even for relatively high risk levels. This leads us to believe that (α-β-γ) preferences are not fully appropriate to compute the VoR. One should explicitly represent risk aversion through the use of adequate utility functions (such as CARA or CRRA).

Several elements remain to be addressed. In several metropolitan areas, dynamic traffic information is available online and people can adjust their departure time based on this information. Integrating this point would increase the complexity of the model, but could provide interesting results concerning the value of this information (which could be negative, as shown by previous works).

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Secondly, one could use more detailed scheduling preferences than (α-β-γ) preferences and include some flexibility on the preferred arrival time or a discrete penalty for being late at destination (which was empirically estimated by Small, 1982). Last but not least, the assumption of a constant volatility of the random delay is strong. Multiplicative random shocks are more realistic, considering that variability increases for larger levels of congestion (Mott MacDonald, 2009). Future research will try to lift these assumptions.

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Annex A – Proof of Lemma 2

We note I the set $\{t \,/\, r(t) > 0\}$ (the rush hour) and C_{eq}^{v} the equilibrium cost of the trip. The proof consists of five results: a) I is bounded, b) the bottleneck is congested throughout the whole rush hour, except possibly at its boundaries, c) there is no congestion in t_q^v and $t_{q'}^v$, d) I is an open-right segment and e) the length of this segment is N/s.

First, Eq. (4) implies that the congestion level is bounded: $\forall t \in \Re$, $\bar{T}(t) \le C_{eq}^{v} / \alpha$. The limits of the normalized safety margin $m(t)$ are $\lim_{t \to \mp\infty} (m(t)) = \pm\infty$ and those of the expected cost of the trip are $\lim_{t \to \pm\infty} \bar{C}(t) = +\infty$. Considering that $\forall t \in I, \bar{C}(t) = C_{eq}^{v}$, I is necessarily bounded (point a).

To show b), let us assume that $\exists t_0 \in I \,/\, Q(t_0) = 0$. Because there is no congestion in t_0, $\forall t \in \Re$, $\bar{T}(t) \ge \bar{T}(t_0)$. From there, using both properties of lemma 1 shows that the expected cost of the trip is strictly higher than in t_0 on either $]-\infty, t_0[$ or on $]t_0, +\infty[$. Again, $\bar{C}(t) = C_{eq}^{v}$ $\forall t \in I$ implies that if $\exists t_0 \in I \,/\, Q(t_0) = 0$, then t_0 is either the lower or the upper boundary of I.

We note t_q^v and $t_{q'}^v$ the lower and upper boundaries of I. $\bar{C}$ is continuous, hence $\bar{C}\left(t_q^v\right) = \bar{C}\left(t_{q'}^v\right) = C_{eq}^{v}$. In addition, t_q^v being the start of the rush hour, we have $Q(t_q^v) = 0$. Let us assume $Q(t_{q'}^v) > 0$. $\forall t > t_{q'}^v, r(t) = 0 \Rightarrow \bar{T}_r\left(t_{q'}^v\right) = -1$ and

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$m_r\left(t_{q'}^{v}\right)=0$, hence $\left(\bar{C}_r\right)_{t=t_{q'}^{v}}=-\alpha$, where subscript r denotes the right-hand derivative.

The expected cost of leaving a little after $t_{q'}^{v}$ is lower than C_{eq}^{v}, in contradiction with the equilibrium condition. This proves $Q(t_{q'}^{v}) = 0$ (point c).

To show d), let us assume $\exists t_1 \in \left]t_q^v, t_{q'}^v\right[/ r\left(t_1\right)=0$. Using the same reasoning as before, $Q(t_1) > 0$ is absurd. This leaves the possibility $Q(t_1) = 0$. However, it would imply $\bar{C}\left(t_q^v\right)=\bar{C}\left(t_1\right)=\bar{C}\left(t_{q'}^v\right)=C_{eq}^v$ with $t_q^v<t_1<t_{q'}^v$ and $\bar{T}\left(t_q^v\right)=\bar{T}\left(t_1\right)=\bar{T}\left(t_{q'}^v\right)$. This is also impossible given lemma 1, so I must be a right-open segment.

Last, based on the previous results, the queue starts at t_q^v and ends at $t_{q'}^v$. When the bottleneck operates at full capacity, the time needed for N people to pass through the bottleneck is N/s, hence point e.

Annex B – Proof of Proposition 1

Using lemma 1, one can show that the equation $H(t+N/s,T_0+\mu) = H(t,T_0+\mu)$admits a unique solution in t. We note t_q^v the solution and $t_{q'}^v = t_q^v+N/s$.

We will now show that the following departure rate function r^v corresponds to a Nash equilibrium:

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$$\begin{cases} r^{v}(t) = \dfrac{\alpha s}{\alpha - \left(\beta F\left(m(t)\right) - \gamma\left(1 - F\left(m(t)\right)\right)\right)} & \text{if } t \in \left[t_q^v, t_{q'}^v\right[\\ r^{v}(t) = 0 & \text{if } t \notin \left[t_q^v, t_{q'}^v\right[\end{cases} \tag{24}$$

where $m(t) = \dfrac{t^* - t - \left(T_0 + \mu + \int_{t_q^v}^{t}\left(r^v(u) - s\right)du\right)}{\sigma}$.

On $]t_q^v, t_{q'}^v[$, this solution solves:

$$\bar{T}'(t) = \frac{r^v(t) - s}{s} = \frac{\beta F\left(m(t)\right) - \gamma\left(1 - F\left(m(t)\right)\right)}{\alpha - \left(\beta F\left(m(t)\right) - \gamma\left(1 - F\left(m(t)\right)\right)\right)} \Rightarrow \bar{C}'(t) = 0 \tag{25}$$

The expected cost of the trip is therefore constant on $\left[t_q^v, t_{q'}^v\right]$. Let C_{eq}^v be the corresponding value.

The next step involves showing that there is no congestion in both t_q^v and $t_{q'}^v$. This point is obvious for t_q^v, and $\bar{C}\left(t_q^v\right) = H\left(t_q^v, T_0 + \mu\right)$. For $t_{q'}^v$, demonstration is as follows:

$$H\left(t_q^v, T_0 + \mu\right) = H\left(t_{q'}^v, T_0 + \mu\right) \qquad \text{(definition of } t_q^v \text{ and } t_{q'}^v\text{)} \tag{26}$$

$$\Rightarrow \bar{C}\left(t_q^v\right) = H\left(t_{q'}^v, T_0 + \mu\right) \tag{27}$$

$$\Rightarrow \bar{C}\left(t_{q'}^v\right) = H\left(t_{q'}^v, T_0 + \mu\right) \qquad \text{(equilibrium condition: } \bar{C}\left(t_{q'}^v\right) = \bar{C}\left(t_q^v\right)\text{)} \tag{28}$$

$$\Rightarrow H\left(t_{q'}^v, \bar{T}\left(t_{q'}^v\right)\right) = H\left(t_{q'}^v, T_0 + \mu\right) \tag{29}$$

$$\Rightarrow \bar{T}\left(t_{q'}^v\right) = T_0 + \mu \text{ and } Q\left(t_{q'}^v\right) = 0 \qquad \text{(using the 2nd property of lemma 1).} \tag{30}$$

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t_q^v being the solution of the equation $H(t+N/s,T_0+\mu) = H(t,T_0+\mu)$, lemma 1 also ensures that $\bar{C}(t) > C_{eq}^v \ \forall t \notin [t_q^v, t_{q'}^v]$. In sum, the expected cost of the trip is constant and minimum on $[t_q^v, t_{q'}^v]$.

Last, $\bar{C}(t) = C_{eq}^v$ on $[t_q^v, t_{q'}^v]$ implies that $Q(t) > 0 \ \forall t \in]t_q^v, t_{q'}^v[$ (see proof of Lemma 2).

Because $Q\left(t_{q'}^v\right) = 0$ and $\forall t \in]t_q^v, t_{q'}^v[, Q(t) > 0$, we can use Eq. (2):

$$Q\left(t_{q'}^v\right) = \int_{t_q^v}^{t_{q'}^v} \left(r^v(u) - s\right) du = 0 \quad \Rightarrow \quad \int_{t_q^v}^{t_{q'}^v} r^v(u)\, du = s\left(t_{q'}^v - t_q^v\right) = N. \tag{31}$$

At the end of the rush hour, the total number of departures exactly adds up to N.

We have proved that the departure rate function r^v is a Nash equilibrium. We must now prove that it is unique.

If z is a departure rate function corresponding to a Nash equilibrium, using Lemma 2, the set $\{t \ / \ z(t) > 0\}$ must be of the form $[a,b[$ with the conditions $b = a + N/s$ and $Q(a) = Q(b) = 0$. Equilibrium condition $\bar{C}(b) = \bar{C}(a)$ implies $H(a+N/s,T_0+\mu) = H(a,T_0+\mu)$, hence $a = t_q^v$ and $b = t_{q'}^v$. Then Lemma 3 implies:

$$\forall t \in]t_q^v, t_{q'}^v[\ , \ \bar{T}'(t) = \frac{\beta F\left(m(t)\right) - \gamma\left(1 - F\left(m(t)\right)\right)}{\alpha - \left(\beta F\left(m(t)\right) - \gamma\left(1 - F\left(m(t)\right)\right)\right)} \ \Rightarrow \ z(t) = r^v(t). \tag{32}$$

The equilibrium solution is unique indeed.

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Annex C – Proof of Proposition 3

Using Eq. (11), the expected costs of leaving at t_q and $t_{q'}$ are:

$$\bar{C}\left(t_q\right)=\alpha\left(T_0+\mu\right)+\gamma\frac{N}{s}F\left(\frac{\delta}{\beta}\frac{N}{s\sigma}\right)+\left(\beta+\gamma\right)\sigma G\left(\frac{\delta}{\beta}\frac{N}{s\sigma}\right)-\frac{\gamma\delta}{\beta}\frac{N}{s} \tag{33}$$

$$\bar{C}\left(t_{q'}\right)=\alpha\left(T_0+\mu\right)-\beta\frac{N}{s}F\left(-\frac{\delta}{\gamma}\frac{N}{s\sigma}\right)+\left(\beta+\gamma\right)\sigma G\left(-\frac{\delta}{\gamma}\frac{N}{s\sigma}\right)+\delta\frac{N}{s} \tag{34}$$

Subtracting the two yields:

$$\bar{C}\left(t_{q'}\right)-\bar{C}\left(t_q\right)=\left(\beta+\gamma\right)\sigma\left(G\left(-\frac{\delta}{\gamma}\frac{N}{s\sigma}\right)-G\left(\frac{\delta}{\beta}\frac{N}{s\sigma}\right)\right)+\gamma\frac{N}{s}\left(1-F\left(\frac{\delta}{\beta}\frac{N}{s\sigma}\right)\right)-\beta\frac{N}{s}F\left(-\frac{\delta}{\gamma}\frac{N}{s\sigma}\right) \tag{35}$$

$$=\left(\beta+\gamma\right)\sigma\left(\int_{\frac{\gamma}{\beta+\gamma}\frac{N}{s\sigma}}^{+\infty}\left(\frac{\gamma}{\beta+\gamma}\frac{N}{s\sigma}-u\right)f\left(u\right)du+\int_{-\infty}^{-\frac{\beta}{\beta+\gamma}\frac{N}{s\sigma}}\left(-\frac{\beta}{\beta+\gamma}\frac{N}{s\sigma}-u\right)f\left(u\right)du\right) \tag{36}$$

$$=\left(\beta+\gamma\right)\sigma\left(J_L\left(-\frac{\beta}{\beta+\gamma}\frac{N}{s\sigma}\right)-J_R\left(\frac{\gamma}{\beta+\gamma}\frac{N}{s\sigma}\right)\right) \tag{37}$$

If $J_L\left(-\frac{\beta}{\beta+\gamma}\frac{N}{s\sigma}\right)>J_R\left(\frac{\gamma}{\beta+\gamma}\frac{N}{s\sigma}\right)$, $\bar{C}\left(t_{q'}\right)>\bar{C}\left(t_q\right)$. The equilibrium condition $\bar{C}\left(t_{q'}^{v}\right)=\bar{C}\left(t_q^{v}\right)$ implies $t_q^{v} < t_q$. Conversely, if $J_L\left(-\frac{\beta}{\beta+\gamma}\frac{N}{s\sigma}\right)<J_R\left(\frac{\gamma}{\beta+\gamma}\frac{N}{s\sigma}\right)$ then $\bar{C}\left(t_{q'}\right)<\bar{C}\left(t_q\right)$ hence $t_q^{v} > t_q$.

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Annex D – Proof of Proposition 4

Let us consider H as a function of t, T and σ instead of t and T only:

$$H(t,T,\sigma)=\alpha T+(\beta+\gamma)\sigma\left(\hat{m}(t,T,\sigma)F\left(\hat{m}(t,T,\sigma)\right)+G\left(\hat{m}(t,T,\sigma)\right)\right)-\gamma\sigma\hat{m}(t,T,\sigma)$$
$$\text{with } \hat{m}(t,T,\sigma)=\frac{t^*-t-T}{\sigma} \tag{38}$$

We have:

$$\frac{\partial H(t,T,\sigma)}{\partial\sigma}=(\beta+\gamma)G\left(\frac{t^*-t-T}{\sigma}\right)\geq 0 \tag{39}$$

If the support of f is unbounded on both sides, $G(x)>0\ \forall x\in\Re$ and $\frac{\partial H(t,T,\sigma)}{\partial\sigma}>0$.

The equilibrium cost of the trip $C_e(\sigma)$ is given by:

$$C_e(\sigma)=H\left(t_q(\sigma);T_0+\mu;\sigma\right)=H\left(t_{q'}(\sigma);T_0+\mu;\sigma\right) \text{ with } t_{q'}(\sigma)-t_q(\sigma)=\frac{N}{s} \tag{40}$$

where the rush hour $[t_q(\sigma), t_{q'}(\sigma)]$ is a function of σ.

The derivative of $C_e(\sigma)$ is:

$$\begin{aligned}\frac{dC_e(\sigma)}{d\sigma}&=\frac{\partial H}{\partial t}\left(t_q(\sigma);T_0+\mu;\sigma\right)\frac{dt_q(\sigma)}{d\sigma}+\frac{\partial H}{\partial\sigma}\left(t_q(\sigma);T_0+\mu;\sigma\right)\\&=\frac{\partial H}{\partial t}\left(t_{q'}(\sigma);T_0+\mu;\sigma\right)\frac{dt_{q'}(\sigma)}{d\sigma}+\frac{\partial H}{\partial\sigma}\left(t_{q'}(\sigma);T_0+\mu;\sigma\right)\end{aligned} \tag{41}$$

First, we have: $t_{q'}(\sigma)-t_q(\sigma)=N/s\Rightarrow\frac{dt_{q'}(\sigma)}{d\sigma}=\frac{dt_q(\sigma)}{d\sigma}$. From lemma 1 and the condition $H\left(t_q(\sigma);T_0+\mu;\sigma\right)=H\left(t_{q'}(\sigma);T_0+\mu;\sigma\right)$, we also know that $\frac{\partial H}{\partial t}\left(t_{q'}(\sigma);T_0+\mu;\sigma\right)$ and

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$\frac{\partial H}{\partial t}\left(t_{q'}(\sigma);T_0+\mu;\sigma\right)$ are of opposite signs. This holds true for $\frac{\partial H}{\partial t}\left(t_q(\sigma);T_0+\mu;\sigma\right)\frac{dt_q(\sigma)}{d\sigma}$ and $\frac{\partial H}{\partial t}\left(t_{q'}(\sigma);T_0+\mu;\sigma\right)\frac{dt_{q'}(\sigma)}{d\sigma}$, so one of the two must be positive. Considering that the term in $\frac{\partial H}{\partial \sigma}$ is also positive (strictly positive if *f* is unbounded on both sides), we have $\frac{dC_e(\sigma)}{d\sigma}\geq 0$ (>0 if *f* is unbounded), which is Proposition 4.

Annex E – Proof of Proposition 5

Let us assume $\sigma M \leq \min\left(\frac{\delta}{\gamma}\frac{N}{s},\frac{\delta}{\beta}\frac{N}{s}\right)$, where M is the maximum value of the normalized random delay (in absolute terms). In the deterministic case $t^*-t_q-T_0-\mu=\frac{\delta}{\beta}\frac{N}{s}$ and $t^*-t_{q'}-T_0-\mu=\frac{-\delta}{\gamma}\frac{N}{s}$. Given the previous assumption, one can easily show that $F(m(t_q)) = 1$, $F(m(t_{q'})) = 0$, and that $G(m(t_q)) = G(m(t_{q'})) = 0$. Using this, it is equally easy to show that t_q and $t_{q'}$ verify the condition for t_q^v and $t_{q'}^v$, hence $t_q^v = t_q$ and $t_{q'}^v = t_{q'}$.

Using Eq.(11), the expected cost of the trip for an individual leaving at t_q^v is $\bar{C}\left(t_q^v\right)=\alpha\left(T_0+\mu\right)+\delta\frac{N}{s}$, which is the same cost as in the deterministic case.